11. Special Theory of Relativity

1. The Situation Before 1900

- **11.3-5**

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}\, t = \boldsymbol{x}'(\boldsymbol{x}, t)$$
$$t' = t = t'(\boldsymbol{x}, t)$$

$\Longrightarrow$

$$\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{v}\, t' = \boldsymbol{x}(\boldsymbol{x}', t')$$
$$t = t' = t(\boldsymbol{x}', t')$$

or $\quad x^\alpha = x'^\alpha + v^\alpha t' \qquad \alpha, \beta = 1, 2, 3$

$$\frac{\partial x^\beta}{\partial x'^\alpha} = \delta_{\alpha\beta} \qquad \frac{\partial t}{\partial x'^\alpha} = 0$$

$$\frac{\partial x^\beta}{\partial t'} = v^\beta \qquad \frac{\partial t}{\partial t'} = 1$$

Thus

$$\frac{\partial}{\partial x'^\alpha} = \sum_\beta \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta} + \frac{\partial t}{\partial x'^\alpha} \frac{\partial}{\partial t} = \frac{\partial}{\partial x^\alpha}$$

or

$$\nabla' = \nabla$$

and

$$\frac{\partial}{\partial t'} = \sum_\beta \frac{\partial x^\beta}{\partial t'} \frac{\partial}{\partial x^\beta} + \frac{\partial t}{\partial t'} \frac{\partial}{\partial t}$$

$$= \sum_\beta v^\beta \frac{\partial}{\partial x^\beta} + \frac{\partial}{\partial t}$$

$$= \boldsymbol{v} \cdot \nabla + \frac{\partial}{\partial t}$$

$$\frac{\partial^2}{\partial t'^2} = \left(\boldsymbol{v} \cdot \nabla + \frac{\partial}{\partial t}\right)\left(\boldsymbol{v} \cdot \nabla + \frac{\partial}{\partial t}\right)$$

$$= \boldsymbol{v} \cdot \nabla\, \boldsymbol{v} \cdot \nabla + 2\, \boldsymbol{v} \cdot \nabla \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}$$

$$\frac{d x'^\alpha}{d t'} = \frac{d x^\alpha - v^i\, d t}{d t} = \frac{d x^\alpha}{d t} - v^\alpha$$

or

$$\frac{d \boldsymbol{x}'}{d t'} = \frac{d \boldsymbol{x}}{d t} - \boldsymbol{v}$$

For a set of N particles:

$$\frac{d\,\boldsymbol{x}_i\,'}{d\,t\,'} = \frac{d\,\boldsymbol{x}_i}{d\,t} - \boldsymbol{v} \qquad i,\ j = 1,\ 2,\ \ldots N$$

or

$$\boldsymbol{v}_i\,' = \boldsymbol{v}_i - \boldsymbol{v}$$

Thus

$$\frac{d\,\boldsymbol{v}_i\,'}{d\,t\,'} = \frac{d\,\boldsymbol{v}_i}{d\,t\,'} = \frac{d\,\boldsymbol{v}_i}{d\,t}$$

$$\boldsymbol{x}_i\,' - \boldsymbol{x}_j\,' = \boldsymbol{x}_i - \boldsymbol{x}_j$$

$$\nabla_i\,' = \nabla_i$$

Thus, under a Galilean transformation:

$$m_i\,\frac{d\,\boldsymbol{v}_i\,'}{d\,t\,'} = -\nabla_i\,'\sum_j V_{i\,j}(\boldsymbol{x}_i\,' - \boldsymbol{x}_j\,') \qquad (11.2)$$

becomes

$$m_i\,\frac{d\,\boldsymbol{v}_i}{d\,t} = -\nabla_i\sum_j V_{i\,j}(\,|\,\boldsymbol{x}_i - \boldsymbol{x}_j\,|) \qquad (11.3)$$

while

$$\left[\nabla\,'^2 - \frac{1}{c^2}\,\frac{\partial^2}{\partial\,t\,'^2}\right]\psi = 0 \qquad (11.4)$$

becomes

$$\left[\nabla^2 - \frac{1}{c^2}\left(\boldsymbol{v}\cdot\nabla\ \boldsymbol{v}\cdot\nabla + 2\,\boldsymbol{v}\cdot\nabla\,\frac{\partial}{\partial\,t} + \frac{\partial^2}{\partial\,t^2}\right)\right]\psi = 0 \qquad (11.5)$$

A plane wave solution $\psi = C\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x} - i\,\omega\,t}$ to (11.5) requires

$$-\boldsymbol{k}^2 + \frac{1}{c^2}\left[(\,\boldsymbol{v}\cdot\boldsymbol{k})^2 + 2\,(\boldsymbol{v}\cdot\boldsymbol{k}\,)\,\omega + \omega^2\right] = 0$$

or

$$(\,\omega + \boldsymbol{v}\cdot\boldsymbol{k}\,)^2 = c^2\,\boldsymbol{k}^2$$

$$\omega = -\boldsymbol{v}\cdot\boldsymbol{k} \pm c\,k \qquad k = \,|\,\boldsymbol{k}\,|$$

$$\frac{\omega}{c} = \pm k\left(1 \mp \frac{v}{c}\cos\theta\right)$$

where θ is the angle between $\boldsymbol{v}$ & $\boldsymbol{k}$ while the upper & lower sign denotes waves traveling to the + and $-$ $\hat{\boldsymbol{k}}$ direction, respectively.

Thus the phase & group velocity are both equal to

$$c \mp v\cos\theta = c \mp \hat{\boldsymbol{k}}\cdot\boldsymbol{v}$$

- **Schrodinger Eq**

From 11.3-5, we see that

$$-\frac{\hbar^2}{2\,m}\,\nabla\,'^2\,\psi\,' + V\,\psi\,' = i\,\hbar\,\frac{\partial\,\psi\,'}{\partial\,t\,'}$$

becomes

$$-\frac{\hbar^2}{2\,m}\,\nabla^2\psi\,' + V\,\psi\,' = i\,\hbar\left(\boldsymbol{v}\cdot\nabla + \frac{\partial}{\partial\,t}\right)\psi\,'$$

Let $\psi' = \psi\,\chi$

$\Longrightarrow$

$$\nabla\,\psi' = (\nabla\psi)\,\chi + \psi\,\nabla\chi$$
$$\nabla^2\,\psi' = \left(\nabla^2\psi\right)\chi + 2\,\nabla\psi\cdot\nabla\chi + \psi\,\nabla^2\chi$$
$$\frac{\partial}{\partial t}\,\psi' = \left(\frac{\partial}{\partial t}\psi\right)\chi + \psi\,\frac{\partial}{\partial t}\chi$$

$$-\frac{\hbar^2}{2\,m}\,\nabla^2\psi' + V\,\psi' = -\frac{\hbar^2}{2\,m}\left[\left(\nabla^2\psi\right)\chi + 2\,\nabla\psi\cdot\nabla\chi + \psi\,\nabla^2\chi\right] + V\,\psi\,\chi$$
$$i\,\hbar\left(\boldsymbol{v}\cdot\nabla + \frac{\partial}{\partial t}\right)\psi' = i\,\hbar\left[(\boldsymbol{v}\cdot\nabla\psi)\,\chi + \psi\,\boldsymbol{v}\cdot\nabla\chi + \left(\frac{\partial}{\partial t}\psi\right)\chi + \psi\,\frac{\partial}{\partial t}\chi\right]$$

Thus, if we wish to have

$$-\frac{\hbar^2}{2\,m}\,\nabla^2\psi + V\,\psi = i\,\hbar\,\frac{\partial\,\psi}{\partial\,t}$$

we must require

$$-\frac{\hbar^2}{2\,m}\left[2\,\nabla\psi\cdot\nabla\chi + \psi\,\nabla^2\chi\right] = i\,\hbar\left[(\boldsymbol{v}\cdot\nabla\psi)\,\chi + \psi\,\boldsymbol{v}\cdot\nabla\chi + \psi\,\frac{\partial}{\partial t}\chi\right]$$

in such a way that a χ can be found independent of ψ.

Dividing the last eq with ψ, we have

$$-\frac{\hbar^2}{2\,m}\left[2\,\frac{\nabla\psi}{\psi}\cdot\nabla\chi + \nabla^2\chi\right] = i\,\hbar\left[\left(\boldsymbol{v}\cdot\frac{\nabla\psi}{\psi}\right)\chi + \boldsymbol{v}\cdot\nabla\chi + \frac{\partial}{\partial t}\chi\right]$$

or

$$-\frac{\nabla\psi}{\psi}\cdot\left[\frac{\hbar^2}{m}\,\nabla\chi + i\,\hbar\,\boldsymbol{v}\,\chi\right] = \frac{\hbar^2}{2\,m}\,\nabla^2\chi + i\,\hbar\left[\boldsymbol{v}\cdot\nabla\chi + \frac{\partial}{\partial t}\chi\right]$$

For χ to be found independent of ψ, the left side must vanishes:

$$\frac{\hbar^2}{m}\,\nabla\chi + i\,\hbar\,\boldsymbol{v}\,\chi = 0 = \left(\frac{\hbar^2}{m}\,\nabla + i\,\hbar\,\boldsymbol{v}\right)\chi$$

so that

$$\frac{\hbar^2}{2\,m}\,\nabla^2\chi + i\,\hbar\left[\boldsymbol{v}\cdot\nabla\chi + \frac{\partial}{\partial t}\chi\right] = 0$$

The 1st eq gives

$$\chi = C(t)\,e^{-i\,\frac{m}{\hbar}\,\boldsymbol{v}\cdot\boldsymbol{x}}$$

so that

$$\nabla\chi = -i\,\frac{m}{\hbar}\,\boldsymbol{v}\,\chi$$
$$i\,\hbar\,\boldsymbol{v}\cdot\nabla\chi = m\,\boldsymbol{v}^2\,\chi$$
$$\frac{\hbar^2}{2\,m}\,\nabla^2\chi = -\frac{1}{2}\,m\,\boldsymbol{v}^2\,\chi$$
$$i\,\hbar\,\frac{\partial}{\partial t}\,\chi = i\,\hbar\,\frac{\partial\,C}{C\,\partial\,t}\,\chi$$

Thus, the 2nd eq becomes

$$\frac{1}{2} m \mathbf{v}^2 \chi + i \hbar \frac{\partial C}{C \partial t} \chi = 0$$

or

$$\frac{1}{2} m \mathbf{v}^2 + i \hbar \frac{\partial C}{C \partial t} = 0$$

Hence

$$C(t) \propto e^{\frac{i}{2\hbar} m \mathbf{v}^2 t}$$

and

$$\chi \propto e^{-i \frac{m}{\hbar} \mathbf{v} \cdot \mathbf{x} + \frac{i}{2\hbar} m \mathbf{v}^2 t}$$

$$\psi' = \psi \chi \propto \psi e^{-i \frac{m}{\hbar} \mathbf{v} \cdot \mathbf{x} + \frac{i}{2\hbar} m \mathbf{v}^2 t}$$

$$\psi \propto \psi' e^{i \frac{m}{\hbar} \mathbf{v} \cdot \mathbf{x} - \frac{i}{2\hbar} m \mathbf{v}^2 t}$$

- Fizeau's investigation [Becker §72.]

Along with the purely electromagnetic experiments on moving bodies with which we have become acquainted in the forego-ing paragraphs, we consider now a purely optical investigation related to the propagation of light in moving media. Accord-ing to the theory the light must, as we shall see later, travel faster in the direction of motion of the medium than in the opposite direction, i.e. the light is, so to speak, carried along with the moving medium.

The first experimental investigation of the magnitude of this transportation was carried out by Fizeau. His experimental arrangement is shown in figure 63. Light from the source L, divided into two parts by the half- silvered mirror PP, pursues paths in opposite directions around the circuit as shown in the figure. The two beams are again united in the half- silvered mirror and arrive at an interference apparatus B for observation. Along their way the two light beams pass through two tubes R_1 and R_2 through which water flows in opposite directions. One light beam goes through both tubes in the same direction as the water flow; the other beam in the opposite direction. This has the result that the first beam requires less time than the second for the journey from the half- silvered mirror through the tube system and back again. Thus the two beams arrive at the interference apparatus with a difference of phase. From the position in the apparatus of the interference fringes obtained, we can determine this time difference and thereby also the amount of speed increase.

Fig. 63. —Measurement of the Fizeau entrainment coefficient.

We wish now to obtain a clear picture of the expected result on the basis of the formulae developed in §70. The Maxwell equations for the electromagnetic field in an uncharged non-magnetic insulating medium ($\rho = 0$, $\boldsymbol{g} = 0$, $\mu = 1$) moving with velocity v are:

$$\text{curl}\, \boldsymbol{H} = \frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{D} \qquad \text{div}\, \boldsymbol{D} = 0 \qquad \boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P}$$

$$\text{curl}\, \boldsymbol{E} = -\frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{B} \qquad \text{div}\, \boldsymbol{B} = 0 \qquad \boldsymbol{H} = \boldsymbol{B} - 4\pi \boldsymbol{P} \times \frac{\boldsymbol{v}}{c} \qquad (72.1)$$

According to (70.5), P is given by

$$\boldsymbol{P} = \frac{\epsilon - 1}{4\pi} \boldsymbol{E}^* = \frac{\epsilon - 1}{4\pi} \left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \right) \qquad (72.2)$$

Owing to the dependence of ϵ on frequency, care is required in applying the foregoing relation to a process involving light propagation. If we consider the propagation of a plane wave, as described in (57.10), given by the expressions

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp i (\omega t - \boldsymbol{k} \cdot \boldsymbol{r}) \qquad \boldsymbol{B} = \boldsymbol{B}_0 \exp i (\omega t - \boldsymbol{k} \cdot \boldsymbol{r}) \qquad (72.3)$$

there then acts at the location of an atom moving with the matter, the location of the atom being designated by $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}\, t$, the apparent field intensity

$$E^* = E_0^* \exp i\,(\omega\, t - \boldsymbol{k}\cdot\boldsymbol{r}_0 - \boldsymbol{k}\cdot\boldsymbol{v}\, t)$$

The field's apparent frequency at the location of this atom is given by

$$\omega^* = \omega - \boldsymbol{k}\cdot\boldsymbol{v} \qquad (72.4)$$

We have thus to put into (72.2) the permittivity appropriate to this "Doppler- shifted" frequency ω*; thus we have in the following that

$$\epsilon = \epsilon(\omega^*) \qquad (72.5)$$

We return now to equations (72.1). From these we obtain, with the unit phase factor from (72.3) [compare (57.13) or (58.9)]

$$-\boldsymbol{k}\times\boldsymbol{H} = \frac{\omega}{c}\,\boldsymbol{D} \qquad \boldsymbol{k}\cdot\boldsymbol{D} = 0$$

$$-\boldsymbol{k}\times\boldsymbol{E} = -\frac{\omega}{c}\,\boldsymbol{B} \qquad \boldsymbol{k}\cdot\boldsymbol{B} = 0 \qquad (72.6)$$

From these equations we see that D and B are perpendicular to k, but that in general (and in contrast to the case of media at rest), this is no longer true for the electric vector E. We have instead

$$0 = \boldsymbol{k}\cdot\boldsymbol{D} = \boldsymbol{k}\cdot\boldsymbol{E} + 4\pi\,\boldsymbol{k}\cdot\boldsymbol{P} = \epsilon\,\boldsymbol{k}\cdot\boldsymbol{E} + (\epsilon - 1)\,\boldsymbol{k}\cdot\left(\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)$$

Accordingly, $\boldsymbol{k}\cdot\boldsymbol{E}$ is always different from zero whenever the three vectors k, v, and B do not lie in a plane. Analogously, we have for the magnetic field strength $\boldsymbol{H} = \boldsymbol{B} - 4\pi\,\boldsymbol{P}\times\boldsymbol{v}/c$. Here then, as a consequence, the Poynting vector does not in general have the direction of k.

For our further calculations it will be expedient to express E and H in equations (72.6) in terms of D and B. We have obviously, on account of (72.2),

$$\boldsymbol{E} = \frac{\boldsymbol{D}}{\epsilon} - \frac{\epsilon - 1}{\epsilon}\,\frac{\boldsymbol{v}}{c}\times\boldsymbol{B} \qquad \boldsymbol{P} = \frac{\epsilon - 1}{4\pi\epsilon}\left(\boldsymbol{D} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right) \qquad (72.7)$$

Limiting ourselves to the case of small velocities so that we may neglect terms of the order $(v/c)^2$ compared to 1, and remembering that $\boldsymbol{k}\cdot\boldsymbol{D} = 0$ and $\boldsymbol{k}\cdot\boldsymbol{B} = 0$, we obtain from (72.6) the relationships

$$-\boldsymbol{k}\times\boldsymbol{B} = \left(\frac{\omega}{c} - \frac{\epsilon - 1}{\epsilon}\,\boldsymbol{k}\cdot\frac{\boldsymbol{v}}{c}\right)\boldsymbol{D}$$

$$-\frac{1}{\epsilon}\,\boldsymbol{k}\times\boldsymbol{D} = -\left(\frac{\omega}{c} - \frac{\epsilon - 1}{\epsilon}\,\boldsymbol{k}\cdot\frac{\boldsymbol{v}}{c}\right)\boldsymbol{B}$$

Making use of the familiar vector formula

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$$

and eliminating B and D respectively by substitution, we arrive at the expression

$$\left(\frac{\omega}{c} - \frac{\epsilon - 1}{\epsilon}\,\boldsymbol{k}\cdot\frac{\boldsymbol{v}}{c}\right)^2 = \frac{k^2}{\epsilon}$$

Upon introducing in addition the angle α between the direction of propagation of the waves and the direction of motion of the material, we find for the wave velocity

$$w = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon}} + \frac{\epsilon - 1}{\epsilon}\, v\cos\alpha \qquad (72.8)$$

Further, we make use of the refractive index formula $\sqrt{\epsilon} = n$, taking for n the value appropriate to the frequency ω^* as given by (72.4). In the first approximation we have for ω^*

$$\omega^* = \omega\left(1 - \frac{v}{w}\cos\alpha\right) \approx \omega\left(1 - \frac{n\,v}{c}\cos\alpha\right)$$

thus we have, also approximately,

$$\sqrt{\epsilon(\omega^*)} = n(\omega^*) = n(\omega) - \frac{d\,n(\omega)}{d\,\omega}\cdot\frac{n\,v\,\omega}{c}\cos\alpha$$

From (72.8) then we obtain as the general law for the propagation of light in slowly moving media (with $\omega_0 = c/n$)

$$w = w_0 + \left(1 - \frac{1}{n^2} + \frac{\omega}{n}\frac{d\,n}{d\,\omega}\right)v\cos\alpha \qquad (72.9)$$

We might picture this result as if the light, in its propagation in the direction of the motion, were "blown" along. It is astonishing, however, that this entrainment of the light by the moving matter is not a complete speeding up but, by contrast with the case of the propagation of sound in a moving liquid, is only partial. The factor

$$\eta = 1 - \frac{1}{n^2} + \frac{\omega}{n}\frac{d\,n}{d\,\omega} \qquad (72.10)$$

characterizing the degree of speeding up is designated the Fizeau entrainment coefficient. In Fizeau's apparatus one light beam went through the two tubes (total length $2l$) in the direction of flow of the water, while the other beam went through in the opposite direction. The difference in travel time is thus

$$\Delta\,t = \frac{2\,l}{w_0 - \eta\,v} - \frac{2\,l}{w_0 + \eta\,v} \approx \frac{4\,l\,\eta\,v}{{w_0}^2}$$

For the case of water Fizeau was able to make an interferometric determination of $\Delta\,t$, thus quantitativeiy confifining the entrainment coefficient formula (72.10).

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2. Some Recent Experiments

■ 11.8

With

$$\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{v}\,t'$$
$$t = t'$$

we have

$$t - \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c} = t' - \left(\frac{\boldsymbol{n}\cdot\boldsymbol{x}'}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\,t'\right)$$

$$= \; t'\left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right) - \frac{\boldsymbol{n}\cdot\boldsymbol{x}'}{c}$$

Thus,

$$\omega\left(t-\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{c}\right)=\omega'\left(t'-\frac{\boldsymbol{n}'\cdot\boldsymbol{x}'}{c'}\right) \qquad (11.7)$$

becomes

$$\omega\left[t'\left(1-\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right)-\frac{\boldsymbol{n}\cdot\boldsymbol{x}'}{c}\right]=\omega'\left(t'-\frac{\boldsymbol{n}'\cdot\boldsymbol{x}'}{c'}\right)$$

Since this must be true for all $\boldsymbol{x}'$ & t', we have

$$\omega\left(1-\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right)=\omega'$$

$$\omega\frac{\boldsymbol{n}}{c}=\omega'\frac{\boldsymbol{n}'}{c'}$$

The second eq implies

$$\boldsymbol{n}=\boldsymbol{n}'$$

$$\frac{\omega}{c}=\frac{\omega'}{c'}$$

The last gives

$$\frac{\omega}{c}=\frac{\omega}{c'}\left(1-\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right)$$

or

$$c'=c\left(1-\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{c}\right)=c-\boldsymbol{n}\cdot\boldsymbol{v}$$

- **11.10**

$$\boldsymbol{m}=\frac{c\,\boldsymbol{n}-\boldsymbol{v}_0}{|\,c\,\boldsymbol{n}-\boldsymbol{v}_0\,|}$$

$\longrightarrow$

$$\boldsymbol{n}=\frac{1}{c}\left|\,c\,\boldsymbol{n}-\boldsymbol{v}_0\,\right|\boldsymbol{m}+\frac{1}{c}\boldsymbol{v}_0$$

$$=\left|\,\boldsymbol{n}-\frac{\boldsymbol{v}_0}{c}\right|\boldsymbol{m}+\frac{\boldsymbol{v}_0}{c}$$

Now

$$\left| \boldsymbol{n} - \frac{\boldsymbol{v}_0}{c} \right| = \sqrt{1 - 2\,\boldsymbol{n} \cdot \frac{\boldsymbol{v}_0}{c} + \left(\frac{\boldsymbol{v}_0}{c}\right)^2}$$

$$= \sqrt{1 - 2\left| \boldsymbol{n} - \frac{\boldsymbol{v}_0}{c} \right| \boldsymbol{m} \cdot \frac{\boldsymbol{v}_0}{c} - \left(\frac{\boldsymbol{v}_0}{c}\right)^2}$$

$$\simeq 1 - \left| \boldsymbol{n} - \frac{\boldsymbol{v}_0}{c} \right| \boldsymbol{m} \cdot \frac{\boldsymbol{v}_0}{c} + O\left[\left(\frac{\boldsymbol{v}_0}{c}\right)^2\right]$$

$$= \frac{1}{1 + \boldsymbol{m} \cdot \frac{\boldsymbol{v}_0}{c}} + O\left[\left(\frac{\boldsymbol{v}_0}{c}\right)^2\right]$$

$$\simeq 1 - \boldsymbol{m} \cdot \frac{\boldsymbol{v}_0}{c} + O\left[\left(\frac{\boldsymbol{v}_0}{c}\right)^2\right]$$

Hence

$$\boldsymbol{n} \simeq \left(1 - \boldsymbol{m} \cdot \frac{\boldsymbol{v}_0}{c}\right)\boldsymbol{m} + \frac{\boldsymbol{v}_0}{c} \qquad (11.10)$$

▪ 11.11

$$\omega_1 = \omega\left(1 - \frac{\boldsymbol{n} \cdot \boldsymbol{v}_1}{c}\right) \qquad \omega_0 = \omega\left(1 - \frac{\boldsymbol{n} \cdot \boldsymbol{v}_0}{c}\right)$$

$\longrightarrow$

$$\omega_1 = \omega_0 \left(\frac{1 - \frac{\boldsymbol{n} \cdot \boldsymbol{v}_1}{c}}{1 - \frac{\boldsymbol{n} \cdot \boldsymbol{v}_0}{c}}\right)$$

$$\simeq \omega_0 \left(1 - \frac{\boldsymbol{n} \cdot \boldsymbol{v}_1}{c}\right)\left(1 + \frac{\boldsymbol{n} \cdot \boldsymbol{v}_0}{c}\right) + O\left[\left(\frac{\boldsymbol{v}_0}{c}\right)^2\right]$$

$$\simeq \omega_0 \left[1 - \boldsymbol{n} \cdot \left(\frac{\boldsymbol{v}_1}{c} - \frac{\boldsymbol{v}_0}{c}\right)\right] + O\left[\left(\frac{\boldsymbol{v}_0}{c}, \frac{\boldsymbol{v}_1}{c}\right)^2\right]$$

$$\simeq \omega_0 \left\{ 1 - \left[\left(1 - \boldsymbol{m} \cdot \frac{\boldsymbol{v}_0}{c}\right)\boldsymbol{m} + \frac{\boldsymbol{v}_0}{c}\right] \cdot \frac{\boldsymbol{u}_1}{c}\right\} + O\left[\left(\frac{\boldsymbol{v}_0}{c}, \frac{\boldsymbol{v}_1}{c}\right)^2\right]$$

where

$$\boldsymbol{u}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_0$$

$$\boldsymbol{n} \simeq \left(1 - \boldsymbol{m} \cdot \frac{\boldsymbol{v}_0}{c}\right)\boldsymbol{m} + \frac{\boldsymbol{v}_0}{c}$$

Similarly

$$\omega_2 = \omega\left(1 - \frac{\boldsymbol{n}\cdot\boldsymbol{v_2}}{c}\right)$$

$$\simeq \omega_0\left\{1 - \left[\left(1 - \boldsymbol{m}\cdot\frac{\boldsymbol{v}_0}{c}\right)\boldsymbol{m} + \frac{\boldsymbol{v}_0}{c}\right]\cdot\frac{\boldsymbol{u_2}}{c}\right\} + O\left[\left(\frac{\boldsymbol{v}_0}{c}, \frac{\boldsymbol{v}_2}{c}\right)^2\right]$$

where

$$\boldsymbol{u}_2 = \boldsymbol{v}_2 - \boldsymbol{v}_0$$

Hence

$$\frac{\omega_1 - \omega_2}{\omega_0} \simeq \frac{1}{c}(\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot\left[\left(1 - \boldsymbol{m}\cdot\frac{\boldsymbol{v}_0}{c}\right)\boldsymbol{m} + \frac{\boldsymbol{v}_0}{c}\right]$$

■ 11.12

For the configuration in Fig.11.2, $\boldsymbol{m}$ points from 1 to 2 and

$$\boldsymbol{u}_2 - \boldsymbol{u}_1 = 2\,\boldsymbol{u}_2$$

$$(\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot\boldsymbol{m} = 0$$

$$(\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot\boldsymbol{v}_0 = (\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot(\boldsymbol{v}_0)_\perp$$

Let the unit vector along $(\boldsymbol{v}_0)_\perp$ be $\hat{\boldsymbol{x}}$, the axis of rotation be $\hat{\boldsymbol{z}}$,

$$\boldsymbol{u}_1 = u_1\left(-\hat{\boldsymbol{x}}\sin\Omega\,t + \hat{\boldsymbol{y}}\cos\Omega\,t\right)$$

where $u_1 = R\,\Omega$ is the magnitude of $\boldsymbol{u}_1$, we have

$$(\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot(\boldsymbol{v}_0)_\perp = 2\,R\,\Omega\,\sin\Omega\,t\cdot|(\boldsymbol{v}_0)_\perp|$$

so that

$$\frac{\omega_1 - \omega_2}{\omega_0} \simeq \frac{1}{c}(\boldsymbol{u}_2 - \boldsymbol{u}_1)\cdot\frac{\boldsymbol{v}_0}{c}$$

$$= \frac{2\,\Omega\,R}{c^2}\sin\Omega\,t\cdot\left|(\boldsymbol{v}_0)_\perp\right| \qquad (11.12)$$

X

For high frequencies,

$$\epsilon(\omega) = 1 - \frac{{\omega_p}^2}{\omega^2} \qquad (7.95)$$

where

$$\omega_p = \frac{4\,\pi\,N\,e^2}{m}$$

is the plasma frequency.

$$n \simeq \sqrt{\epsilon} = \sqrt{1 - \frac{4\pi N e^2}{m\omega^2}} \simeq 1 - \frac{2\pi N e^2}{m\omega^2}$$
$$= \quad 1 - \frac{2\pi N e^2}{k^2 m c^2}$$

where

$$\omega = k c$$

- **$c(\omega)$**

Photon with mass m satisfies the Klein- Gordon equation

$$\left[\Box + \left(\frac{m c}{\hbar}\right)^2\right]\psi = 0$$

where

$$\Box = \frac{\partial^2}{c^2 \partial t^2} - \nabla^2$$

A plane wave solution $\psi = C e^{i \boldsymbol{k}\cdot\boldsymbol{x} - i\omega t}$ requires

$$-\frac{\omega^2}{c^2} + k^2 + \left(\frac{m c}{\hbar}\right)^2 = 0 \qquad\qquad k = |\boldsymbol{k}| \geq 0$$

or

$$\frac{\omega}{c} = \pm\sqrt{k^2 + \left(\frac{m c}{\hbar}\right)^2}$$

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m c}{\hbar}\right)^2}$$

The group velocity is therefore

$$c(\omega) = \frac{d\omega}{d k} = \pm\frac{k c}{\sqrt{k^2 + \left(\frac{m c}{\hbar}\right)^2}}$$

$$= \quad \frac{k c^2}{\omega} = \frac{c^2}{\omega}\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m c}{\hbar}\right)^2}$$

$$= \quad c\sqrt{1 - \left(\frac{m c^2}{\hbar\omega}\right)^2}$$

$$= \quad c\sqrt{1 - \left(\frac{\omega_0}{\omega}\right)^2}$$

where

$$\hbar\omega_0 = m c^2$$

is the rest energy of the photon.

3. Lorentz Transformations

▪ 11.19

(11.16) can be written as

$$x_0' = \gamma\left(x_0 - \beta\, x_{||}\right) = \gamma\,(x_0 - \boldsymbol{\beta}\cdot\boldsymbol{x})$$
$$\boldsymbol{x}_{||}' = \gamma\left(\boldsymbol{x}_{||} - \boldsymbol{\beta}\, x_0\right)$$
$$\boldsymbol{x}_{\perp}' = \boldsymbol{x}_{\perp} = \boldsymbol{x} - \boldsymbol{x}_{||}$$

$$\begin{aligned}\boldsymbol{x}' &= \boldsymbol{x}_{||}' + \boldsymbol{x}_{\perp}' \\ &= \gamma\left(\boldsymbol{x}_{||} - \boldsymbol{\beta}\, x_0\right) + \boldsymbol{x} - \boldsymbol{x}_{||} \\ &= \boldsymbol{x} + (\gamma - 1)\,\boldsymbol{x}_{||} - \gamma\,\boldsymbol{\beta}\, x_0\end{aligned}$$

By definition:

$$\boldsymbol{x}_{||} = \hat{\beta}\left(\hat{\beta}\cdot\boldsymbol{x}\right) = \frac{1}{\beta^2}\,\boldsymbol{\beta}\,(\beta\cdot\boldsymbol{x})$$

Hence

$$\boldsymbol{x}' = \boldsymbol{x} + \frac{(\gamma - 1)}{\beta^2}\,\boldsymbol{\beta}\,(\beta\cdot\boldsymbol{x}) - \gamma\,\boldsymbol{\beta}\, x_0 \qquad (11.19)$$

▪ 11.30

$$k = (k_0, \boldsymbol{k}) = \left(\frac{\omega}{c}, \boldsymbol{k}\right) = \frac{\omega}{c}\left(1, \hat{\boldsymbol{k}}\right)$$

Thus

$$k_0' = \gamma\,(k_0 - \boldsymbol{\beta}\cdot\boldsymbol{k})$$

becomes

$$\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \boldsymbol{\beta}\cdot\boldsymbol{k}\right) = \gamma\,\frac{\omega}{c}\left(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{k}}\right)$$

or

$$\omega' = \gamma\,(\omega - c\,\boldsymbol{\beta}\cdot\boldsymbol{k}) = \gamma\,\omega\,(1 - \beta\cos\theta)$$

while

$$k_{||}' = \gamma\,(k_{||} - \beta\, k_0)$$
$$\boldsymbol{k}_{\perp}' = \boldsymbol{k}_{\perp}$$

gives

$$\tan\theta' = \frac{k_{\perp}'}{k_{||}'} = \frac{k_{\perp}}{\gamma\,(k_{||} - \beta\, k_0)}$$

$$= \frac{\frac{k_\perp}{k_\parallel}}{\gamma\left(1 - \beta \frac{k_0}{k_\parallel}\right)}$$

$$= \frac{\tan\theta}{\gamma\left(1 - \frac{\beta}{\cos\theta}\right)}$$

$$= \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \qquad (11.30)$$

where

$$\frac{k_0}{k_\parallel} = \frac{\frac{\omega}{c}}{\frac{\omega}{c}\cos\theta} = \frac{1}{\cos\theta}$$

---

4. 4-Velocity

■ 11.31

With

$$d x_0 = \gamma_v \left(d x_0' + \beta\, d x_\parallel' \right)$$
$$d x_\parallel = \gamma_v \left(d x_\parallel' + \beta\, d x_0' \right)$$
$$d \boldsymbol{x}_\perp = d \boldsymbol{x}_\perp'$$

we have

$$u_\parallel = c\, d \frac{x_\parallel}{d x_0} = c\, d x_\parallel' + \frac{\beta\, d x_0'}{d x_0' + \beta\, d x_\parallel'}$$

$$= c \frac{\frac{u_\parallel'}{c} + \beta}{1 + \beta \frac{u_\parallel'}{c}}$$

$$= \frac{u_\parallel' + v}{1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}}$$

$$\boldsymbol{u}_\perp = c \frac{d \boldsymbol{x}_\perp}{d x_0} = c \frac{d \boldsymbol{x}_\perp'}{\gamma_v \left(d x_0' + \beta\, d x_\parallel' \right)}$$

$$= c \frac{\frac{\boldsymbol{u}_\perp'}{c}}{\gamma_v \left(1 + \beta \frac{u_\parallel'}{c}\right)}$$

$$= \frac{\boldsymbol{u}_\perp'}{\gamma_v \left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\right)} \qquad (11.31)$$

■ 11.32

$$\tan\theta = \frac{u_\perp}{u_{||}} = \frac{u_\perp'}{\gamma_v\,(u_{||}' + v)} = \frac{u'\sin\theta'}{\gamma_v\,(u'\cos\theta' + v)}$$

where by definition

$$u_\perp' = u'\sin\theta'$$
$$u_{||}' = u'\cos\theta'$$

$$u^2 = u_{||}^2 + \boldsymbol{u}_\perp^2$$
$$= \left(\frac{u_{||}' + v}{1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}}\right)^2 + \left(\frac{\boldsymbol{u}_\perp'}{\gamma_v\left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\right)}\right)^2$$
$$= \left[(u'\cos\theta' + v)^2 + \frac{(u'\sin\theta')^2}{\gamma_v^2}\right] \Big/ \left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\right)^2$$

with

$$\gamma_v^2 = \frac{1}{1 - \left(\frac{v}{c}\right)^2}$$

the numerator becomes

$$(u'\cos\theta' + v)^2 + (u'\sin\theta')^2\left[1 - \left(\frac{v}{c}\right)^2\right]$$
$$= u'^2 + 2u'v\cos\theta' + v^2 - \left(\frac{u'v\sin\theta'}{c}\right)^2$$

Hence

$$u = \frac{\sqrt{u'^2 + v^2 + 2u'v\cos\theta' - \left(\frac{u'v\sin\theta'}{c}\right)^2}}{1 + \frac{u'v}{c^2}\cos\theta'}$$

where

$$\boldsymbol{v}\cdot\boldsymbol{u}' = u'v\cos\theta'$$

■ 11.34

From 11.31 we have

$$\gamma_v\left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\right) = \frac{u_\perp'}{u_\perp}$$

Now

$$\frac{u_\perp'}{u_\perp} = \frac{\frac{d x_\perp'}{d x_0'}}{\frac{d x_\perp}{d x_0}} = \frac{d x_0}{d x_0'} = \frac{\gamma_u\, d\tau}{\gamma_{u'}\, d\tau} = \frac{\gamma_u}{\gamma_{u'}}$$

Hence

$$\gamma_v\left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\right) = \frac{\gamma_u}{\gamma_{u'}}$$

or

$$\gamma_u = \gamma_v\,\gamma_{u'}\left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\right) \qquad (11.34)$$

11.35

When 11.34 is substituted into 11.31,

$$u_{\parallel} = \frac{u_{\parallel}' + v}{1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}}$$

$$\boldsymbol{u}_{\perp} = \frac{\boldsymbol{u}_{\perp}'}{\gamma_v\left(1 + \frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\right)}$$

we have

$$u_{\parallel} = (u_{\parallel}' + v)\,\frac{\gamma_v\,\gamma_{u'}}{\gamma_u}$$

ie

$$\gamma_u\, u_{\parallel} = (u_{\parallel}' + v)\,\gamma_v\,\gamma_{u'}$$

and

$$\boldsymbol{u}_{\perp} = \frac{\boldsymbol{u}_{\perp}'}{\gamma_v}\,\frac{\gamma_v\,\gamma_{u'}}{\gamma_u} = \boldsymbol{u}_{\perp}'\,\frac{\gamma_{u'}}{\gamma_u}$$

ie

$$\gamma_u\,\boldsymbol{u}_{\perp} = \gamma_{u'}\,\boldsymbol{u}_{\perp}'$$

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5. Relativistic Momentum & Energy

■ 11.41-55

In the K' (CM) system,

$$\boldsymbol{u}_a' = -\boldsymbol{u}_b' = \boldsymbol{v}$$

Momentum conservation means

$$\boldsymbol{u}_c' = -\boldsymbol{u}_d' = \boldsymbol{v}'$$

Energy conservation means

$$v = v'$$

Let the angle between $\boldsymbol{v}$ & $\boldsymbol{v}'$ be θ', we have

$$u_{c\parallel}' = v\cos\theta' \qquad u_{c\perp}' = v\sin\theta'$$

$$u_{d\parallel}' = -v\cos\theta' \qquad u_{d\perp}' = -v\sin\theta'$$

In the K (lab) system,

$$\boldsymbol{u}_b = 0$$

which means K' should move with velocity $\boldsymbol{v}$ with respect to K.

Using the tranform of 4-velocities

$$\gamma_u \boldsymbol{u}_{\parallel} = \gamma_v \gamma_{u'} (\boldsymbol{u}_{\parallel}' + \boldsymbol{v})$$

$$\gamma_u \boldsymbol{u}_{\perp} = \gamma_{u'} \boldsymbol{u}_{\perp}'$$

$$\frac{\gamma_v \gamma_{u'}}{\gamma_u} = \frac{1}{1 + \frac{\boldsymbol{u}' \cdot \boldsymbol{v}}{c^2}}$$

we have

$$\gamma_{u_a} \boldsymbol{u}_a = \gamma_v \gamma_v 2 \boldsymbol{v} = \frac{2 \boldsymbol{v}}{1 - \beta^2} \qquad \beta = v/c$$

the square of which gives

$$\frac{1}{1 - \left(\frac{u_a}{c}\right)^2} \left(\frac{u_a}{c}\right)^2 = \frac{4 \beta^2}{\left(1 - \beta^2\right)^2}$$

$$\frac{c^2}{u_a{}^2} - 1 = \frac{\left(1 - \beta^2\right)^2}{4 \beta^2}$$

$$\frac{c^2}{u_a{}^2} = \frac{\left(1 - \beta^2\right)^2 + 4 \beta^2}{4 \beta^2} = \frac{\left(1 + \beta^2\right)^2}{4 \beta^2}$$

$$\frac{u_a}{c} = \frac{2 \beta}{1 + \beta^2}$$

so that

$$\boldsymbol{u}_a = \frac{1}{\gamma_{u_a}} \frac{2 \boldsymbol{v}}{1 - \beta^2} = \frac{2 \boldsymbol{\beta} c}{1 + \beta^2} \qquad (11.41)$$

namely

$$\gamma_{u_a} = \frac{1 + \beta^2}{1 - \beta^2}$$

Also

$$\gamma_{u_c} u_{c\parallel} = \gamma_v{}^2 \left(u_{c\parallel}' + v \right) = \gamma_v{}^2 v (\cos\theta' + 1)$$

$$\gamma_{u_c} u_{c\perp} = \gamma_v u_{c\perp}' = \gamma_v v \sin\theta'$$

$$\gamma_{u_d} u_{d\parallel} = \gamma_v{}^2 \left(u_{d\parallel}' + v \right) = \gamma_v{}^2 v (-\cos\theta' + 1)$$

$$\gamma_{u_d} u_{d\perp} = \gamma_v u_{d\perp}' = -\gamma_v v \sin\theta'$$

These become (11.42) if we make use of the identities

$$\frac{\gamma_v{}^2}{\gamma_{u_c}} = \frac{1}{1 + \frac{\boldsymbol{u}_c \cdot \boldsymbol{v}}{c^2}} = \frac{1}{1 + \beta^2 \cos\theta'}$$

$$\frac{\gamma_v{}^2}{\gamma_{u_d}} = \frac{1}{1 + \frac{\boldsymbol{u}_d \cdot \boldsymbol{v}}{c^2}} = \frac{1}{1 - \beta^2 \cos\theta'}$$

so that

$$u_{c||} = \frac{v(\cos\theta' + 1)}{1 + \beta^2 \cos\theta'} \qquad u_{c\perp} = \frac{v \sin\theta'}{\gamma_v \left(1 + \beta^2 \cos\theta'\right)}$$

$$u_{d||} = \frac{v(-\cos\theta' + 1)}{1 - \beta^2 \cos\theta'} \qquad u_{d\perp} = -\frac{v \sin\theta'}{\gamma_v \left(1 - \beta^2 \cos\theta'\right)} \qquad (11.42)$$

Momentum conservation of the scattering process then implies

$$\mathcal{M}(u_c)\, u_{c||} + \mathcal{M}(u_d)\, u_{d||} = \mathcal{M}(u_a)\, u_a$$

and

$$\mathcal{M}(u_c)\, u_{c\perp} + \mathcal{M}(u_d)\, u_{d\perp} = 0$$

The 2nd eq gives

$$\mathcal{M}(u_c) \cdot \frac{v \sin\theta'}{\gamma_v \left(1 + \beta^2 \cos\theta'\right)} - \mathcal{M}(u_d) \cdot \frac{v \sin\theta'}{\gamma_v \left(1 - \beta^2 \cos\theta'\right)} = 0$$

or

$$\mathcal{M}(u_c) = \mathcal{M}(u_d) \left(\frac{1 + \beta^2 \cos\theta'}{1 - \beta^2 \cos\theta'} \right)$$

which is valid for all θ '.

For θ ' = 0 (no scattering), we have

$$\boldsymbol{u}_c = \boldsymbol{u}_a \, \boldsymbol{u}_d = 0$$

so that

$$\mathcal{M}(u_a) = \mathcal{M}(0) \left(\frac{1 + \beta^2}{1 - \beta^2} \right) \qquad (11.44)$$

$$= \mathcal{M}(0)\, \gamma_{u_a}$$

Thus, we have

$$\boldsymbol{p} = \gamma\, m\, \boldsymbol{u} \qquad (11.46)$$

Now

$$\frac{\gamma_v^2}{\gamma_{u_c}} = \frac{1}{1 + \beta^2 \cos\theta'}$$

gives

$$1 - \frac{u_c^2}{c^2} = \left(\frac{1 - \beta^2}{1 + \beta^2 \cos\theta'} \right)^2$$

Similarly

$$\gamma_{u_a} = \frac{1 + \beta^2}{1 - \beta^2}$$

gives

$$1 - \frac{u_a^2}{c^2} = \left(\frac{1 - \beta^2}{1 + \beta^2} \right)^2$$

Thus

$$\frac{u_c^2}{c^2} - \frac{u_a^2}{c^2} = \left(\frac{1 - \beta^2}{1 + \beta^2}\right)^2 - \left(\frac{1 - \beta^2}{1 + \beta^2 \cos\theta'}\right)^2$$

$$= \frac{1}{\gamma_{u_a}{}^2}\left[1 - \left(\frac{1 + \beta^2}{1 + \beta^2 \cos\theta'}\right)^2\right]$$

$$= \frac{1}{\gamma_{u_a}{}^2}\left[1 - \left(\frac{1}{1 + \frac{\beta^2}{1+\beta^2}(\cos\theta' - 1)}\right)^2\right]$$

For small θ',

$$\cos\theta' - 1 \simeq -\frac{1}{2}\theta'^2 + O(\theta'^4)$$

is small so that

$$\frac{u_c^2}{c^2} - \frac{u_a^2}{c^2} = \frac{1}{\gamma_{u_a}{}^2}\left[1 - \left(1 - 2\,\frac{\beta^2}{1 + \beta^2}(\cos\theta' - 1) + \ldots\right)\right]$$

$$= -\frac{1}{\gamma_{u_a}{}^2}\,\frac{\beta^2}{1 + \beta^2}\,\theta'^2 + O(\theta'^4)$$

$$= -\frac{1}{\gamma_{u_a}{}^3}\,\frac{\beta^2}{1 - \beta^2}\,\theta'^2 + O(\theta'^4)$$

or

$$u_c^2 = u_a^2 - \frac{\eta}{\gamma_{u_a}{}^3} + \ldots$$

where

$$\eta = \frac{c^2\,\beta^2}{1 - \beta^2}\,\theta'^2$$

Similarly

$$\frac{\gamma_v^2}{\gamma_{u_d}} = \frac{1}{1 - \beta^2 \cos\theta'}$$

so that

$$1 - \frac{u_d{}^2}{c^2} = \left(\frac{1 - \beta^2}{1 - \beta^2 \cos\theta'} \right)^2$$

$$= \left(\frac{1}{1 - \frac{\beta^2}{1-\beta^2} (\cos\theta' - 1)} \right)^2$$

$$= 1 + 2 \frac{\beta^2}{1 - \beta^2} (\cos\theta' - 1) + \ldots$$

or

$$\frac{u_d{}^2}{c^2} \simeq \frac{\beta^2}{1 - \beta^2} \theta'^2 + \ldots$$

ie.

$$u_d{}^2 \simeq \eta + \ldots$$

Thus

$$\mathcal{E}(u_c) = \mathcal{E}\left(u_c{}^2\right) \simeq \mathcal{E}\left(u_a{}^2\right) + \left(\frac{d\,\mathcal{E}\left(u_c{}^2\right)}{d\,u_c{}^2} d \frac{u_c{}^2}{d\,\eta} \right)_{\eta=0} \eta + \ldots$$

$$\simeq \mathcal{E}\left(u_a{}^2\right) + \frac{d\,\mathcal{E}\left(u_a{}^2\right)}{d\,u_a{}^2} \left(-\frac{1}{\gamma_{u_a}{}^3} \right) \eta + \ldots$$

$$\mathcal{E}(u_d) = \mathcal{E}\left(u_d{}^2\right) \simeq \mathcal{E}(0) + \left(\frac{d\,\mathcal{E}\left(u_d{}^2\right)}{d\,u_d{}^2} d \frac{u_d{}^2}{d\,\eta} \right)_{\eta=0} \eta + \ldots$$

$$\simeq \mathcal{E}(0) + \left(\frac{d\,\mathcal{E}\left(u^2\right)}{d\,u^2} \right)_{u=0} \eta + \ldots$$

The energy conservation eq thus becomes

$$\frac{d\,\mathcal{E}\left(u_a{}^2\right)}{d\,u_a{}^2} \left(-\frac{1}{\gamma_{u_a}{}^3} \right) + \left(\frac{d\,\mathcal{E}\left(u^2\right)}{d\,u^2} \right)_{u=0} = 0$$

or

$$\frac{d\,\mathcal{E}\left(u^2\right)}{d\,u^2} = \gamma_u{}^3 \left(\frac{d\,\mathcal{E}\left(u^2\right)}{d\,u^2} \right)_{u=0} = \gamma_u{}^3 \frac{m}{2}$$

$$= \frac{m}{2\left[1-\left(\frac{u}{c}\right)^2\right]^{3/2}}$$

so that

$$\mathcal{E}(u^2) = \mathcal{E}(0) + \int_0^{u^2} d\,x\,\frac{m}{2\left[1-\frac{x}{c^2}\right]^{3/2}} \qquad x = u^2$$

$$= \mathcal{E}(0) + m\,c^2\left[\frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} - 1\right]$$

$$= \mathcal{E}(0) + m\,c^2\,(\gamma - 1)$$

$$= \gamma\,\mathrm{m}\,\mathrm{c}^2 + \left[\mathcal{E}(0) - m\,c^2\right]$$

Setting $\mathcal{E}(0) = m\,c^2$, we have

$$\mathcal{E}(u) = \gamma\,m\,c^2 \qquad (11.51)$$

The kinetic energy is defined as

From

$$\boldsymbol{p} = \gamma\,m\,\boldsymbol{u}$$

we have

$$\boldsymbol{p} = \frac{\mathcal{E}(u)}{c^2}\,\boldsymbol{u}$$

or

$$\boldsymbol{u} = \frac{c^2}{\mathcal{E}}\,\boldsymbol{p} \qquad (11.53)$$

The 4-momentum vector is defined as

$$p^\mu = (E/c,\ \boldsymbol{p}) = \gamma\,m\,(c,\ \boldsymbol{u}) = m\,u^\mu = m\,\frac{d\,x^\mu}{d\,\tau}$$

Its length is an invariant:

$$p^2 = \left(\frac{E}{c}\right)^2 - \boldsymbol{p}^2 = \gamma^2\,m^2\left(c^2 - \boldsymbol{u}^2\right) = m^2\,c^2 \qquad (11.54)$$

which also gives

$$E = \sqrt{\boldsymbol{p}^2\,c^2 + m^2\,c^4} \qquad (11.55)$$

■ 11.56-58

Setting $\boldsymbol{p}_{||}' = 0$, we have

$$\boldsymbol{p}_{||}' = 0 = \gamma\left(\boldsymbol{p}_{||} - \boldsymbol{\beta}\,\frac{E}{c}\right)$$

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \boldsymbol{\beta}\cdot\boldsymbol{p}\right) = \gamma\left(\frac{E}{c} - \beta\,p_{||}\right)$$

$$\boldsymbol{p}_\perp' = \boldsymbol{p}_\perp$$

The 1st eq gives

$$p_{||} = \beta \frac{E}{c}$$

so that the 2nd becomes

$$\frac{E'}{c} = \gamma \left(\frac{E}{c} - \beta^2 \frac{E}{c} \right) = \frac{E}{\gamma c}$$

Alternatively

$$\begin{aligned} E' &= \sqrt{\boldsymbol{p}'^2 c^2 + m^2 c^4} \\ &= \sqrt{\boldsymbol{p}_\perp'^2 c^2 + m^2 c^4} \\ &= \sqrt{\boldsymbol{p}_\perp^2 c^2 + m^2 c^4} \\ &= \Omega\, \mathrm{c} \end{aligned} \qquad (11.56)$$

where

$$\begin{aligned} \Omega &= \sqrt{\boldsymbol{p}_\perp^2 + m^2 c^2} \\ &= \frac{E'}{c} = \frac{E}{\gamma c} \end{aligned}$$

Thus

$$\begin{aligned} \frac{E}{c} &= \gamma\, \Omega = \Omega \cosh\zeta \\ p_{||} &= \beta \frac{E}{c} = \beta\, \gamma\, \Omega = \Omega \sinh\zeta \end{aligned} \qquad (11.57)$$

where

$$\cosh\zeta = \gamma \qquad \sinh\zeta = \beta\,\gamma$$

If the particle is at rest in K',

$$\boldsymbol{p}_\perp = \boldsymbol{p}_\perp' = 0$$

so that

$$\Omega = m\, c$$

and

$$\begin{aligned} \frac{E}{c} &= m\, c \cosh\zeta \\ p = p_{||} &= m\, c \sinh\zeta \end{aligned} \qquad (11.58)$$

- **Rapidity**

Consider 2 successive Lorentz transformations in the same direction:

$$K \xrightarrow{v_1} K' \xrightarrow{v_2} K''$$

This should be equivalent to the transform

$$K \xrightarrow{v} K''$$

where

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

In terms of the rapidities:

$$\zeta_1 = \cosh^{-1}\gamma_1 = \cosh^{-1}\frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}}$$

$$\zeta_2 = \cosh^{-1}\gamma_2 = \cosh^{-1}\frac{1}{\sqrt{1-\frac{v_2^2}{c^2}}}$$

$$\zeta = \cosh^{-1}\gamma = \cosh^{-1}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

In terms of the transformation matrices (the transverse parts are left out for convenience):

$$\begin{pmatrix} \cosh\zeta_2 & -\sinh\zeta_2 \\ -\sinh\zeta_2 & \cosh\zeta_2 \end{pmatrix}\begin{pmatrix} \cosh\zeta_1 & -\sinh\zeta_1 \\ -\sinh\zeta_1 & \cosh\zeta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\zeta_2\cosh\zeta_1 + \sinh\zeta_2\sinh\zeta_1 & -\cosh\zeta_2\sinh\zeta_1 - \sinh\zeta_2\cosh\zeta_1 \\ -\sinh\zeta_2\cosh\zeta_1 - \cosh\zeta_2\sinh\zeta_1 & \sinh\zeta_2\sinh\zeta_1 + \cosh\zeta_2\cosh\zeta_1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\zeta_2+\zeta_1) & -\sinh(\zeta_2+\zeta_1) \\ -\sinh(\zeta_2+\zeta_1) & \cosh(\zeta_2+\zeta_1) \end{pmatrix}$$

$$\equiv \begin{pmatrix} \cosh\zeta & -\sinh\zeta \\ -\sinh\zeta & \cosh\zeta \end{pmatrix}$$

so that

$$\zeta = \zeta_1 + \zeta_2$$

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7. Matrix Representation

■ 11.87-90

Consider $A = e^L$ (11.87)

If L can be diagonalized by a similarity transformation S, we have

$$S\,L\,S^{-1} = \text{diag}(\,l_1\,,\,\ldots\,,\,l_n\,)$$

where l_i are the eigenvalues of L.

Since e^L is defined as the (convergent) Taylor series

$$e^L = \sum_{m=0}^{\infty} \frac{L^m}{m!}$$

A must also be diagonalized by S:

$$\begin{aligned} S A S^{-1} &= \sum_{m=0}^{\infty} \frac{1}{m!} S L^m S^{-1} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} [\operatorname{diag}(l_1 , \ldots , l_n)]^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \operatorname{diag}(l_1{}^m , \ldots , l_n{}^m) \\ &= \operatorname{diag}\left(\sum_{m=0}^{\infty} \frac{1}{m!} l_1{}^m , \ldots , \sum_{m=0}^{\infty} \frac{1}{m!} l_n{}^m \right) \\ &= \operatorname{diag}\left(e^{l_1} , \ldots , e^{l_n} \right) \\ &= \operatorname{diag}(a_1 , \ldots , a_n) \end{aligned}$$

where $a_i = e^{l_i}$ are the eigenvalues of A.

Now

$$\det A = \det \left| S A S^{-1} \right| = \det \left| \operatorname{diag}\left(e^{l_1} , \ldots , e^{l_n} \right) \right|$$

$$= e^{l_1} \ldots e^{l_n} = e^{l_1 + \ldots + l_n} = e^{\operatorname{tr} L}$$

If L is real, so is $\det L = \prod_i l_i$. Hence, l_i must either be real or form complex conjugates pairs.

Since the sum of a complex conjugate pair is real, $\operatorname{tr} L$ is real so that $e^{\operatorname{tr} L} \geq 0$. That is, $\det A = -1$ is excluded.

If L is traceless, ie., $\operatorname{tr} L = 0$, we have

$$\det A = e^0 = 1$$

Hence, for proper Lorentz transformations ($\det A = +1$), L is real & traceless.

Since

$$A^{-1} = g A^T g \qquad g^2 = I \qquad g = g^T \qquad (11.88)$$

we have

$$e^{-L} = g e^{L^T} g = e^{g L^T g}$$

so that

$$-L = g L^T g$$

or

$$-g L = L^T g = (g L)^T \qquad (11.89)$$

ie. $g L$ is antisymmetric.

Writing

$$L = \begin{pmatrix} a & b^T \\ c & \mathcal{D} \end{pmatrix}$$

where a is a number, $\boldsymbol{b}$, $\boldsymbol{c}$ are 3-(column) vectors, and $\mathcal{D}$ is a 3×3 matrix. We have

$$g L = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} a & b^T \\ c & \mathcal{D} \end{pmatrix} = \begin{pmatrix} a & b^T \\ -c & -\mathcal{D} \end{pmatrix}$$

$$(g L)^T = \begin{pmatrix} a & -c^T \\ b & -\mathcal{D}^T \end{pmatrix}$$

Hence, the antisymmetry of $g\,L$ means

$$\begin{pmatrix} -a & -b^T \\ c & \mathcal{D} \end{pmatrix} = \begin{pmatrix} a & -c^T \\ b & -\mathcal{D}^T \end{pmatrix}$$

or

$$a = 0 \quad \boldsymbol{b} = \boldsymbol{c} \quad \mathcal{D} = -\mathcal{D}^T$$

ie.

$$L = \begin{pmatrix} 0 & b^T \\ b & \mathcal{D} \end{pmatrix} \qquad \mathcal{D} \text{ antisymmetric}$$

$$= \left(\begin{array}{c|ccc} 0 & b_1 & b_2 & b_3 \\ \hline b_1 & 0 & D_{12} & -D_{31} \\ b_2 & -D_{12} & 0 & D_{23} \\ b_2 & D_{31} & -D_{23} & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|ccc} 0 & L_{01} & L_{02} & L_{03} \\ \hline L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{array}\right) \qquad (11.90)$$

■ 11.91-99

■ ✸

All 4×4 matrices in space-time can be partitioned as:

$$\begin{pmatrix} \mathrm{a} & \mathrm{b}^\mathrm{T} \\ \mathrm{c} & \mathcal{D} \end{pmatrix}$$

where a is a number, $\boldsymbol{b}$, $\boldsymbol{c}$ are 3-(column) vectors, and $\mathcal{D}$ is a 3×3 matrix.

The multiplication of 2 such matrices is

$$\begin{pmatrix} \mathrm{a} & \mathrm{b}^\mathrm{T} \\ \mathrm{c} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \mathrm{e} & \mathrm{f}^\mathrm{T} \\ \mathrm{g} & \mathcal{H} \end{pmatrix} = \begin{pmatrix} \mathrm{a\,e} + \mathrm{b}^\mathrm{T}\mathrm{g} & \mathrm{a\,f}^\mathrm{T} + \mathrm{b}^\mathrm{T}\mathcal{H} \\ \mathrm{e\,c} + \mathcal{D}\,\mathrm{g} & \mathrm{c\,f}^\mathrm{T} + \mathcal{D}\,\mathcal{H} \end{pmatrix}$$

Thus,

$$S_i = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & J_i \end{pmatrix}$$

with

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$K_i = \begin{pmatrix} 0 & \boldsymbol{e}_i^T \\ \boldsymbol{e}_i & 0 \end{pmatrix}$$

with

$$\boldsymbol{e}_1^T = (\,1 \times 0 \times 0\,) \qquad \boldsymbol{e}_2^T = (\,0 \times 1 \times 0\,) \qquad \boldsymbol{e}_3^T = (\,0 \times 0 \times 1\,)$$

The elements of J_i can be written as

$$(J_i)_{j\,k} = -\epsilon_{i\,j\,k}$$

and those of $\boldsymbol{e}_i$ as

$$(\boldsymbol{e}_i)_j = \delta_{i\,j}$$

$\left[\mathbf{S_i}, \mathbf{K_j} \right] = \boldsymbol{\epsilon_{ijk}} \mathbf{K_k}$

$$S_i K_j = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & J_i \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{e}_j{}^T \\ \boldsymbol{e}_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ J_i \boldsymbol{e}_j & 0 \end{pmatrix}$$

with

$$\begin{aligned} \left(J_i \boldsymbol{e}_j\right)_k &= (J_i)_{k m} \left(\boldsymbol{e}_j\right)_m \\ &= -\epsilon_{i k m} \delta_{j m} \\ &= -\epsilon_{i k j} \\ &= \epsilon_{i j k} \end{aligned}$$

so that

$$S_i K_j = \begin{pmatrix} 0 & 0 \\ \epsilon_{[i j]} & 0 \end{pmatrix}$$

where $\epsilon_{[i j]}$ is a column 3-vector with components $\left(\epsilon_{[i j]}\right)_k = \epsilon_{i j k}$. That is

$$\epsilon_{[i j]} = \boldsymbol{e}_k \qquad (i,\ j,\ k \text{ cyclic})$$

Similarly

$$K_j S_i = \begin{pmatrix} 0 & \boldsymbol{e}_j{}^T \\ \boldsymbol{e}_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & J_i \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{e}_j{}^T J_i \\ 0 & 0 \end{pmatrix}$$

where

$$\begin{aligned} \left(\boldsymbol{e}_j{}^T J_i\right)_k &= \left(\boldsymbol{e}_j\right)_m (J_i)_{m k} \\ &= -\delta_{j m} \epsilon_{i m k} \\ &= -\epsilon_{i j k} \end{aligned}$$

so that

$$K_j S_i = \begin{pmatrix} 0 & -\epsilon_{[i j]}{}^T \\ 0 & 0 \end{pmatrix}$$

Thus

$$\left[S_i, K_j \right] = \begin{pmatrix} 0 & \epsilon_{[i j]}{}^T \\ \epsilon_{[i j]} & 0 \end{pmatrix} = \epsilon_{i j k} K_k$$

- $\left[\mathbf{S_i}, \mathbf{S_j} \right] = \boldsymbol{\epsilon_{ijk}} \mathbf{S_k}$

$$S_i S_j = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & J_i \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & J_j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & J_i J_j \end{pmatrix}$$

with

$$\begin{aligned} \left(J_i J_j\right)_{k l} &= (J_i)_{k m} \left(J_j\right)_{m l} \\ &= \epsilon_{i k m} \epsilon_{j m l} \\ &= -\delta_{i j} \delta_{k l} + \delta_{i l} \delta_{k j} \end{aligned}$$

Thus

$$\left[S_i, S_j \right] = \begin{pmatrix} 0 & 0 \\ 0 & \left[J_i, J_j \right] \end{pmatrix}$$

with

$$\begin{aligned}[J_i\,,\,J_j]_{k\,l} &= -\delta_{i\,j}\,\delta_{k\,l} + \delta_{i\,l}\,\delta_{k\,j} + \delta_{j\,i}\,\delta_{k\,l} - \delta_{j\,l}\,\delta_{k\,i} \\ &= \delta_{i\,l}\,\delta_{k\,j} - \delta_{j\,l}\,\delta_{k\,i} \\ &= \epsilon_{m\,i\,j}\,\epsilon_{m\,l\,k} \\ &= -\epsilon_{i\,j\,m}\,\epsilon_{m\,k\,l} \\ &= \epsilon_{i\,j\,m}\,(J_m)_{k\,l}\end{aligned}$$

Hence

$$[\,S_i\,,\,S_j\,] = \epsilon_{i\,j\,m}\begin{pmatrix} 0 & 0 \\ 0 & J_m \end{pmatrix} = \epsilon_{i\,j\,m}\,S_m$$

- $\boldsymbol{S_i^2}$

For $i = j$,

$$(J_i\,J_i)_{k\,l} = -\delta_{k\,l} + \delta_{i\,l}\,\delta_{k\,i} = \delta_{k\,l}\,(\,-1 + \delta_{i\,k}\,)$$

ie., J_i^2 is diagonal with $\left(J_i^2\right)_{i\,i} = 0$ & $\left(J_i^2\right)_{j\,j} = -1$ for $j \neq i$.

$$J_1^2 = \begin{pmatrix} 0 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad J_2^2 = \begin{pmatrix} -1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad J_1^2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

Thus

$$S_i^2 = \begin{pmatrix} 0 & 0 \\ 0 & J_i^2 \end{pmatrix}$$

ie

$$S_1^2 = \begin{pmatrix} 0 & & 0 & \\ & 0 & & \\ 0 & & -1 & \\ & & & -1 \end{pmatrix} \qquad S_2^2 = \begin{pmatrix} 0 & & 0 & \\ & -1 & & \\ 0 & & 0 & \\ & & & -1 \end{pmatrix} \qquad S_3^2 = \begin{pmatrix} 0 & & 0 & \\ & -1 & & \\ 0 & & -1 & \\ & & & 0 \end{pmatrix} \tag{11.92}$$

- $\boldsymbol{\epsilon \cdot S}$

$$\boldsymbol{\epsilon}\cdot\boldsymbol{S} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\epsilon}\cdot\boldsymbol{J} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_3 & \epsilon_2 \\ 0 & \epsilon_3 & 0 & -\epsilon_1 \\ 0 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix}$$

- $\boldsymbol{(\,\epsilon \cdot S\,)^2}$

$$S_i\,S_j = \begin{pmatrix} 0 & 0 \\ 0 & J_i\,J_j \end{pmatrix}$$

with

$$\left(J_i\,J_j\right)_{k\,l} = -\delta_{i\,j}\,\delta_{k\,l} + \delta_{i\,l}\,\delta_{k\,j}$$

Hence

$$\left((\boldsymbol{\epsilon}\cdot\boldsymbol{J})^2 \right)_{k\,l} = \epsilon_i\,\epsilon_j\left(J_i\,J_j\right)_{k\,l}$$
$$= \epsilon_i\,\epsilon_j\left(-\delta_{i\,j}\,\delta_{k\,l} + \delta_{i\,l}\,\delta_{k\,j}\right)$$
$$= -\epsilon^2\,\delta_{k\,l} + \epsilon_k\,\epsilon_l$$

where

$$\epsilon^2 = \epsilon_i\,\epsilon_i = \boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^T\,\boldsymbol{\epsilon}$$

Therefore

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{J})^2 = -\epsilon^2\,I + \boldsymbol{\epsilon}\,\boldsymbol{\epsilon}^T$$

$$= \begin{pmatrix} -\epsilon_2{}^2 - \epsilon_3{}^2 & \epsilon_1\,\epsilon_2 & \epsilon_1\,\epsilon_3 \\ \epsilon_2\,\epsilon_1 & -\epsilon_1{}^2 - \epsilon_3{}^2 & \epsilon_2\,\epsilon_3 \\ \epsilon_3\,\epsilon_1 & \epsilon_3\,\epsilon_2 & -\epsilon_1{}^2 - \epsilon_2{}^2 \end{pmatrix}$$

Hence

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{S})^2 = \begin{pmatrix} 0 & 0 \\ 0 & (\boldsymbol{\epsilon}\cdot\boldsymbol{J})^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon^2\,I + \boldsymbol{\epsilon}\,\boldsymbol{\epsilon}^T \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\epsilon_2{}^2 - \epsilon_3{}^2 & \epsilon_1\,\epsilon_2 & \epsilon_1\,\epsilon_3 \\ 0 & \epsilon_2\,\epsilon_1 & -\epsilon_1{}^2 - \epsilon_3{}^2 & \epsilon_2\,\epsilon_3 \\ 0 & \epsilon_3\,\epsilon_1 & \epsilon_3\,\epsilon_2 & -\epsilon_1{}^2 - \epsilon_2{}^2 \end{pmatrix}$$

- **$(\boldsymbol{\epsilon}\cdot\boldsymbol{S})^3$**

$$S_i\,S_j\,S_k = \begin{pmatrix} 0 & 0 \\ 0 & J_i\,J_j\,J_k \end{pmatrix}$$

with

$$\left(J_i\,J_j\,J_k\right)_{m\,n} = (J_i)_{m\,a}\left(J_j\right)_{a\,b}(J_k)_{b\,n}$$
$$= -\epsilon_{i\,m\,a}\,\epsilon_{j\,a\,b}\,\epsilon_{k\,b\,n}$$
$$= \left(\delta_{i\,j}\,\delta_{m\,b} - \delta_{i\,b}\,\delta_{m\,j}\right)\epsilon_{k\,b\,n}$$
$$= \delta_{i\,j}\,\epsilon_{k\,m\,n} - \delta_{m\,j}\,\epsilon_{k\,i\,n}$$
$$= -\delta_{i\,j}\,(J_k)_{m\,n} + \delta_{m\,j}\,(J_k)_{i\,n}$$

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{J})^3 = \epsilon_i\,\epsilon_j\,\epsilon_k\,J_i\,J_j\,J_k$$

$$\left((\boldsymbol{\epsilon}\cdot\boldsymbol{J})^3 \right)_{m\,n} = \epsilon_i\,\epsilon_j\,\epsilon_k\,\left(J_i\,J_j\,J_k\right)_{m\,n}$$
$$= \epsilon_i\,\epsilon_j\,\epsilon_k\left(\delta_{i\,j}\,\epsilon_{k\,m\,n} - \delta_{m\,j}\,\epsilon_{k\,i\,n}\right)$$

The 2nd term vanishes since

$$\epsilon_i\,\epsilon_k\,\epsilon_{k\,i\,n} = -\epsilon_i\,\epsilon_k\,\epsilon_{i\,k\,n} \underset{i\leftrightarrow k}{=} -\epsilon_k\,\epsilon_i\,\epsilon_{k\,i\,n} = -\epsilon_i\,\epsilon_k\,\epsilon_{k\,i\,n} = 0$$

Thus

$$\left((\boldsymbol{\epsilon}\cdot\boldsymbol{J})^3 \right)_{m\,n} = \epsilon^2\,\epsilon_k\,\epsilon_{k\,m\,n} = -\epsilon^2\,\epsilon_k\,(J_k)_{m\,n} = -\epsilon^2\,[\,\boldsymbol{\epsilon}\cdot\boldsymbol{J}\,]_{m\,n}$$

or

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{J})^3 = -\epsilon^2\,(\boldsymbol{\epsilon}\cdot\boldsymbol{J})$$

so that

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{S})^3 = \begin{pmatrix} 0 & 0 \\ 0 & (\boldsymbol{\epsilon}\cdot\boldsymbol{J})^3 \end{pmatrix} = -\epsilon^2\begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\epsilon}\cdot\boldsymbol{J} \end{pmatrix} = -\epsilon^2\,(\boldsymbol{\epsilon}\cdot\boldsymbol{S})$$

■ $e^{-\boldsymbol{\epsilon}\cdot\boldsymbol{S}}$

Thus for $n \geq 1$

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{S})^n = \begin{cases} (-)^{\frac{n-1}{2}}\,\epsilon^n\,(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S}) & \text{for } n = \text{odd} \\ (-)^{\frac{n}{2}-1}\,\epsilon^n\,(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^2 & \text{for } n = \text{even} \end{cases}$$

For example:

$$e^{-\boldsymbol{\epsilon}\cdot\boldsymbol{S}} = \sum_{n=0} \frac{1}{n!}(-\boldsymbol{\epsilon}\cdot\boldsymbol{S})^n$$

$$= I + \sum_{m=1}\left[\frac{1}{(2m)!}(\boldsymbol{\epsilon}\cdot\boldsymbol{S})^{2m} - \frac{1}{(2m-1)!}(\boldsymbol{\epsilon}\cdot\boldsymbol{S})^{2m-1}\right]$$

$$= I + \sum_{m=1}\left[\frac{1}{(2m)!}(-)^{m-1}\epsilon^{2m}(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^2 - \frac{1}{(2m-1)!}(-)^{m-1}\epsilon^{2m-1}(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})\right]$$

$$= I + (\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^2(1-\cos\epsilon) - (\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})\sin\epsilon$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \epsilon_1^2 + (\epsilon_2^2+\epsilon_3^2)\cos\epsilon & \epsilon_1\epsilon_2(1-\cos\epsilon)+\epsilon_3\sin\epsilon & \epsilon_1\epsilon_3(1-\cos\epsilon)-\epsilon_2\sin\epsilon \\ 0 & \epsilon_2\epsilon_1(1-\cos\epsilon)-\epsilon_3\sin\epsilon & \epsilon_2^2+(\epsilon_1^2+\epsilon_3^2)\cos\epsilon & \epsilon_2\epsilon_3(1-\cos\epsilon)+\epsilon_1\sin\epsilon \\ 0 & \epsilon_3\epsilon_1(1-\cos\epsilon)+\epsilon_2\sin\epsilon & \epsilon_3\epsilon_2(1-\cos\epsilon)-\epsilon_1\sin\epsilon & \epsilon_3^2+(\epsilon_2^2+\epsilon_1^2)\cos\epsilon \end{pmatrix}$$

with

$$\epsilon_1^2+\epsilon_2^2+\epsilon_3^2 = 1$$

■ $e^{-\epsilon_3 S_3}$

For $\epsilon^T = (0,\,0,\,\epsilon)$, we have

$$(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$e^{-\epsilon_3 S_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}(1-\cos\epsilon) - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\sin\epsilon$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon & 0 \\ 0 & -\sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (11.96)$$

which is the rotation about the z axis by angle ϵ.

- $\left[\mathbf{K_i}, \mathbf{K_j}\right] = -\epsilon_{ijk}\,\mathbf{S_k}$

$$K_i\,K_j = \begin{pmatrix} 0 & \boldsymbol{e}_i{}^T \\ \boldsymbol{e}_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{e}_j{}^T \\ \boldsymbol{e}_j & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_i{}^T\,\boldsymbol{e}_j & 0 \\ 0 & \boldsymbol{e}_i\,\boldsymbol{e}_j{}^T \end{pmatrix}$$

with

$$\boldsymbol{e}_i{}^T\,\boldsymbol{e}_j = \delta_{ij}$$
$$\left(\boldsymbol{e}_i\,\boldsymbol{e}_j{}^T\right)_{kl} = (\boldsymbol{e}_i)_k\left(\boldsymbol{e}_j{}^T\right)_l = \delta_{ik}\,\delta_{jl}$$

Thus

$$\left[K_i, K_j\right] = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$$

where

$$\begin{aligned} A_{kl} &= \delta_{ik}\,\delta_{jl} - \delta_{jk}\,\delta_{il} \\ &= \epsilon_{mij}\,\epsilon_{mkl} \\ &= \epsilon_{ijm}\,\epsilon_{mkl} \\ &= -\epsilon_{ijm}\,(J_m)_{kl} \end{aligned}$$

Hence

$$\left[K_i, K_j\right] = -\epsilon_{ijm}\begin{pmatrix} 0 & 0 \\ 0 & J_m \end{pmatrix} = -\epsilon_{ijm}\,S_m$$

- $\boldsymbol{K_i^2}$

For $i = j$,

$$\left(\boldsymbol{e}_i\,\boldsymbol{e}_i{}^T\right)_{kl} = \delta_{ik}\,\delta_{il} = \delta_{kl}\,\delta_{ik}$$

so that

$$K_i^2 = \begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{e}_i\,\boldsymbol{e}_i{}^T \end{pmatrix}$$

where

$$\boldsymbol{e}_1\,\boldsymbol{e}_1{}^T = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad \boldsymbol{e}_2\,\boldsymbol{e}_2{}^T = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \quad \boldsymbol{e}_3\,\boldsymbol{e}_3{}^T = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

ie

$$K_1{}^2 = \begin{pmatrix} 1 & & 0 & \\ & 1 & & \\ 0 & & 0 & \\ & & & 0 \end{pmatrix} \quad K_2{}^2 = \begin{pmatrix} 1 & & 0 & \\ & 0 & & \\ 0 & & 1 & \\ & & & 0 \end{pmatrix} \quad K_3{}^2 = \begin{pmatrix} 1 & & 0 & \\ & 0 & & \\ 0 & & 0 & \\ & & & 1 \end{pmatrix} \tag{11.92}$$

- $\boldsymbol{\epsilon \cdot K}$

$$\boldsymbol{\epsilon}\cdot\boldsymbol{K} = \boldsymbol{\epsilon}_i\,K_i = \begin{pmatrix} 0 & \epsilon_i\,\boldsymbol{e}_i{}^T \\ \epsilon_i\,\boldsymbol{e}_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\epsilon}^T \\ \boldsymbol{\epsilon} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 & 0 & 0 & 0 \\ \epsilon_2 & 0 & 0 & 0 \\ \epsilon_3 & 0 & 0 & 0 \end{pmatrix}$$

$(\boldsymbol{\epsilon}\cdot\boldsymbol{K})^2$

$$K_i K_j = \begin{pmatrix} \boldsymbol{e}_i^T \boldsymbol{e}_j & 0 \\ 0 & \boldsymbol{e}_i \boldsymbol{e}_j^T \end{pmatrix}$$

with

$$\boldsymbol{e}_i^T \boldsymbol{e}_j = \delta_{ij}$$
$$\left(\boldsymbol{e}_i \boldsymbol{e}_j^T\right)_{kl} = (\boldsymbol{e}_i)_k \left(\boldsymbol{e}_j^T\right)_l = \delta_{ik}\,\delta_{jl}$$

Hence

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{K})^2 = \epsilon_i\,\epsilon_j\,K_i\,K_j$$

$$= \begin{pmatrix} \epsilon_i\,\epsilon_j\,\boldsymbol{e}_i^T \boldsymbol{e}_j & 0 \\ 0 & \epsilon_i\,\epsilon_j\,\boldsymbol{e}_i \boldsymbol{e}_j^T \end{pmatrix}$$

Since

$$\epsilon_i\,\epsilon_j \left(\boldsymbol{e}_i \boldsymbol{e}_j^T\right)_{kl} = \epsilon_i\,\epsilon_j\,\delta_{ik}\,\delta_{jl} = \epsilon_k\,\epsilon_l = \left(\boldsymbol{\epsilon}\,\boldsymbol{\epsilon}^T\right)_{kl}$$

we have

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{K})^2 = \begin{pmatrix} \epsilon^2 & 0 \\ 0 & \boldsymbol{\epsilon}\,\boldsymbol{\epsilon}^T \end{pmatrix} = \epsilon^2 \begin{pmatrix} 1 & 0 \\ 0 & \hat{\boldsymbol{\epsilon}}\,\hat{\boldsymbol{\epsilon}}^T \end{pmatrix}$$

$$= \begin{pmatrix} \epsilon^2 & 0 & 0 & 0 \\ 0 & \epsilon_1^2 & \epsilon_1\,\epsilon_2 & \epsilon_1\,\epsilon_3 \\ 0 & \epsilon_1\,\epsilon_2 & \epsilon_2^2 & \epsilon_2\,\epsilon_3 \\ 0 & \epsilon_1\,\epsilon_3 & \epsilon_3\,\epsilon_2 & \epsilon_3^2 \end{pmatrix}$$

where

$$\epsilon^2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$$

- **$(\boldsymbol{\epsilon}\cdot\mathbf{K})^3$**

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{K})^3 = \begin{pmatrix} \epsilon^2 & 0 \\ 0 & \boldsymbol{\epsilon}\,\boldsymbol{\epsilon}^T \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\epsilon}^T \\ \boldsymbol{\epsilon} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \epsilon^2\,\epsilon^T \\ \boldsymbol{\epsilon}\;\boldsymbol{\epsilon}^2 & 0 \end{pmatrix}$$

$$= \epsilon^2 \begin{pmatrix} 0 & \epsilon^T \\ \boldsymbol{\epsilon} & 0 \end{pmatrix}$$

$$= \epsilon^2\,(\boldsymbol{\epsilon}\cdot\boldsymbol{K}) = \epsilon^3 \left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{K}\right)$$

- **$e^{-\boldsymbol{\epsilon}\cdot\boldsymbol{K}}$**

Thus for $n \geq 1$

$$(\boldsymbol{\epsilon}\cdot\boldsymbol{K})^n = \begin{cases} \epsilon^n \left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S}\right) \\ \epsilon^n \left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{S}\right)^2 \end{cases} \quad \text{for } n = \begin{cases} \text{odd} \\ \text{even} \end{cases}$$

For example:

$$e^{-\boldsymbol{\epsilon}\cdot\boldsymbol{K}} = \sum_{n=0} \frac{1}{n!} (-\boldsymbol{\epsilon}\cdot\boldsymbol{K})^n$$

$$= I + \sum_{m=1} \left[\frac{1}{(2m)!} (\boldsymbol{\epsilon}\cdot\boldsymbol{K})^{2m} - \frac{1}{(2m-1)!} (\boldsymbol{\epsilon}\cdot\boldsymbol{K})^{2m-1} \right]$$

$$= I + \sum_{m=1} \left[\frac{1}{(2m)!} \epsilon^{2m} \left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{K}\right)^2 - \frac{1}{(2m-1)!} \epsilon^{2m-1} \left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{K}\right) \right]$$

$$= I + \left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{K}\right)^2 (\cosh\epsilon - 1) - \left(\hat{\boldsymbol{\epsilon}}\cdot\boldsymbol{K}\right) \sinh\epsilon$$

$$= \begin{pmatrix} \cosh\epsilon & -\epsilon_1 \sinh\epsilon & -\epsilon_2 \sinh\epsilon & -\epsilon_3 \sinh\epsilon \\ -\epsilon_1 \sinh\epsilon & 1 + {\epsilon_1}^2 (\cosh\epsilon - 1) & \epsilon_1 \epsilon_2 (\cosh\epsilon - 1) & \epsilon_1 \epsilon_3 (\cosh\epsilon - 1) \\ -\epsilon_2 \sinh\epsilon & \epsilon_1 \epsilon_2 (\cosh\epsilon - 1) & 1 + {\epsilon_2}^2 (\cosh\epsilon - 1) & \epsilon_2 \epsilon_3 (\cosh\epsilon - 1) \\ -\epsilon_3 \sinh\epsilon & \epsilon_1 \epsilon_3 (\cosh\epsilon - 1) & \epsilon_3 \epsilon_2 (\cosh\epsilon - 1) & 1 + {\epsilon_3}^2 (\cosh\epsilon - 1) \end{pmatrix}$$

with

$${\epsilon_1}^2 + {\epsilon_2}^2 + {\epsilon_3}^2 = 1$$

Setting

$$\gamma = \cosh\epsilon \qquad \gamma\,\beta = \sinh\epsilon \qquad \beta = \tanh\epsilon$$

$$\beta_i = \beta\,\epsilon_i$$

we have

$$e^{-\boldsymbol{\epsilon}\cdot\boldsymbol{K}} = \begin{pmatrix} \gamma & -\gamma\,\beta_1 & -\gamma\,\beta_2 & -\gamma\,\beta_3 \\ -\gamma\,\beta_1 & 1 + {\epsilon_1}^2 (\gamma - 1) & \epsilon_1 \epsilon_2 (\gamma - 1) & \epsilon_1 \epsilon_3 (\gamma - 1) \\ -\gamma\,\beta_2 & \epsilon_1 \epsilon_2 (\gamma - 1) & 1 + {\epsilon_2}^2 (\gamma - 1) & \epsilon_2 \epsilon_3 (\gamma - 1) \\ -\gamma\,\beta_3 & \epsilon_1 \epsilon_3 (\gamma - 1) & \epsilon_3 \epsilon_2 (\gamma - 1) & 1 + {\epsilon_3}^2 (\gamma - 1) \end{pmatrix} \qquad (11.98)$$

$$= \mathrm{e}^{-\hat{\boldsymbol{\beta}}\cdot\boldsymbol{K} \tanh^{-1}\beta}$$

8. Thomas Precession

For a charge e with mass m and orbital angular momentum $\boldsymbol{L}$, there is associated a magnetic moment (see Section 5.6)

$$\boldsymbol{\mu} = \frac{e\hbar}{2mc} \boldsymbol{L} = \mu_B \boldsymbol{L}$$

where $\mu_B = \dfrac{e\hbar}{2mc}$ is the Bohr magneton.

This gives rise to the normal Zeeman effect (each spectral line of atoms in a magnetic field are splitted into $2L + 1$ components; $L = 0, 1, 2, \ldots$)

When the number of splitted components is even, it is called the anomalous Zeeman effect.
Uhlenbeck & Goudsmit described it as due to an intrinsic angular momentum (spin) $\boldsymbol{s}$ with an associated magnetic moment

$$\boldsymbol{\mu} = g \frac{e\hbar}{2mc} \boldsymbol{s} = g\,\mu_B\, \boldsymbol{s}$$

where

$$s = \frac{1}{2}, \ \frac{3}{2}, \ \ldots$$

so that the interaction energy in the rest frame of the electron is

$$U' = -\boldsymbol{\mu}\cdot\boldsymbol{B}' = -g\,\mu_B\,\boldsymbol{s}\cdot\boldsymbol{B}'$$

where $g = 2$ is required to describe the magnitude of the splitting correctly.

U' implies an equation of motion

$$\left(\frac{d\boldsymbol{s}}{dt}\right)_{\text{rest frame}} = \boldsymbol{\mu}\times\boldsymbol{B}' \qquad (11.101)$$

For non-relativistic motion, $\boldsymbol{B}'$ is related to the fields in the laboratory frame by

$$\boldsymbol{B}' \simeq \boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E} \qquad (11.102)$$

Hence

$$\left(\frac{d\boldsymbol{s}}{dt}\right)_{\text{rest frame}} \simeq \boldsymbol{\mu}\times(\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E}) \qquad (11.103)$$

and

$$U' = -\boldsymbol{\mu}\cdot(\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E}) \qquad (11.104)$$

For an atom at rest in the lab frame, $\boldsymbol{E}$ can be approximated by

$$e\boldsymbol{E} = -\hat{\boldsymbol{r}}\frac{dV}{dr} \qquad (11.105)$$

where V is, say, the screened coulomb potential of the nucleus.

Hence

$$\begin{aligned} U' &= -\boldsymbol{\mu}\cdot\left(\boldsymbol{B} + \boldsymbol{\beta}\times\hat{\boldsymbol{r}}\frac{dV}{dr}\right) \\ &= -\boldsymbol{\mu}\cdot\boldsymbol{B} - \frac{1}{c}\boldsymbol{\mu}\cdot(\boldsymbol{v}\times\boldsymbol{r})\frac{dV}{r\,dr} \\ &= -\boldsymbol{\mu}\cdot\boldsymbol{B} + \frac{1}{mc}\boldsymbol{\mu}\cdot\boldsymbol{L}\frac{dV}{r\,dr} \qquad (\boldsymbol{L} = m\,\boldsymbol{r}\times\boldsymbol{v}) \\ &= -g\,\mu_B\,\boldsymbol{s}\cdot\boldsymbol{B} + \frac{1}{mc}g\,\mu_B\,\boldsymbol{s}\cdot\boldsymbol{L}\frac{dV}{r\,dr} \\ &= -g\frac{e\hbar}{2mc}\boldsymbol{s}\cdot\boldsymbol{B} + g\frac{e\hbar}{2m^2c^2}\boldsymbol{s}\cdot\boldsymbol{L}\frac{dV}{r\,dr} \qquad (11.106) \end{aligned}$$

where the 2nd term is called the spin- orbit interaction.

Let K be the lab frame.
At time t, the electron is traveling with velocity $c\,\boldsymbol{\beta}(t)$; its rest frame being K'.
At time $t + \delta t$, the electron is traveling with velocity $c\,[\,\boldsymbol{\beta}(t) + \delta\boldsymbol{\beta}\,]$; its rest frame being K''.

These frames are related by Lorentz boosts:

$$x' = \Lambda(\boldsymbol{\beta})\, x$$

$$x'' = \Lambda(\boldsymbol{\beta} + \delta\, \boldsymbol{\beta})\, x$$

$$= \Lambda(\boldsymbol{\beta} + \delta\, \boldsymbol{\beta})\, \Lambda(-\boldsymbol{\beta})\, x'$$

$$\equiv \Lambda_T\, x'$$

where

$$\Lambda_T \equiv \Lambda(\beta + \delta\, \beta)\, \Lambda(-\beta)$$

Without loss of generality, we can set $\boldsymbol{\beta}$ to be along the x^1-axis, $\delta\, \beta$ to be in the $x^1 - x^2$ plane (see fig 11.7). Using 11.98, we have

$$\boldsymbol{\beta} = (\,\beta\,,\ 0,\ 0\,)$$

$$\boldsymbol{\beta} + \delta\, \boldsymbol{\beta} = (\,\beta + \delta\, \beta_1\,,\ \delta\, \beta_2\,,\ 0\,)$$

$$\Lambda(\beta + \delta\, \beta) = \begin{pmatrix} \gamma'' & -\gamma''(\,\beta + \delta\, \beta_1\,) & -\gamma''\, \delta\, \beta_2 & 0 \\ -\gamma''(\,\beta + \delta\, \beta_1\,) & 1 + \dfrac{(\gamma'' - 1)}{\beta''^2}(\,\beta + \delta\, \beta_1\,)^2 & \dfrac{(\gamma'' - 1)}{\beta''^2}(\,\beta + \delta\, \beta_1\,)\, \delta\, \beta_2 & 0 \\ -\gamma''\, \delta\, \beta_2 & \dfrac{(\gamma'' - 1)}{\beta''^2}(\,\beta + \delta\, \beta_1\,)\, \delta\, \beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\beta'' = |\,\boldsymbol{\beta} + \delta\, \boldsymbol{\beta}\,| \simeq \beta + \delta\, \beta_1$$

$$\gamma'' = \frac{1}{\sqrt{1 - \beta''^2}} \simeq \frac{1}{\sqrt{1 - (\,\beta + \delta\, \beta_1)^2}}$$

$$\simeq \frac{1}{\sqrt{1 - \beta^2 - 2\, \beta\, \delta\, \beta_1}}$$

$$\simeq \gamma\left(1 + \gamma^2\, \beta\, \delta\, \beta_1\right) \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

So that

$$\gamma''(\,\beta + \delta\, \beta_1\,) \simeq \gamma\left(1 + \gamma^2\, \beta\, \delta\, \beta_1\right)(\,\beta + \delta\, \beta_1\,)$$

$$\simeq \gamma\left[\,\beta + \left(1 + \gamma^2\, \beta^2\right) \delta\, \beta_1\,\right]$$

$$= \gamma\left(\beta + \gamma^2\, \delta\, \beta_1\right)$$

$$\gamma''\, \delta\, \beta_2 \simeq \gamma\left(1 + \gamma^2\, \beta\, \delta\, \beta_1\right) \delta\, \beta_2 \simeq \gamma\, \delta\, \beta_2$$

$$\frac{(\gamma''-1)}{\beta''^2}(\beta+\delta\beta_1) \simeq \frac{(\gamma''-1)}{\beta''}$$

$$\simeq \frac{1}{\beta}\left(\gamma - 1 + \gamma^3\beta\,\delta\beta_1\right)\left(1 - \frac{1}{\beta}\,\delta\beta_1\right)$$

Hence

$$\frac{(\gamma''-1)}{\beta''^2}(\beta+\delta\beta_1)\,\delta\beta_2 \simeq \frac{1}{\beta}(\gamma-1)\,\delta\beta_2$$

$$\frac{(\gamma''-1)}{\beta''^2}(\beta+\delta\beta_1)^2 = \gamma''-1 \simeq \left(\gamma - 1 + \gamma^3\beta\,\delta\beta_1\right)$$

$$\Lambda(\beta+\delta\beta) = \begin{pmatrix} \gamma\left(1+\gamma^2\beta\,\delta\beta_1\right) & -\gamma\left(\beta+\gamma^2\,\delta\beta_1\right) & -\gamma\,\delta\beta_2 & 0 \\ -\gamma\left(\beta+\gamma^2\,\delta\beta_1\right) & \gamma+\gamma^3\beta\,\delta\beta_1 & \frac{1}{\beta}(\gamma-1)\,\delta\beta_2 & 0 \\ -\gamma\,\delta\beta_2 & \frac{1}{\beta}(\gamma-1)\,\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (11.114)$$

$$= \Lambda(\beta) + \begin{pmatrix} \gamma^3\beta\,\delta\beta_1 & -\gamma^3\,\delta\beta_1 & -\gamma\,\delta\beta_2 & 0 \\ -\gamma^3\,\delta\beta_1 & \gamma^3\beta\,\delta\beta_1 & \frac{1}{\beta}(\gamma-1)\,\delta\beta_2 & 0 \\ -\gamma\,\delta\beta_2 & \frac{1}{\beta}(\gamma-1)\,\delta\beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\Lambda(\beta) = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus

$$\Lambda_T = I + \begin{pmatrix} \gamma^3\beta\,\delta\beta_1 & -\gamma^3\,\delta\beta_1 & -\gamma\,\delta\beta_2 & 0 \\ -\gamma^3\,\delta\beta_1 & \gamma^3\beta\,\delta\beta_1 & \frac{1}{\beta}(\gamma-1)\,\delta\beta_2 & 0 \\ -\gamma\,\delta\beta_2 & \frac{1}{\beta}(\gamma-1)\,\delta\beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= I + \begin{pmatrix} 0 & (\gamma^4 \beta^2 - \gamma^4)\,\delta\beta_1 & -\gamma\,\delta\beta_2 & 0 \\ (\gamma^4 \beta^2 - \gamma^4)\,\delta\beta_1 & 0 & \frac{1}{\beta}(\gamma - 1)\,\delta\beta_2 & 0 \\ \left[-\gamma^2 + \gamma(\gamma-1)\right]\delta\beta_2 & \left[-\gamma^2\beta + \frac{1}{\beta}(\gamma - 1)\gamma\right]\delta\beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\gamma^2\,\delta\beta_1 & -\gamma\,\delta\beta_2 & 0 \\ -\gamma^2\,\delta\beta_1 & 1 & \frac{1}{\beta}(\gamma - 1)\,\delta\beta_2 & 0 \\ -\gamma\,\delta\beta_2 & -\frac{1}{\beta}(\gamma - 1)\,\delta\beta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (11.115)$$

Using (11.91), we have

$$\Lambda_T = I - \gamma^2\,\delta\beta_1\,K_1 - \gamma\,\delta\beta_2\,K_2 - \frac{1}{\beta}(\gamma - 1)\,\delta\beta_2\,S_3$$

This can be thrown into a general form by setting x^1 to be the || direction, x^2 to be the $\perp$ direction, ie,

$$\delta\beta_1 = \delta\boldsymbol{\beta}_{||} \qquad \delta\beta_2 = \delta\boldsymbol{\beta}_{\perp}$$

so that

$$\gamma^2\,\delta\beta_1\,K_1 = \gamma^2\,\delta\boldsymbol{\beta}_{||}\cdot\boldsymbol{K}$$

$$\gamma\,\delta\beta_2\,K_2 = \gamma\,\delta\boldsymbol{\beta}_{\perp}\cdot\boldsymbol{K}$$

Furthermore, x^3 is the direction $\hat{\boldsymbol{\beta}}\times\delta\hat{\boldsymbol{\beta}}_{\perp} = \hat{\boldsymbol{\beta}}\times\delta\hat{\boldsymbol{\beta}}$, hence

$$\delta\beta_2\,S_3 = \delta\boldsymbol{\beta}_{\perp}\left(\hat{\boldsymbol{\beta}}\times\delta\hat{\boldsymbol{\beta}}_{\perp}\right)\cdot\boldsymbol{S} = \left(\hat{\boldsymbol{\beta}}\times\delta\boldsymbol{\beta}_{\perp}\right)\cdot\boldsymbol{S}$$

$$= \left(\hat{\boldsymbol{\beta}}\times\delta\boldsymbol{\beta}\right)\cdot\boldsymbol{S}$$

$$= \frac{1}{\beta}(\boldsymbol{\beta}\times\delta\boldsymbol{\beta})\cdot\boldsymbol{S}$$

Therefore

$$\Lambda_T = I - \gamma^2\,\delta\boldsymbol{\beta}_{||}\cdot\boldsymbol{K} - \gamma\,\delta\boldsymbol{\beta}_{\perp}\cdot\boldsymbol{K} - \frac{1}{\beta^2}(\gamma - 1)(\boldsymbol{\beta}\times\delta\boldsymbol{\beta})\cdot\boldsymbol{S}$$

$$= I - \left(\gamma^2\,\delta\boldsymbol{\beta}_{||} + \gamma\,\delta\boldsymbol{\beta}_{\perp}\right)\cdot\boldsymbol{K} - \frac{1}{\beta^2}(\gamma - 1)(\boldsymbol{\beta}\times\delta\boldsymbol{\beta})\cdot\boldsymbol{S} \qquad (11.116)$$

$$= I - \Delta\boldsymbol{\beta}\cdot\boldsymbol{K} - \Delta\boldsymbol{\Omega}\cdot\boldsymbol{S}$$

$$\simeq \Lambda(\Delta\boldsymbol{\beta})\,\Lambda(\Delta\boldsymbol{\Omega})$$

$$= \Lambda(\Delta\boldsymbol{\Omega})\,\Lambda(\Delta\boldsymbol{\beta}) \qquad (11.117)$$

where

$$\Delta \boldsymbol{\beta} = \gamma^2\, \delta \boldsymbol{\beta}_{||} + \gamma\, \delta \boldsymbol{\beta}_{\perp}$$

$$\Delta \boldsymbol{\Omega} = \frac{1}{\beta^2} (\gamma - 1)(\boldsymbol{\beta} \times \delta \boldsymbol{\beta})$$

$$\Lambda(\Delta \boldsymbol{\beta}) \simeq I - \Delta \boldsymbol{\beta} \cdot \boldsymbol{K}$$

$$\Lambda(\Delta \boldsymbol{\Omega}) \simeq I - \Delta \boldsymbol{\Omega} \cdot \boldsymbol{S}$$

What (11.117) means is that the rest frames K' and K'' are related by a boost $\Lambda(\Delta \boldsymbol{\beta})$ and a rotation $\Lambda(\Delta \boldsymbol{\Omega})$. Namely, K'' is rotated by an amount $\Delta \boldsymbol{\Omega}$ with respect to K'.

Thus, in the non-rotating rest frame,

$$\left(\frac{d \boldsymbol{s}}{d t}\right)_{\text{non-rot}} = \left(\frac{d \boldsymbol{s}}{d t}\right)_{\text{rest frame}} - \frac{\Delta \boldsymbol{\Omega}}{d t} \times \boldsymbol{s}$$

$$\simeq g\, \mu_B\, \boldsymbol{s} \times (\boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{E}) - \frac{\Delta \boldsymbol{\Omega}}{d t} \times \boldsymbol{s}$$

$$= \boldsymbol{s} \times [g\, \mu_B (\boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{E}) - \omega_T]$$

where

$$\begin{aligned} \omega_T &= -\lim_{d t \to 0} \frac{\Delta \boldsymbol{\Omega}}{d t} \\ &= -\lim_{d t \to 0} \frac{1}{\beta^2} (\gamma - 1)\left(\boldsymbol{\beta} \times \frac{\delta \boldsymbol{\beta}}{d t}\right) \\ &= -\frac{1}{\beta^2 c} (\gamma - 1)(\boldsymbol{\beta} \times \boldsymbol{a}) \\ &= \frac{\gamma^2}{(\gamma + 1) c} (\boldsymbol{a} \times \boldsymbol{\beta}) \end{aligned}$$

with $\boldsymbol{a}$ being the acceleration.

For the spin-orbit problem,

$$m\, \boldsymbol{a} \simeq -\hat{\mathbf{r}} \frac{d V}{d r}$$

so that

$$\begin{aligned} \omega_T &= -\frac{\gamma^2}{(\gamma + 1) m c} \frac{d V}{d r} (\hat{\mathbf{r}} \times \boldsymbol{\beta}) \\ &= -\frac{\gamma^2}{(\gamma + 1) m c^2} \frac{d V}{r\, d r} (\boldsymbol{r} \times \boldsymbol{v}) \\ &= -\frac{\gamma^2}{(\gamma + 1) m^2 c^2} \frac{d V}{r\, d r} \boldsymbol{L} \\ &\xrightarrow[\text{Non-Rel}]{} -\frac{1}{2 m^2 c^2} \frac{d V}{r\, d r} \boldsymbol{L} \end{aligned}$$

Hence

$$U = U' + \boldsymbol{s}\cdot\boldsymbol{\omega}_T$$
$$= -g \frac{e\hbar}{2mc} \boldsymbol{s}\cdot\boldsymbol{B} + (g-1)\frac{1}{2m^2c^2}\boldsymbol{s}\cdot\boldsymbol{L}\frac{dV}{r\,dr}$$

For nucleons, we still have

$$U' = -g \frac{e\hbar}{2mc} \boldsymbol{s}\cdot\boldsymbol{B} + g\frac{1}{2m^2c^2}\boldsymbol{s}\cdot\boldsymbol{L}\frac{dV}{r\,dr}$$

where V is the screened coulomb potential.
But the acceleration giving ω_T is not due to the strong interactions V_N between nucleons.
Hence

$$U = U' + \boldsymbol{s}\cdot\boldsymbol{\omega}_T \simeq \boldsymbol{s}\cdot\boldsymbol{\omega}_T = -\frac{1}{2M^2c^2}\boldsymbol{s}\cdot\boldsymbol{L}\frac{dV}{r\,dr}$$

where M is the mass of the nucleon. Thus, the levels are inverted.

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9. Covariance

Consider a particle of charge q moving in fields $\boldsymbol{E}$ & $\boldsymbol{B}$.

The Lorentz force is

$$\frac{d}{dt}\boldsymbol{p} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) \qquad (11.124)$$

and the rate of change of energy

$$\frac{d}{dt}U = q\,\boldsymbol{v}\cdot\boldsymbol{E} \qquad (6.110)$$

The 4-force is defined as

$$\frac{d}{d\tau}p^\alpha = m\frac{d}{d\tau}v^\alpha = m\frac{d^2}{d\tau^2}x^\alpha$$

with

$$p^\alpha = \left(p^0,\ \boldsymbol{p}\right) = m\,u^\alpha = m\left(u^0,\ \boldsymbol{u}\right) = m\,\gamma\,(c,\boldsymbol{v})$$

The Lorentz force can be written as

$$\frac{d}{dt}\boldsymbol{p} = \frac{d}{\gamma\,d\tau}\boldsymbol{p} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) = \frac{q}{c}(c\,\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

or

$$\frac{d}{d\tau}\boldsymbol{p} = \frac{q}{c}\gamma(c\,\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$
$$= \frac{q}{c}\left(u^0\,\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\right) \qquad (11.125)$$

Likewise, the rate of change of energy is

$$\frac{d}{\gamma\, d\tau} U = q\, \boldsymbol{v}\cdot\boldsymbol{E}$$

or

$$\frac{d}{d\tau} U = c\, \frac{d}{d\tau} p^0$$
$$= \gamma\, q\, \boldsymbol{v}\cdot\boldsymbol{E} = q\, \boldsymbol{u}\cdot\boldsymbol{E}$$

ie

$$\frac{d}{d\tau} p^0 = \frac{q}{c}\, \boldsymbol{u}\cdot\boldsymbol{E} \qquad (11.126)$$

Thus

$$\frac{d}{d\tau} p^\alpha = \frac{q}{c} \left(\boldsymbol{u}\cdot\boldsymbol{E}\, ,\, u^0\, \boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} \right)$$

The right side will be shown later to be simply $\frac{q}{c} F^{\alpha\beta} u_\beta$.

The 4-current density is defined as

$$j^\alpha = (\, c\, \rho,\ \boldsymbol{j}\,) \qquad (11.128)$$

Thus, the equation of continuity

$$\frac{\partial}{\partial t} \rho + \nabla\cdot\boldsymbol{j} = 0 \qquad (11.127)$$

can be written as

$$\partial_\alpha\, j^\alpha = 0 \qquad (11.129)$$

Now, the 4-volume element $d^4 x$ is a Lorentz invariant, ie.

$$d^4 x' = \frac{\partial\left(x'^0,\ x'^1,\ x'^2,\ x'^3 \right)}{\partial\left(x^0,\ x^1,\ x^2,\ x^3 \right)} = \det A \cdot d^4 x = d^4 x$$

The charge $\rho\, d^3 x$ in the volume element $d^3 x$ is therefore a Lorentz invariant, ie,

$$\rho\, d^3 x = \rho'\, d^3 x'$$

since ρ is the time component of a 4-vector.

In the Lorentz gauge

$$\frac{1}{c} \frac{\partial}{\partial t} \Phi + \nabla\cdot\boldsymbol{A} = 0 \qquad (11.131)$$

The inhomogeneous Maxwell's equations become

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi - \nabla^2 \Phi = 4\pi\rho$$

$$(11.130)$$

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{A} - \nabla^2 \boldsymbol{A} = \frac{4\pi}{c}\, \boldsymbol{j}$$

Defining the 4-vector- potential by

$$A^\alpha = (\, \Phi,\ \boldsymbol{A}\,)$$

(11.131) & (11.130) become

$$\partial_\alpha A^\alpha = 0$$

$$\Box\, A^\alpha = \partial_\mu \partial^\mu A^\alpha = \frac{4\pi}{c}\, j^\alpha$$

On the other hand, the fields

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\Phi$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad (11.134)$$

can be written as

$$E^i = -\partial^0 A^i + \partial^i A^0$$

$$B^i = -\epsilon^{ijk}\,\partial^j A^k$$

$$= -\left(\partial^j A^k - \partial^k A^j\right) \qquad (i,\ j,\ k \text{ cyclic})$$

Define the field-strength tensor $F^{\alpha\beta}$ as

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \qquad (11.136)$$

We have

$$F^{\alpha\beta} = \left(\begin{array}{c|c} 0 & -\boldsymbol{E} \\ \hline \boldsymbol{E} & \mathbb{B} \end{array}\right) = \left(\begin{array}{c|ccc} 0 & -E^1 & -E^2 & -E^3 \\ \hline E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{array}\right) \qquad (11.137)$$

where

$$\mathbb{B} = \begin{pmatrix} 0 & -B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix}$$

Obviously,

$$E^i = -\partial^0 A^i + \partial^i A^0 = -F^{0i} = F^{i0}$$

$$B^i = -\epsilon^{ijk}\,\partial^j A^k$$

$$= -\frac{1}{2}\epsilon^{ijk}\left(\partial^j A^k - \partial^k A^j\right) = -\frac{1}{2}\epsilon^{ijk} F^{jk}$$

The covariant version is

$$F_{\alpha\beta} = g_{\alpha\mu}\, g_{\beta\nu}\, F^{\mu\nu} = g\, F\, g^T$$

$$= \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$= \left(\begin{array}{c|c} 0 & \boldsymbol{E} \\ \hline -\boldsymbol{E} & \mathbb{B} \end{array}\right) = \left(\begin{array}{c|ccc} 0 & E^1 & E^2 & E^3 \\ \hline -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{array}\right) \qquad (11.138)$$

From the definition

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

we have

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha$$

In the Lorentz gauge

$$\partial_\alpha A^\alpha = 0$$

we have

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta = \Box A^\beta$$

so that the inhomogeneous Maxwell eqs can be written as

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}\, j^\beta \qquad (11.141)$$

The simplest way to rewrite the homogeneous eqs

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B} = 0$$

in covariant form is to note that they can be obtained from the inhomogeneous ones by setting $\boldsymbol{E} \to \boldsymbol{B}$, $\boldsymbol{B} \to -\boldsymbol{E}$ and setting $j^{\alpha} = 0$.

Thus, by defining the dual field strength tensor as

$$\mathcal{F}^{\alpha\beta}(\boldsymbol{E}, \boldsymbol{B}) = F^{\alpha\beta}(\boldsymbol{E} \to \boldsymbol{B}, \boldsymbol{B} \to -\boldsymbol{E})$$

$$= \left(\begin{array}{c|c} 0 & -\boldsymbol{B} \\ \hline \boldsymbol{B} & -\mathbb{E} \end{array}\right) = \left(\begin{array}{c|ccc} 0 & -B^1 & -B^2 & -B^3 \\ \hline B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{array}\right)$$

where

$$\mathbb{E} = \begin{pmatrix} 0 & -E^3 & E^2 \\ E^3 & 0 & -E^1 \\ -E^2 & E^1 & 0 \end{pmatrix}$$

The homogeneous eqs are subsumed as

$$\partial_{\alpha} \mathcal{F}^{\alpha\beta} = 0$$

Note that

$$\mathcal{F}^{i0} = -\mathcal{F}^{0i} = B^i = -\frac{1}{2}\epsilon^{ijk} F^{jk}$$

$$\mathcal{F}^{ij} = \epsilon^{ijk} E^k = \epsilon^{ijk} F^{k0}$$

Using $\epsilon^{ijk} = \epsilon^{0ijk} = -\epsilon^{ijk0}$, these can written as

$$\mathcal{F}^{i0} = -\mathcal{F}^{0i} = -\frac{1}{2}\epsilon^{0ijk} F^{jk}$$

$$\mathcal{F}^{ij} = -\epsilon^{ijk0} F^{k0}$$

Using the fact that $\epsilon^{\alpha\beta\gamma\delta} = 0$ if any 2 of its indices are equal, we see that

$$\epsilon^{0ijk} F^{jk} = \epsilon^{0i\beta\gamma} F^{\beta\gamma}$$

since all terms with either β or $\gamma = 0$ vanish due to the $\epsilon^{0i\beta\gamma}$ factor. Also

$$\epsilon^{ijk0} F^{k0} = \epsilon^{ij0k} F^{0k} = \frac{1}{2}\epsilon^{ij\alpha\beta} F^{\alpha\beta}$$

Thus, we can write, with $\mathcal{F}^{00} = 0$,

$$\mathcal{F}^{\alpha 0} = -\mathcal{F}^{0\alpha} = -\frac{1}{2}\epsilon^{0\alpha\beta\gamma} F^{\beta\gamma} = -\frac{1}{2}\epsilon^{0\alpha\beta\gamma} F_{\beta\gamma} \qquad (\beta, \gamma \text{ restricted to } 1.3)$$

$$\mathcal{F}^{ij} = -\frac{1}{2}\epsilon^{ij\alpha\beta} F^{\alpha\beta} = \frac{1}{2}\epsilon^{ij\alpha\beta} F_{\alpha\beta} \qquad (\alpha, \beta \text{ restricted to } (0, k))$$

which can be combined as

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \qquad (11.140)$$

In a media, fields in the inhomogeneous Maxwell eqs should be replaced according to the rules

$$\boldsymbol{E} \to \boldsymbol{D} \qquad \boldsymbol{B} \to \boldsymbol{H}$$

Defining

$$G^{\alpha\beta}(\boldsymbol{D}, \boldsymbol{H}) = F^{\alpha\beta}(\boldsymbol{E} \to \boldsymbol{D}, \boldsymbol{B} \to \boldsymbol{H})$$

$$= \left(\begin{array}{c|c} 0 & -\boldsymbol{D} \\ \hline \boldsymbol{D} & \mathbb{H} \end{array}\right) = \left(\begin{array}{c|ccc} 0 & -D^1 & -D^2 & -D^3 \\ \hline D^1 & 0 & -H^3 & H^2 \\ D^2 & H^3 & 0 & -H^1 \\ D^3 & -H^2 & H^1 & 0 \end{array}\right)$$

we have

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} j^\beta \qquad (11.145)$$

10. Transformation of Electromagnetic Fields

■ 11.149

The field tensor $F^{\alpha\beta}$ transforms as

$$F^{\alpha'\beta'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^{\beta'}}{\partial x^\beta} F^{\alpha\beta} = \Lambda^{\alpha'}{}_\alpha \Lambda^{\beta'}{}_\beta F^{\alpha\beta}$$

or, in matrix form

$$F' = \Lambda F \Lambda^T$$

For a boost in the x^1 direction

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda^T$$

so that

$$F' = \left(\begin{array}{c|ccc} 0 & -E^{1'} & -E^{2'} & -E^{3'} \\ \hline E^{1'} & 0 & -B^{3'} & B^{2'} \\ E^{2'} & B^{3'} & 0 & -B^{1'} \\ E^{3'} & -B^{2'} & B^{1'} & 0 \end{array}\right)$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{c|ccc} 0 & -E^1 & -E^2 & -E^3 \\ \hline E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{array}\right) \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{c|ccc} \gamma\beta E^1 & -\gamma E^1 & -E^2 & -E^3 \\ \hline \gamma E^1 & -\gamma\beta E^1 & -B^3 & B^2 \\ \gamma E^2 - \gamma\beta B^3 & -\gamma\beta E^2 + \gamma B^3 & 0 & -B^1 \\ \gamma E^3 + \gamma\beta B^2 & -\gamma\beta E^3 - \gamma B^2 & B^1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|ccc} 0 & \gamma^2\beta^2 E^1 - \gamma^2 E^1 & -\gamma E^2 + \gamma\beta B^3 & -\gamma E^3 - \gamma\beta B^2 \\ \hline -\gamma^2\beta^2 E^1 + \gamma^2 E^1 & 0 & \gamma\beta E^2 - \gamma B^3 & \gamma\beta E^3 + \gamma B^2 \\ \gamma E^2 - \gamma\beta B^3 & -\gamma\beta E^2 + \gamma B^3 & 0 & -B^1 \\ \gamma E^3 + \gamma\beta B^2 & -\gamma\beta E^3 - \gamma B^2 & B^1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|ccc} 0 & -E^1 & -\gamma\left(E^2 - \beta B^3\right) & -\gamma\left(E^3 + \beta B^2\right) \\ \hline E^1 & 0 & -\gamma\left(-\beta E^2 + B^3\right) & \gamma\left(\beta E^3 + B^2\right) \\ \gamma\left(E^2 - \beta B^3\right) & \gamma\left(-\beta E^2 + B^3\right) & 0 & -B^1 \\ \gamma\left(E^3 + \beta B^2\right) & -\gamma\left(\beta E^3 + B^2\right) & B^1 & 0 \end{array}\right)$$

Hence

$$\begin{aligned} E^{1'} &= E^1 & B^{1'} &= B^1 \\ E^{2'} &= \gamma\left(E^2 - \beta B^3\right) & B^{2'} &= \gamma\left(\beta E^3 + B^2\right) \\ E^{3'} &= \gamma\left(E^3 + \beta B^2\right) & B^{3'} &= \gamma\left(-\beta E^2 + B^3\right) \end{aligned} \qquad (11.148)$$

Writing x^1 & $x^{1'}$ as the || direction, (11.148) can be written as

$$E_{||}' = E_{||} \qquad B_{||}' = B_{||}$$

$$\boldsymbol{E}_{\perp}' = \gamma(\boldsymbol{E}_{\perp} + \boldsymbol{\beta}\times\boldsymbol{B}) \qquad \boldsymbol{B}_{\perp}' = \gamma(\boldsymbol{B}_{\perp} - \boldsymbol{\beta}\times\boldsymbol{E})$$

These can be combined as

$$\begin{aligned} \boldsymbol{E}' &= \boldsymbol{E}_{||}' + \boldsymbol{E}_{\perp}' \\ &= \boldsymbol{E}_{||} + \gamma(\boldsymbol{E}_{\perp} + \boldsymbol{\beta}\times\boldsymbol{B}) \\ &= (1-\gamma)\boldsymbol{E}_{||} + \gamma(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}) \\ &= (1-\gamma)\frac{1}{\beta^2}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E}) + \gamma(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}) \end{aligned}$$

Since

$$\gamma^2 = \frac{1}{1-\beta^2}$$

or

$$\beta^2 = 1 - \frac{1}{\gamma^2} \qquad \frac{1}{\beta^2} = \frac{\gamma^2}{\gamma^2 - 1}$$

we have

$$(1-\gamma)\frac{1}{\beta^2} = (1-\gamma)\frac{\gamma^2}{\gamma^2-1} = -\frac{\gamma^2}{\gamma+1}$$

Hence

$$\boldsymbol{E}' = -\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{E}) + \gamma(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}) \qquad (11.149)$$

The corresponding expression for $\boldsymbol{B}$ can be obtained by the rule ($\boldsymbol{E}\to\boldsymbol{B}$, $\boldsymbol{B}\to-\boldsymbol{E}$):

$$\boldsymbol{B}' = -\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B}) + \gamma(\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E})$$

The inverse of this transformation is obtained by setting $\boldsymbol{\beta} \to -\boldsymbol{\beta}$ & interchanging primed & unprimed quantities, ie.

$$\boldsymbol{E} = -\frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \boldsymbol{E}') + \gamma(\boldsymbol{E}' - \boldsymbol{\beta} \times \boldsymbol{B}')$$

$$\boldsymbol{B} = -\frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta} (\boldsymbol{\beta} \cdot \boldsymbol{B}') + \gamma(\boldsymbol{B}' + \boldsymbol{\beta} \times \boldsymbol{E}')$$

In the nonrelativistic limit,

$$\gamma \simeq 1$$

$$\boldsymbol{B}' \simeq \boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{E} + O(\beta^2)$$

This was used in the discussion of the Thomas precession.

■ 11.150-4

Consider now a charged particle q moving with constant $\boldsymbol{v}$ along the x^1 axis and passing by an observation point $P = (0, b, 0)$.
[see fig 11.8]

Let the rest frame of the particle be K'.
For convenience, we set

$$\boldsymbol{x} = \boldsymbol{x}' \qquad \text{at} \qquad t = t' = 0$$

In K', the charge is stationary so that

$$\boldsymbol{B}' = 0$$

$$\boldsymbol{E}' = \frac{q}{r'^2} \hat{\boldsymbol{r}}'$$

where $\boldsymbol{r}'$ is the vector from the charge to the observation point P.

Obviously, the co-ordinates of P in K' are

$$\boldsymbol{r}' = (-v t', b, 0)$$

so that

$$r' = \sqrt{(v t')^2 + b^2}$$

$$\hat{\boldsymbol{r}}' = \left(-\frac{v t'}{\sqrt{(v t')^2 + b^2}}, \frac{b}{\sqrt{(v t')^2 + b^2}}, 0 \right)$$

$$\boldsymbol{E}' = \left(-\frac{q v t'}{\left[(v t')^2 + b^2 \right]^{3/2}}, \frac{q b}{\left[(v t')^2 + b^2 \right]^{3/2}}, 0 \right)$$

We wish to find the fields at P in the coordinate system K where P is stationary.

Eq (11.148) [or, rather, its inverse] then gives

$$E^1 = E^{1'} = -\frac{q v t'}{\left[(v t')^2 + b^2 \right]^{3/2}}$$

$$E^2 = \gamma \left(E^{2'} + \beta B^{3'} \right) = \frac{\gamma q b}{\left[(v t')^2 + b^2 \right]^{3/2}}$$

$$E^3 = \gamma \left(E^{3'} - \beta B^{2'} \right) = 0$$

$$B^1 = B^{1'} = 0$$
$$B^2 = \gamma\left(-\beta E^{3'} + B^{2'}\right) = 0$$
$$B^3 = \gamma\left(\beta E^{2'} + B^{3'}\right) = \frac{\gamma \beta q b}{\left[(v t')^2 + b^2\right]^{3/2}}$$

Obviously, it's more convenient to have these quantities as a function of t instead of t'.
Now, the coordinates of P in K are fixed:

$$\mathbf{r}_P = (0, b, 0)$$

Using

$$c t' = \gamma\left(c t - \frac{v}{c} x^1\right)$$

we see that t' is related to the time t at P by

$$c t' = \gamma\, c t$$

Hence

$$E^1 = E^{1'} = -\frac{\gamma q v t}{\left[(\gamma v t)^2 + b^2\right]^{3/2}}$$
$$E^2 = \gamma\left(E^{2'} + \beta B^{3'}\right) = \frac{\gamma q b}{\left[(\gamma v t)^2 + b^2\right]^{3/2}}$$
$$E^3 = \gamma\left(E^{3'} - \beta B^{2'}\right) = 0$$

(11.152)

$$B^1 = B^{1'} = 0$$
$$B^2 = \gamma\left(-\beta E^{3'} + B^{2'}\right) = 0$$
$$B^3 = \gamma\left(\beta E^{2'} + B^{3'}\right) = \frac{\gamma \beta q b}{\left[(\gamma v t)^2 + b^2\right]^{3/2}} = \beta E^2$$

Note that in K, the position of the charge at time t' is

$$\mathbf{r}_q = (v t', 0, 0) = (\gamma v t, 0, 0)$$

so that

$$\mathbf{r} = \mathbf{r}_P - \mathbf{r}_q = (-\gamma v t, b, 0)$$

For the non-relativistic case, $\gamma \simeq 1$, so that

$$\mathbf{r} \simeq (-v t, b, 0)$$
$$\mathbf{E} \simeq \frac{q}{r^3}\mathbf{r}$$
$$\mathbf{B} \simeq \left(0, 0, q\frac{v}{c}\frac{b}{r^3}\right) = \frac{q}{c r^3}\mathbf{v}\times\mathbf{r}$$

where

$$\mathbf{v} = (v, 0, 0)$$

For ultra-relativistic motion, $\beta \simeq 1$, $\gamma \gg 1$, so that

$$B^3 \simeq E^2$$

so that the fields are transverse.
Furthermore, E^2 peaks at $t = 0$ with value

$$E^2 = \gamma\frac{q}{b^2}$$

It falls off (to $\frac{1}{2\sqrt{2}}$ peak value) after a time $\Delta t = \frac{b}{\gamma v}$.

11. Relativistic Equation of Motion for Spin

In the rest frame K' of the particle,

$$\frac{d}{dt'}\boldsymbol{s}' = \frac{g\,e}{2\,m\,c}\,\boldsymbol{s}'\times B' \qquad (11.155)$$

a) Covariant Equation of Motion

The 4-spin vector in the rest frame is defined as

$$s^{\alpha'} = (0,\ \boldsymbol{s}')$$

ie

$$s^{0'} = 0$$

In a frame K where the particle move with velocity $c\,\boldsymbol{\beta}$ (K' moves with $c\,\boldsymbol{\beta}$ relative to K), the 4-spin vector is s^{α}. Obviously

$$s^{0'} = 0 = \gamma\left(s^0 - \boldsymbol{\beta}\cdot\boldsymbol{s}\right) = \frac{1}{c}\,u_\alpha\,s^{\alpha}$$

where $u^\alpha = c\,\gamma(1,\ \boldsymbol{\beta})$ is the 4-velocity of the particle in K.

In other words, $s^{0'} = 0$ implies

$$u\cdot s = u_\alpha\,s^\alpha = 0 \qquad (11.156)$$

in every inertial frame. Thus

$$s^\alpha = \left(s^0,\ \boldsymbol{s}\right) = (\boldsymbol{\beta}\cdot\boldsymbol{s},\ \boldsymbol{s}) \qquad (11.157)$$

Now

$$\begin{aligned}
s_{||}' &= \gamma\left(s_{||} - \beta\,s_0\right) = \gamma\left[s_{||} - \beta\,(\boldsymbol{\beta}\cdot\boldsymbol{s})\right] \\
&= \gamma\left(1-\beta^2\right)\left(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{s}\right) \\
&= \frac{1}{\gamma}\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{s} \\
&= \frac{1}{\gamma}\,s_{||}
\end{aligned}$$

$$\boldsymbol{s}_\perp' = \boldsymbol{s}_\perp$$

so that

$$\begin{aligned}
\boldsymbol{s}' &= \boldsymbol{s}_{||}' + \boldsymbol{s}_\perp' \\
&= \frac{1}{\gamma}\,\boldsymbol{s}_{||} + \boldsymbol{s} - \boldsymbol{s}_{||} \\
&= \boldsymbol{s} + \left(\frac{1}{\gamma} - 1\right)\left(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{s}\right)\hat{\boldsymbol{\beta}}
\end{aligned}$$

$$= \; \boldsymbol{s} + \left(\frac{1}{\gamma} - 1\right) \frac{1}{\beta^2} (\boldsymbol{\beta} \cdot \boldsymbol{s})\, \boldsymbol{\beta}$$

$$= \; \boldsymbol{s} + \left(\frac{1}{\gamma} - 1\right) \frac{1}{\beta^2} (\boldsymbol{\beta} \cdot \boldsymbol{s})\, \boldsymbol{\beta}$$

Using

$$\frac{1}{\beta^2} = \frac{\gamma^2}{\gamma^2 - 1}$$

we have

$$\boldsymbol{s}' = \boldsymbol{s} - \frac{\gamma}{\gamma + 1} (\boldsymbol{\beta} \cdot \boldsymbol{s})\, \boldsymbol{\beta} \qquad (11.158)$$

The inverse is

$$\boldsymbol{s} = \boldsymbol{s}_{||} + \boldsymbol{s}_{\perp}$$

$$= \; \gamma\, \boldsymbol{s}_{||}' + \boldsymbol{s}_{\perp}'$$

$$= \; \boldsymbol{s}' + (\gamma - 1)\, \boldsymbol{s}_{||}'$$

$$= \; \boldsymbol{s}' + (\gamma - 1) \frac{1}{\beta^2} (\boldsymbol{\beta} \cdot \boldsymbol{s}')\, \boldsymbol{\beta}$$

$$= \; \boldsymbol{s}' + \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \boldsymbol{s}')\, \boldsymbol{\beta} \qquad (11.159)$$

$$s^0 = \boldsymbol{\beta} \cdot \boldsymbol{s} = \boldsymbol{\beta} \cdot \left[\boldsymbol{s}' + \frac{\gamma^2}{\gamma + 1} (\boldsymbol{\beta} \cdot \boldsymbol{s}')\, \boldsymbol{\beta} \right]$$

$$= \; (\boldsymbol{\beta} \cdot \boldsymbol{s}') \left[1 + \frac{\gamma^2}{\gamma + 1} \beta^2 \right]$$

$$= \; (\boldsymbol{\beta} \cdot \boldsymbol{s}')\, \gamma$$

Note: (11.159) cannot be obtained from (11.158) by the usual rule $\beta \to -\beta$, & primed $\leftrightarrow$ unprimed. Why?

One way to generalize (11.155) or (11.103) is to write the left side as $\frac{d}{d\tau} s^\alpha$; the right side must then be a combination of s^α, $F^{\alpha\beta}$, U^α and $\frac{d}{d\tau} u^\alpha$ that results in a 4-vector. ($\frac{d}{d\tau} u^\alpha$ is included since $\frac{d}{d\tau} u^\alpha = \frac{q}{m\,c} F^{\alpha\beta} u_\beta$). Some possible terms linear in both s and F are:

$$F^{\alpha\beta} s_\beta \qquad \left(s_\mu F^{\mu\nu} u_\nu \right) u^\alpha \qquad \left(s_\mu \frac{d}{d\tau} u^\mu \right) u^\alpha$$

Other combinations can be shown to reduce to the linear combinations of these 3. Thus

$$\frac{d}{d\tau} s^\alpha = \; A_1 F^{\alpha\beta} s_\beta + \frac{A_2}{c^2} \left(s_\mu F^{\mu\nu} u_\nu \right) u^\alpha + \frac{A_3}{c^2} \left(s_\mu \frac{d}{d\tau} u^\mu \right) u^\alpha \qquad (11.160)$$

where A_i are constants of the same dimension (time / field).

Since (11.156) holds for all τ, we have

$$\frac{d}{d\tau}(u_\alpha s^\alpha) = u_\alpha \frac{d}{d\tau} s^\alpha + s^\alpha \frac{d}{d\tau} u_\alpha = 0$$

Hence

$$-s^\alpha \frac{d}{d\tau} u_\alpha = u_\alpha \frac{d}{d\tau} s^\alpha$$

$$= u_\alpha \left[A_1 F^{\alpha\beta} s_\beta + \frac{A_2}{c^2} \left(s_\mu F^{\mu\nu} u_\nu \right) u^\alpha + \frac{A_3}{c^2} \left(s_\mu \frac{d}{d\tau} u^\mu \right) u^\alpha \right]$$

$$= A_1 u_\alpha F^{\alpha\beta} s_\beta + A_2 \; s_\mu F^{\mu\nu} u_\nu + A_3 s_\mu \frac{d}{d\tau} u^\mu$$

where $u_\alpha u^\alpha = c^2$.

Now

$$s_\mu F^{\mu\nu} u_\nu = -s_\mu F^{\nu\mu} u_\nu = -u_\alpha F^{\alpha\beta} s_\beta$$

$$s^\alpha \frac{d}{d\tau} u_\alpha = s_\mu \frac{d}{d\tau} u^\mu$$

Hence

$$(A_1 - A_2) u_\alpha F^{\alpha\beta} s_\beta + (A_3 + 1) s_\mu \frac{d}{d\tau} u^\mu = 0 \qquad (11.161)$$

If there is some forces other than electromagnetic present, the second term, through $\frac{d}{d\tau} u^\mu$, will depend on it while the first term won't. Hence, these 2 terms must vanish separately, ie.,

$$A_1 = A_2 \qquad\qquad A_3 = -1$$

so that

$$\frac{d}{d\tau} s^\alpha = A_1 \left[F^{\alpha\beta} s_\beta + \frac{1}{c^2} \left(s_\mu F^{\mu\nu} u_\nu \right) u^\alpha \right] - \frac{1}{c^2} \left(s_\mu \frac{d}{d\tau} u^\mu \right) u^\alpha$$

In the rest frame, $u = 0$, so that

$$\frac{d}{dt'} s^\alpha = A_1 F^{\alpha\beta} s_\beta$$

thus

$$\frac{d}{dt'} s^i = A_1 F'^{i\beta} s_\beta = A_1 F'^{ij} s_j$$

$$= -A_1 \epsilon_{ijk} B_k' s_j$$

$$= \mathrm{A}_1 \epsilon_{ijk} B_k' s^j$$

$$= A_1 (\boldsymbol{s} \times \boldsymbol{B}')^i$$

Comparing with (11.155) gives

$$A_1 = \frac{g e}{2 m c}$$

Hence

$$\frac{d}{d\tau} s^\alpha = \frac{g e}{2 m c} \left[F^{\alpha\beta} s_\beta + \frac{1}{c^2} \left(s_\mu F^{\mu\nu} u_\nu \right) u^\alpha \right] - \frac{1}{c^2} \left(s_\mu \frac{d}{d\tau} u^\mu \right) u^\alpha \quad (11.162)$$

If the particle is subject only to a Lorentz force,

$$\frac{d}{d\tau} u^{\alpha} = \frac{e}{m c} F^{\alpha\beta} u_{\beta} \qquad (11.163)$$

we have

$$\frac{1}{c^2}\left(s_{\mu} \frac{d}{d\tau} u^{\mu} \right) u^{\alpha} = \frac{e}{m c^3} \left(s_{\mu} F^{\mu\nu} u_{\nu} \right) u^{\alpha}$$

so that

$$\frac{d}{d\tau} s^{\alpha} = \frac{g e}{2 m c} F^{\alpha\beta} s_{\beta} + \frac{e}{m c^3} \left(\frac{g}{2} - 1 \right) \left(s_{\mu} F^{\mu\nu} u_{\nu} \right) u^{\alpha} \qquad (11.164)$$

b) Connection to Thomas Precession

In the K frame,

$$s^{\alpha} = \left(s^0, \boldsymbol{s} \right) = \left(\boldsymbol{\beta}\cdot\boldsymbol{s}, \boldsymbol{s} \right)$$

$$u^{\alpha} = \left(u^0, \boldsymbol{u} \right) = \gamma c \left(1, \boldsymbol{\beta} \right)$$

Hence

$$\begin{aligned} s_{\alpha} \frac{d}{d\tau} u^{\alpha} &= s_0 \frac{d}{d\tau} u^0 - \boldsymbol{s}\cdot \frac{d}{d\tau} \boldsymbol{u} \\ &= \boldsymbol{\beta}\cdot\boldsymbol{s}\, c \frac{d}{d\tau} \gamma - \boldsymbol{s}\cdot c \frac{d}{d\tau} \left(\gamma \boldsymbol{\beta} \right) \\ &= c\, \boldsymbol{s}\cdot \left[\boldsymbol{\beta} \frac{d}{d\tau} \gamma - \frac{d}{d\tau} \left(\gamma \boldsymbol{\beta} \right) \right] \\ &= -\gamma c\, \boldsymbol{s}\cdot \frac{d}{d\tau} \boldsymbol{\beta} \qquad (11.165) \end{aligned}$$

Writing (11.162) as

$$\frac{d}{d\tau} s^{\alpha} = \mathcal{F}^{\alpha} - \frac{1}{c^2} \left(s_{\mu} \frac{d}{d\tau} u^{\mu} \right) u^{\alpha}$$

where

$$\mathcal{F}^{\alpha} = \frac{g e}{2 m c} \left[F^{\alpha\beta} s_{\beta} + \frac{1}{c^2} \left(s_{\mu} F^{\mu\nu} u_{\nu} \right) u^{\alpha} \right]$$

it becomes, with the help of (11.165),

$$\frac{d}{d\tau} s^{\alpha} = \mathcal{F}^{\alpha} + \frac{1}{c} \left(\gamma\, \boldsymbol{s}\cdot \frac{d}{d\tau} \boldsymbol{\beta} \right) u^{\alpha}$$

or,

$$\frac{d s^0}{d\tau} = \mathcal{F}^0 + \gamma^2 \boldsymbol{s}\cdot \frac{d \boldsymbol{\beta}}{d\tau}$$

$$\frac{d \boldsymbol{s}}{d\tau} = \boldsymbol{\mathcal{F}} + \left(\gamma^2 \boldsymbol{s}\cdot \frac{d \boldsymbol{\beta}}{d\tau} \right) \boldsymbol{\beta} \qquad (A)$$

with

$$\mathcal{F}^{\alpha} = \left(\mathcal{F}^0, \boldsymbol{\mathcal{F}} \right)$$

Note that

$$\begin{aligned} u_\alpha \mathcal{F}^\alpha = u_\alpha \frac{g\,e}{2\,m\,c}\left[F^{\alpha\beta} s_\beta + \frac{1}{c^2}\left(s_\mu F^{\mu\nu} u_\nu\right) u^\alpha \right] \\ = \frac{g\,e}{2\,m\,c}\left[u_\alpha F^{\alpha\beta} s_\beta + s_\mu F^{\mu\nu} u_\nu \right] \\ = \frac{g\,e}{2\,m\,c}\left[u_\alpha F^{\alpha\beta} s_\beta - s_\mu F^{\nu\mu} u_\nu \right] \\ = 0 \end{aligned}$$

ie.

$$u^0 \mathcal{F}^0 - \boldsymbol{u}\cdot\boldsymbol{\mathcal{F}} = \gamma\, c\left(\mathcal{F}^0 - \boldsymbol{\beta}\cdot\boldsymbol{\mathcal{F}}\right) = 0$$

or

$$\mathcal{F}^0 = \boldsymbol{\beta}\cdot\boldsymbol{\mathcal{F}}$$
$$\mathcal{F}^\alpha = (\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}})$$

From

$$\boldsymbol{s}' = \boldsymbol{s} - \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{s})\,\boldsymbol{\beta} \qquad (11.159)$$
$$= \boldsymbol{s} - \frac{\gamma}{\gamma+1} s^0\,\boldsymbol{\beta}$$

we have

$$\frac{d\,\boldsymbol{s}'}{d\,\tau} = \frac{d\,\boldsymbol{s}}{d\,\tau} - \frac{d}{d\,\tau}\left[\frac{\gamma}{\gamma+1} s^0\,\boldsymbol{\beta}\right]$$
$$= \frac{d\,\boldsymbol{s}}{d\,\tau} - \left(\frac{d}{d\,\tau}\frac{\gamma}{\gamma+1}\right) s^0\,\boldsymbol{\beta} - \frac{\gamma}{\gamma+1}\left(\frac{d\,s^0}{d\,\tau}\right)\boldsymbol{\beta} - \frac{\gamma}{\gamma+1} s^0 \frac{d\,\boldsymbol{\beta}}{d\,\tau} \qquad \text{(B)}$$

Now

$$\frac{d}{d\,\tau}\frac{\gamma}{\gamma+1} = \frac{d}{d\,\tau}\left(1 - \frac{1}{\gamma+1}\right) = \frac{1}{(\gamma+1)^2}\left(\frac{d\,\gamma}{d\,\tau}\right) = \frac{\gamma^3}{(\gamma+1)^2}\,\boldsymbol{\beta}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau}$$

where

$$\frac{d\,\gamma}{d\,\tau} = \frac{d}{d\,\tau}\frac{1}{\sqrt{1-\beta^2}} = \gamma^3\,\boldsymbol{\beta}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau}$$

Hence, with the help of (A), (B) becomes

$$\frac{d\,\boldsymbol{s}'}{d\,\tau} = \frac{d\,\boldsymbol{s}}{d\,\tau} - \frac{\gamma^3}{(\gamma+1)^2}\left(\boldsymbol{\beta}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau}\right) s^0\,\boldsymbol{\beta} - \frac{\gamma}{\gamma+1}\left(\frac{d\,s^0}{d\,\tau}\right)\boldsymbol{\beta} - \frac{\gamma}{\gamma+1} s^0\frac{d\,\boldsymbol{\beta}}{d\,\tau}$$
$$= \boldsymbol{\mathcal{F}} + \left(\gamma^2\,\boldsymbol{s}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau}\right)\boldsymbol{\beta} - \frac{\gamma^3}{(\gamma+1)^2}\left(\boldsymbol{\beta}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau}\right) s^0\,\boldsymbol{\beta}$$
$$-\frac{\gamma}{\gamma+1}\left(\mathcal{F}^0 + \gamma^2\,\boldsymbol{s}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau}\right)\boldsymbol{\beta} - \frac{\gamma}{\gamma+1} s^0\frac{d\,\boldsymbol{\beta}}{d\,\tau}$$
$$= \boldsymbol{\mathcal{F}} - \frac{\gamma}{\gamma+1}\mathcal{F}^0\,\boldsymbol{\beta} + \frac{\gamma^2}{\gamma+1}\left[\boldsymbol{s}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau}\right) s^0\right]\boldsymbol{\beta} - \frac{\gamma}{\gamma+1} s^0\frac{d\,\boldsymbol{\beta}}{d\,\tau}$$
$$= \boldsymbol{\mathcal{F}} - \frac{\gamma}{\gamma+1}\mathcal{F}^0\,\boldsymbol{\beta} + \frac{\gamma^2}{\gamma+1}\left[\boldsymbol{s}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\frac{d\,\boldsymbol{\beta}}{d\,\tau}\right)(\boldsymbol{s}\cdot\beta)\right]\boldsymbol{\beta} - \frac{\gamma}{\gamma+1}(\boldsymbol{s}\cdot\beta)\frac{d\,\boldsymbol{\beta}}{d\,\tau}$$

Now,

$$s^0 = \boldsymbol{s}\cdot\boldsymbol{\beta} = \gamma\,\boldsymbol{s}'\cdot\boldsymbol{\beta}$$

$$\boldsymbol{s}\cdot\frac{d\boldsymbol{\beta}}{d\tau} = \left[\boldsymbol{s}' + \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{s}')\,\boldsymbol{\beta}\right]\cdot\frac{d\boldsymbol{\beta}}{d\tau}$$

$$= \boldsymbol{s}'\cdot\frac{d\boldsymbol{\beta}}{d\tau} + \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{s}')\left(\boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right)$$

$$= \boldsymbol{s}'\cdot\frac{d\boldsymbol{\beta}}{d\tau} + \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{s})\left(\boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right)$$

so that

$$\frac{d\boldsymbol{s}'}{d\tau} = \boldsymbol{\mathcal{F}} - \frac{\gamma}{\gamma+1}\mathcal{F}^0\,\boldsymbol{\beta} + \frac{\gamma^2}{\gamma+1}\left[\left(\boldsymbol{s}'\cdot\frac{d\boldsymbol{\beta}}{d\tau}\right)\boldsymbol{\beta} - (\boldsymbol{s}'\cdot\boldsymbol{\beta})\frac{d\boldsymbol{\beta}}{d\tau}\right]$$

$$= \boldsymbol{\mathcal{F}} - \frac{\gamma}{\gamma+1}\mathcal{F}^0\,\boldsymbol{\beta} + \frac{\gamma^2}{\gamma+1}\boldsymbol{s}'\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{d\tau}\right) \qquad (11.166)$$

Now

$$\boldsymbol{\mathcal{F}}' = \boldsymbol{\mathcal{F}} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{F}})\,\boldsymbol{\beta} - \gamma\,\boldsymbol{\beta}\,\mathcal{F}^0$$

$$= \boldsymbol{\mathcal{F}} + \left[\frac{\gamma-1}{\beta^2} - \gamma\right]\boldsymbol{\beta}\,\mathcal{F}^0 \qquad (\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{F}} = \mathcal{F}^0)$$

$$= \boldsymbol{\mathcal{F}} + \left[\frac{\gamma-1}{\beta^2} - \gamma\right]\boldsymbol{\beta}\,\mathcal{F}^0$$

$$= \boldsymbol{\mathcal{F}} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}\,\mathcal{F}^0$$

where we've used

$$\frac{1}{\beta^2} = \frac{\gamma^2}{\gamma^2-1}$$

$$\frac{\gamma-1}{\beta^2} - \gamma = \frac{\gamma^2}{\gamma+1} - \gamma = -\frac{\gamma}{\gamma+1}$$

Thus

$$\frac{d\boldsymbol{s}'}{d\tau} = \boldsymbol{\mathcal{F}}' + \frac{\gamma^2}{\gamma+1}\boldsymbol{s}'\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{d\tau}\right)$$

$$\frac{d\boldsymbol{s}'}{dt} = \frac{1}{\gamma}\boldsymbol{\mathcal{F}}' + \frac{\gamma^2}{\gamma+1}\boldsymbol{s}'\times\left(\boldsymbol{\beta}\times\frac{d\boldsymbol{\beta}}{dt}\right) \qquad (dt = \gamma\,d\tau)$$

$$= \frac{1}{\gamma}\boldsymbol{\mathcal{F}}' - \boldsymbol{s}'\times\omega_T \qquad (11.167)$$

where

$$\boldsymbol{\omega}_T = \frac{\gamma^2}{\gamma+1}\frac{d\boldsymbol{\beta}}{dt}\times\boldsymbol{\beta} \qquad (11.119)$$

For Lorentz forces,

$$\frac{d}{d\tau} u^{\alpha} = \frac{e}{m c} F^{\alpha\beta} u_{\beta} \qquad (11.163)$$

the space part is

$$\gamma \frac{d}{dt}\left(\gamma c \beta^{i}\right) = \frac{e}{m c}\left(F^{i\alpha} u_{\alpha}\right)$$

$$= \frac{e}{m} \gamma \left(F^{i0} - F^{ij} \beta^{j}\right) \qquad [\, u_{\alpha} = \gamma c\,(1, -\boldsymbol{\beta})\,]$$

$$= \frac{e}{m} \gamma \left[E_i + \epsilon_{ijk} B_k \beta^{j}\right]$$

$$= \frac{e}{m} \gamma (\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B})^{i}$$

or

$$\frac{d}{dt}(\gamma\ \boldsymbol{\beta}) = \frac{e}{m c}(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B})$$

$$= \frac{d\gamma}{dt}\boldsymbol{\beta} + \gamma \frac{d\boldsymbol{\beta}}{dt}$$

so that

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{e}{\gamma m c}(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}) - \frac{1}{\gamma}\frac{d\gamma}{dt}\boldsymbol{\beta}$$

Now

$$\frac{d\gamma}{dt} = \frac{d\gamma}{\gamma\, d\tau} = \gamma^{2} \boldsymbol{\beta}\cdot \frac{d\boldsymbol{\beta}}{d\tau} = \gamma^{3} \boldsymbol{\beta}\cdot \frac{d\boldsymbol{\beta}}{dt}$$

Hence

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{e}{\gamma m c}(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B}) - \gamma^{2}\left(\boldsymbol{\beta}\cdot \frac{d\boldsymbol{\beta}}{dt}\right)\boldsymbol{\beta}$$

Taking $\boldsymbol{\beta}\cdot$ on both sides gives

$$\boldsymbol{\beta}\cdot \frac{d\boldsymbol{\beta}}{dt} = \frac{e}{\gamma m c}\boldsymbol{\beta}\cdot \boldsymbol{E} - \gamma^{2}\left(\boldsymbol{\beta}\cdot \frac{d\boldsymbol{\beta}}{dt}\right)\beta^{2}$$

$$= \frac{1}{1 + \gamma^{2}\beta^{2}} \cdot \frac{e}{\gamma m c}\boldsymbol{\beta}\cdot \boldsymbol{E}$$

$$= \frac{e}{\gamma^{3} m c}\boldsymbol{\beta}\cdot \boldsymbol{E}$$

Hence

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{e}{\gamma m c}[\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B} - (\boldsymbol{\beta}\cdot \boldsymbol{E})\boldsymbol{\beta}] \qquad (11.168)$$

From

$$\mathcal{F}^{\alpha} = \frac{g e}{2 m c}\left[F^{\alpha\beta} s_{\beta} + \frac{1}{c^{2}}\left(s_{\mu} F^{\mu\nu} u_{\nu}\right) u^{\alpha}\right]$$

we see that

$$\mathcal{F}^{i} = \frac{g e}{2 m c}\left[F^{i\beta} s_{\beta} + \frac{1}{c^{2}}\left(s_{\mu} F^{\mu\nu} u_{\nu}\right) u^{i}\right]$$

In the rest frame, $\boldsymbol{v} = 0$, so that

$$\mathcal{F}^{i'} = \frac{g\,e}{2\,m\,c} F^{i'\beta'} s_{\beta'}$$
$$= \frac{g\,e}{2\,m\,c} F^{i'j'} s_{j'} \qquad\qquad (s^{0'} = 0)$$
$$= \frac{g\,e}{2\,m\,c} \epsilon_{i'j'k'} B_{k'} s^{j'}$$

or

$$\boldsymbol{\mathcal{F}}' = \frac{g\,e}{2\,m\,c} \boldsymbol{s}'\times\boldsymbol{B}'$$
$$= \frac{g\,e}{2\,m\,c} \boldsymbol{s}'\times\left[\gamma\,(\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\right]$$

where (11.149) was used to get the last expression.

Thus

$$\frac{1}{\gamma}\boldsymbol{\mathcal{F}}' = \frac{g\,e}{2\,m\,c} \boldsymbol{s}'\times\left[\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\right] \qquad (11.169)$$

Using (11.168), (11.119) becomes

$$\boldsymbol{\omega}_T = \frac{\gamma^2}{\gamma+1}\frac{d\,\boldsymbol{\beta}}{d\,t}\times\boldsymbol{\beta}$$
$$= \frac{e\,\gamma}{(\gamma+1)\,m\,c}(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B})\times\boldsymbol{\beta}$$

Thus, (11.167) becomes

$$\frac{d\,\boldsymbol{s}'}{d\,t} = \frac{1}{\gamma}\boldsymbol{\mathcal{F}}' - \boldsymbol{s}'\times\boldsymbol{\omega}_T$$
$$= \frac{g\,e}{2\,m\,c} \boldsymbol{s}'\times\left[\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\right]$$
$$- \frac{e\,\gamma}{(\gamma+1)\,m\,c}\boldsymbol{s}'\times[\,(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B})\times\boldsymbol{\beta}\,]$$

Now

$$\boldsymbol{s}'\times[\,(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B})\times\boldsymbol{\beta}\,] = \boldsymbol{s}'\times(\boldsymbol{E}\times\boldsymbol{\beta}) + \boldsymbol{s}'\times[(\boldsymbol{\beta}\times\boldsymbol{B})\times\boldsymbol{\beta}]$$
$$= \boldsymbol{s}'\times\left[\boldsymbol{E}\times\boldsymbol{\beta} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B}) + \boldsymbol{B}\,\beta^2\right]$$

Hence

$$\frac{d\,\boldsymbol{s}'}{d\,t} = \frac{g\,e}{2\,m\,c} \boldsymbol{s}'\times\left[\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{E} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\right]$$
$$- \frac{\gamma\,e}{(\gamma+1)\,m\,c}\boldsymbol{s}'\times\left[\boldsymbol{B}\,\beta^2 - \boldsymbol{\beta}\times\boldsymbol{E} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\right]$$
$$= \frac{e}{m\,c}\boldsymbol{s}'\times\left[\left(\frac{g}{2} - \frac{\gamma\,\beta^2}{\gamma+1}\right)\boldsymbol{B} - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\boldsymbol{E} - \frac{\gamma}{\gamma+1}\left(\frac{g}{2} - 1\right)\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\right]$$
$$= \frac{e}{m\,c}\boldsymbol{s}'\times\left[\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\boldsymbol{B} - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\boldsymbol{\beta}\times\boldsymbol{E} - \frac{\gamma}{\gamma+1}\left(\frac{g}{2} - 1\right)\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{B})\right]$$

(11.170)

where

$$\frac{\gamma\,\beta^2}{\gamma+1} = \frac{\gamma}{\gamma+1}\,\frac{\gamma^2-1}{\gamma^2} = \frac{\gamma-1}{\gamma} = 1-\frac{1}{\gamma}$$

c) Logitudinal Polarization

Consider the time rate of change of the longitudinal polarization, $\hat{\boldsymbol\beta}\cdot\boldsymbol s\,'$,

$$\frac{d}{dt}\left(\hat{\boldsymbol\beta}\cdot\boldsymbol s'\right) = \hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol s'}{dt} + \boldsymbol s\,'\cdot\frac{d}{dt}\frac{\boldsymbol\beta}{\beta}$$

$$= \hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol s'}{dt} + \boldsymbol s\,'\cdot\left[\frac{1}{\beta}\frac{d\boldsymbol\beta}{dt} - \frac{\boldsymbol\beta}{\beta^2}\frac{d\beta}{dt}\right]$$

$$= \hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol s\,'}{dt} + \frac{1}{\beta}\boldsymbol s\,'\cdot\left[\frac{d\boldsymbol\beta}{dt} - \hat{\boldsymbol\beta}\frac{d\beta}{dt}\right]$$

$$= \hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol s\,'}{dt} + \frac{1}{\beta}\left[\boldsymbol s\,'\cdot\frac{d\boldsymbol\beta}{dt} - \left(\boldsymbol s\,'\cdot\hat{\boldsymbol\beta}\right)\frac{d\beta}{dt}\right]$$

Using

$$\frac{d\beta}{dt} = \frac{d}{dt}\sqrt{\boldsymbol\beta\cdot\boldsymbol\beta} = \frac{1}{\beta}\boldsymbol\beta\cdot\frac{d\boldsymbol\beta}{dt} = \hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol\beta}{dt}$$

we have

$$\frac{d}{dt}\left(\hat{\boldsymbol\beta}\cdot\boldsymbol s\,'\right) = \hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol s\,'}{dt} + \frac{1}{\beta}\left[\boldsymbol s\,' - \left(\boldsymbol s\,'\cdot\hat{\boldsymbol\beta}\right)\hat{\boldsymbol\beta}\right]\cdot\frac{d\boldsymbol\beta}{dt}$$

$$= \hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol s\,'}{dt} + \frac{1}{\beta}\boldsymbol s_\perp{}'\cdot\frac{d\boldsymbol\beta}{dt}$$

$$= \frac{e}{mc}\boldsymbol s'\times\left[\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\boldsymbol B - \left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\boldsymbol\beta\times\boldsymbol E - \frac{\gamma}{\gamma+1}\left(\frac{g}{2}-1\right)\boldsymbol\beta\,(\boldsymbol\beta\cdot\boldsymbol B)\right]$$

From 11.170, we have

$$\hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol s\,'}{dt} = \frac{e}{mc}\left\{\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\hat{\boldsymbol\beta}\cdot(\boldsymbol s'\times\boldsymbol B) - \left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\hat{\boldsymbol\beta}\cdot[\boldsymbol s'\times(\boldsymbol\beta\times\boldsymbol E)]\right\}$$

Now

$$\hat{\boldsymbol\beta}\cdot(\boldsymbol s'\times\boldsymbol B) = -\boldsymbol s'\cdot\left(\hat{\boldsymbol\beta}\times\boldsymbol B\right) = -\boldsymbol s_\perp{}'\cdot\left(\hat{\boldsymbol\beta}\times\boldsymbol B\right)$$

$$\hat{\boldsymbol\beta}\cdot[\boldsymbol s'\times(\boldsymbol\beta\times\boldsymbol E)] = \hat{\boldsymbol\beta}\cdot[\,\boldsymbol\beta(\boldsymbol s'\cdot\boldsymbol E) - \boldsymbol E(\boldsymbol s'\cdot\boldsymbol\beta)]$$

$$= \beta(\boldsymbol s'\cdot\boldsymbol E) - \left(\hat{\boldsymbol\beta}\cdot\boldsymbol E\right)(\boldsymbol s'\cdot\boldsymbol\beta)$$

$$= \beta\left(\boldsymbol s_{||}{}' + \boldsymbol s_\perp{}'\right)\cdot\boldsymbol E - \beta\left(\hat{\boldsymbol\beta}\cdot\boldsymbol E\right)s_{||}{}'$$

$$= \beta\left(\boldsymbol s_{||}{}' + \boldsymbol s_\perp{}'\right)\cdot\boldsymbol E - \beta\left(\boldsymbol s_{||}{}'\cdot\boldsymbol E\right)$$

$$= \beta\,\boldsymbol s_\perp{}'\cdot\boldsymbol E$$

Hence

$$\hat{\boldsymbol\beta}\cdot\frac{d\boldsymbol s\,'}{dt} = \frac{e}{mc}\left\{-\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\boldsymbol s_\perp{}'\cdot\left(\hat{\boldsymbol\beta}\times\boldsymbol B\right) - \left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\beta\,\boldsymbol s_\perp{}'\cdot\boldsymbol E\right\}$$

From 11.168, we have

$$\frac{1}{\beta}\boldsymbol s_\perp{}'\cdot\frac{d\boldsymbol\beta}{dt} = \frac{e}{\gamma mc}\boldsymbol s_\perp{}'\cdot\left[\frac{1}{\beta}\boldsymbol E + \hat{\boldsymbol\beta}\times\boldsymbol B\right]$$

Hence

$$\frac{d}{dt}\left(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{s}'\right) = \frac{e}{mc}\left\{-\left(\frac{g}{2}-1\right)\boldsymbol{s}_\perp'\cdot\left(\hat{\boldsymbol{\beta}}\times\boldsymbol{B}\right)-\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\beta\,\boldsymbol{s}_\perp'\cdot\boldsymbol{E}+\frac{1}{\gamma\beta}\boldsymbol{s}_\perp'\cdot\boldsymbol{E}\right\}$$

Now

$$\begin{aligned}\frac{\gamma\beta}{\gamma+1}+\frac{1}{\gamma\beta} &= \gamma\beta\left(\frac{1}{\gamma+1}+\frac{1}{\gamma^2\beta^2}\right)\\ &= \gamma\beta\left(\frac{1}{\gamma+1}+\frac{1}{\gamma^2-1}\right)\\ &= \gamma\beta\left(\frac{\gamma}{\gamma^2-1}\right)\\ &= \frac{\gamma\beta}{\gamma\beta^2}\\ &= \frac{1}{\beta}\end{aligned}$$

Hence

$$\frac{d}{dt}\left(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{s}'\right) = -\frac{e}{mc}\boldsymbol{s}_\perp'\cdot\left\{\left(\frac{g}{2}-1\right)\left(\hat{\boldsymbol{\beta}}\times\boldsymbol{B}\right)+\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\boldsymbol{E}\right\} \qquad (11.171)$$