12. Dynamics of Relativistic Particles & Electromagnetic Fields

1. Particles Lagrangians

(a) Elementary Approach

In classical mechanics, the action of a single particle is defined as

$$S = \int d\,t\, L(x_i\,,\, v_i\,,\;t) \qquad (12.3)$$

where $v_i \equiv \dfrac{d\,x_i}{d\,t}$, with $i = 1,\ 2,\ 3$; or, in vector form, $\boldsymbol{v} = \dfrac{d\,\boldsymbol{x}}{d\,t}$.

The least action principle, $\delta\,S = 0$, then gives the Euler- Lagrange eq

$$\frac{\partial\,L}{\partial\,x_i} - \frac{d}{d\,t}\left(\frac{\partial\,L}{\partial\,v_i}\right) = 0 \qquad (12.4)$$

To incorporate the relativistic theory, we assume (12.4) to be valid in every inertial frame. This implies S is a Lorentz scalar. Using $d\,t = \gamma\,d\,\tau$, where τ is the proper time, we can write (12.3) as

$$S = \int d\,\tau\,\gamma\,L \qquad (12.5)$$

which means $\gamma\,L$ must be a scalar.

For a free particle, L must be independent of x_i. The simplest scalar that involves only v_i is $u \cdot u = u_\alpha\,u^\alpha$, where $u^\alpha = \gamma\,(\,c,\;\boldsymbol{v}\,)$ is the 4-velocity. We therefore set

$$\gamma\,L = \zeta\,u_\alpha\,u^\alpha = \zeta\,c^2$$

where ζ is a constant to be determined.

Thus

$$L = \frac{1}{\gamma}\,\zeta\,c^2 = \zeta\,c^2\,\sqrt{1 - \frac{v^2}{c^2}}$$

In the non-relativistic limit, $v/c \to 0$, so that

$$L \to \zeta\,c^2\left(1 - \frac{1}{2}\,\frac{v^2}{c^2} + \ldots\right)$$

Comparing with the classical Lagrange

$$L = \frac{1}{2}\,m\,v^2$$

we see that $\zeta = -m$ and

$$L = -\frac{1}{\gamma}\,m\,c^2 = -m\,c^2\,\sqrt{1 - \frac{v^2}{c^2}} \qquad (12.6)$$

Using

$$\frac{\partial \gamma}{\partial \boldsymbol{v}} = \gamma^3 \frac{\boldsymbol{v}}{c^2}$$

$$\frac{\partial}{\partial \boldsymbol{v}} \frac{1}{\gamma} = -\frac{1}{\gamma^2} \frac{\partial \gamma}{\partial \boldsymbol{v}} = -\gamma \frac{\boldsymbol{v}}{c^2}$$

$$\frac{\partial L}{\partial \boldsymbol{v}} = \gamma m \boldsymbol{v}$$

the Euler- Lagrange equation becomes

$$\frac{d}{d t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) = 0 = \frac{d}{d t}(\gamma m \boldsymbol{v})$$

To incorporate the EM forces, we add to L an interaction term L_{int} which reproduces the Lorentz force equation. This is accomplished by setting

$$\gamma L_{\text{int}} = -\frac{q}{c} u \cdot A = -\frac{q}{c} u_\alpha A^\alpha = -\gamma q \left(\Phi - \frac{1}{c} \boldsymbol{v} \cdot \boldsymbol{A}\right)$$

or

$$L_{\text{int}} = q\left(-\Phi + \frac{1}{c} \boldsymbol{v} \cdot \boldsymbol{A}\right) \tag{12.8}$$

Hence

$$\frac{\partial L_{\text{int}}}{\partial \boldsymbol{x}} = -q \nabla \Phi + \frac{q}{c} \nabla(\boldsymbol{v} \cdot \boldsymbol{A})$$

$$\frac{\partial L_{\text{int}}}{\partial \boldsymbol{v}} = \frac{q}{c} \boldsymbol{A}$$

so that

$$\frac{\partial L_{\text{int}}}{\partial \boldsymbol{x}} - \frac{d}{d t}\left(\frac{\partial L_{\text{int}}}{\partial \boldsymbol{v}}\right) = -q \nabla \Phi + \frac{q}{c} \nabla(\boldsymbol{v} \cdot \boldsymbol{A}) - \frac{q}{c} \frac{d \boldsymbol{A}}{d t}$$

Using

$$\frac{d \boldsymbol{A}}{d t} = \frac{\partial A}{\partial t} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{A}$$

$$\begin{aligned}
\nabla(\boldsymbol{v} \cdot \boldsymbol{A}) &= (\boldsymbol{v} \cdot \nabla) \boldsymbol{A} + (\boldsymbol{A} \cdot \nabla) \boldsymbol{v} + \boldsymbol{v} \times (\nabla \times \boldsymbol{A}) + \boldsymbol{A} \times (\nabla \times \boldsymbol{v}) \\
&= (\boldsymbol{v} \cdot \nabla) \boldsymbol{A} + \boldsymbol{v} \times (\nabla \times \boldsymbol{A}) \\
&= (\boldsymbol{v} \cdot \nabla) \boldsymbol{A} + \boldsymbol{v} \times \boldsymbol{B}
\end{aligned}$$

$$\nabla(\boldsymbol{v} \cdot \boldsymbol{A}) - \frac{d \boldsymbol{A}}{d t} = \boldsymbol{v} \times \boldsymbol{B} - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{E} = -\nabla \Phi - \frac{\partial \boldsymbol{A}}{c \partial t}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

we have

$$\frac{\partial L_{\text{int}}}{\partial \boldsymbol{x}} - \frac{d}{d t}\left(\frac{\partial L_{\text{int}}}{\partial \boldsymbol{v}}\right) = q\left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B}\right)$$

With

$$L = -m\,c^2 \sqrt{1 - \frac{v^2}{c^2}} + L_{\text{int}}$$

we have

$$\frac{\partial L}{\partial \boldsymbol{x}} - \frac{d}{d\,t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) = \quad \frac{d}{d\,t}(\gamma\,m\,\boldsymbol{v}) - q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) = 0$$

or

$$m\,\frac{d}{d\,t}\boldsymbol{u} = m\,\frac{d}{d\,t}(\gamma\,\boldsymbol{v}) = q\left(E + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)$$

as promised.

The Hamiltonian is defined by the Legendre transform

$$H = \frac{\partial L}{\partial \boldsymbol{v}}\cdot\boldsymbol{v} - L$$

For free particles,

$$L = -m\,c^2 \sqrt{1 - \frac{v^2}{c^2}} = -\frac{1}{\gamma}\,m\,c^2$$

and

$$H = \gamma\,m\,v^2 + \frac{1}{\gamma}\,m\,c^2 = m\,c^2\,\gamma\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = m\,c^2\,\gamma$$

In terms of the momentum, we have

$$\gamma\,\boldsymbol{v} = \frac{1}{m}\,\boldsymbol{p}$$

$$\gamma^2\,v^2 = \frac{1}{m^2}\,\boldsymbol{p}^2 = \frac{1}{1 - \frac{v^2}{c^2}}\,v^2 = \frac{1}{\frac{1}{v^2} - \frac{1}{c^2}}$$

$$\frac{1}{v^2} - \frac{1}{c^2} = \frac{m^2}{\boldsymbol{p}^2}$$

$$v^2 = \frac{1}{\frac{1}{c^2} + \frac{m^2}{\boldsymbol{p}^2}} = \frac{\boldsymbol{p}^2\,c^2}{\boldsymbol{p}^2 + m^2\,c^2}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{\boldsymbol{p}^2}{\boldsymbol{p}^2 + m^2\,c^2}}} = \frac{1}{m\,c}\sqrt{\boldsymbol{p}^2 + m^2\,c^2}$$

so that

$$H = c\sqrt{\boldsymbol{p}^2 + m^2\,c^2} = \sqrt{\boldsymbol{p}^2\,c^2 + m^2\,c^4}$$

which can be rearranged to give

$$H^2 - \boldsymbol{p}^2\,c^2 = m^2\,c^4$$

so that H/c is the time component of the 4-momentum $p^\alpha = m\,u^\alpha = \left(p^0,\ \boldsymbol{p}\right)$.

In the presence of an EM field,

$$L = -m\,c^2 \sqrt{1 - \frac{v^2}{c^2}} + q\left(-\Phi + \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right)$$

The momentum conjugate to $\boldsymbol{x}$ is

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \gamma m \boldsymbol{v} + \frac{q}{c} \boldsymbol{A}$$

so that

$$\begin{aligned} H &= \gamma m \boldsymbol{v}^2 + \frac{q}{c} \boldsymbol{v}\cdot\boldsymbol{A} + \frac{1}{\gamma} m c^2 - q\left(-\Phi + \frac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}\right) \\ &= m c^2 \gamma + q \Phi \end{aligned}$$

From

$$\boldsymbol{v} = \frac{1}{m\gamma}\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)$$

we see that γ can be obtained from the free particle result by replacing $\boldsymbol{p}$ with $\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}$, so that

$$H = \sqrt{\left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2 c^2 + m^2 c^4} + q\Phi \qquad (12.14)$$

This can be rearranged to give

$$(H - q\Phi)^2 - \left(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\right)^2 c^2 = m^2 c^4$$

Since $\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}$ is the space component of the linear 4-momentum $m u^\alpha$, $(H - q\Phi)/c$ is the corresponding time component, ie, the mechanical energy divided by c.

(b) Covariant Approach

Constraint

To make the least action principle & the Euler- Lagrange equations covariant, we define the action of a single particle to be

$$S = \int d\tau\, \mathbb{L}$$

where τ is the proper time of the particle and $\mathbb{L}$ is a Lorentz scalar.

The definition of τ,

$$c^2 (d\tau)^2 = c^2 (dt)^2 - (d\boldsymbol{x})^2 = d x_\alpha\, d x^\alpha$$

implies the constraint

$$u_\alpha u^\alpha = c^2 \qquad \text{where} \quad u^\alpha = \dot{x}^\alpha = \frac{d x^\alpha}{d\tau} = \gamma(c, \boldsymbol{v})$$

This can be incorporated via a Lagrangian

$$\mathbb{L}' = \mathbb{L} + \frac{\lambda}{2}\left(u_\alpha u^\alpha - c^2\right)$$

where the Lagrange multiplier λ is a function of τ and the factor $\frac{1}{2}$ is for later convenience. The constraint is then embedded in the Euler- Lagrange equation for λ:

$$\frac{\partial}{\partial \lambda}\mathbb{L}' - \frac{d}{d\tau}\frac{\partial}{\partial \dot{\lambda}}\mathbb{L}' = 0$$

since

$$\frac{\partial}{\partial \dot{\lambda}}\mathbb{L}' = 0$$

so that

$$\frac{\partial}{\partial \lambda}\mathbb{L}' = \frac{1}{2}\left(u_\alpha u^\alpha - c^2\right) = 0$$

The Euler- Lagrange equation for x_α is

$$\frac{\partial \mathbb{L}'}{\partial x_\alpha} - \frac{d}{d\tau}\left(\frac{\partial \mathbb{L}'}{\partial u_\alpha}\right) = 0$$

($u_\alpha u^\alpha = c^2$ can and must be applied here after the derivatives are done)

Using

$$\frac{\partial \mathbb{L}'}{\partial u_\alpha} = \frac{\partial \mathbb{L}}{\partial u_\alpha} + \lambda u^\alpha$$

$$\frac{\partial \mathbb{L}'}{\partial x_\alpha} = \frac{\partial \mathbb{L}}{\partial x_\alpha}$$

we have

$$\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\left(\frac{\partial \mathbb{L}}{\partial u_\alpha} + \lambda u^\alpha\right) = 0$$

or

$$\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial u_\alpha} - \frac{d\lambda}{d\tau}u^\alpha - \lambda\frac{du^\alpha}{d\tau} = 0 \qquad \text{(a)}$$

To eliminate λ, we multiply both side by u_α. Since

$$u_\alpha u^\alpha = c^2$$

and hence

$$u_\alpha \frac{du^\alpha}{d\tau} = 0$$

we have

$$u_\alpha \frac{\partial \mathbb{L}}{\partial x_\alpha} - u_\alpha \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial u_\alpha} - \frac{d\lambda}{d\tau}c^2 = 0$$

or

$$\begin{aligned}\frac{d\lambda}{d\tau} &= \frac{1}{c^2}\left[u_\alpha \frac{\partial \mathbb{L}}{\partial x_\alpha} - u_\alpha \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial u_\alpha}\right] \\ &= \frac{1}{c^2}u_\alpha\left[\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial u_\alpha}\right]\end{aligned}$$

For $\mathbb{L} = \mathbb{L}(x, u)$

$$\frac{d\mathbb{L}}{d\tau} = \frac{d x_\alpha}{d\tau}\frac{\partial \mathbb{L}}{\partial x_\alpha} + \frac{d u_\alpha}{d\tau}\frac{\partial \mathbb{L}}{\partial u_\alpha}$$

$$= u_\alpha \frac{\partial \mathbb{L}}{\partial x_\alpha} + \frac{d u_\alpha}{d\tau}\frac{\partial \mathbb{L}}{\partial u_\alpha}$$

$$= u_\alpha \frac{\partial \mathbb{L}}{\partial x_\alpha} + \frac{d}{d\tau}\left(u_\alpha \frac{\partial \mathbb{L}}{\partial u_\alpha}\right) - u_\alpha \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial u_\alpha}$$

so that

$$\frac{d\lambda}{d\tau} = \frac{1}{c^2}\left[\frac{d\mathbb{L}}{d\tau} - \frac{d}{d\tau}\left(u_\alpha \frac{\partial \mathbb{L}}{\partial u_\alpha}\right)\right]$$

$$= \frac{1}{c^2}\frac{d}{d\tau}\left[\mathbb{L} - u_\alpha \frac{\partial \mathbb{L}}{\partial u_\alpha}\right]$$

or

$$\lambda = \frac{1}{c^2}\left[\mathbb{L} - u_\alpha \frac{\partial \mathbb{L}}{\partial u_\alpha}\right] + \text{const}$$

Since the constant can also be absorbed in λ by a redefinition, we can set it to zero so that

$$\lambda = \frac{1}{c^2}\left(\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right)_{u_\alpha u^\alpha = c^2}$$

where the subscript $u_\alpha u^\alpha = c^2$ is used to emphasize the fact that λ is a dynamical variable so that the constraint must be incorporated explicitly.

Also

$$\frac{d\lambda}{d\tau} = \frac{1}{c^2}\frac{d}{d\tau}\left[\mathbb{L} - u_\alpha \frac{\partial \mathbb{L}}{\partial u_\alpha}\right]$$

$$\frac{d\lambda}{d\tau}u^\alpha + \lambda\frac{d u^\alpha}{d\tau} = \frac{1}{c^2}\left\{u^\alpha \frac{d}{d\tau}\left[\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right] + \left[\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right]\frac{d u^\alpha}{d\tau}\right\}$$

$$= \frac{1}{c^2}\frac{d}{d\tau}\left\{u^\alpha \mathbb{L} - u^\alpha u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right\}$$

$$= \frac{1}{c^2}\frac{d}{d\tau}\left[u^\alpha\left(\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right)\right]$$

The Euler Lagrange equation becomes

$$\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\frac{\partial \mathbb{L}}{\partial u_\alpha} - \frac{1}{c^2}\frac{d}{d\tau}\left\{u^\alpha \mathbb{L} - u^\alpha u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right\} = 0$$

$$= \frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\left\{\frac{\partial \mathbb{L}}{\partial u_\alpha} + \frac{1}{c^2}u^\alpha\left(\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right)\right\}$$

$$= \frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\left\{\left(\delta^\alpha{}_\beta - \frac{1}{c^2}u^\alpha u_\beta\right)\frac{\partial \mathbb{L}}{\partial u_\beta} + \frac{1}{c^2}u^\alpha \mathbb{L}\right\}$$

Also

$$\mathbb{L}' = \mathbb{L} + \frac{\lambda}{2}\left(u_\alpha u^\alpha - c^2\right)$$

where

$$\lambda = \frac{1}{c^2}\left(\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right)_{u_\alpha u^\alpha = c^2}$$

Now, $\mathbb{L}$ is a homogeneous function of degree n of u if

$$u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta} = n\mathbb{L}$$

Thus, if $\mathbb{L}$ is a linear homogeneous function of u, ie, $n = 1$,

$$\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta} = 0$$

so that the Euler- Lagrange equation simplifies to

$$\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\left(\frac{\partial \mathbb{L}}{\partial u_\alpha}\right) = 0$$

which is the same as if the constraint $u_\alpha u^\alpha = c^2$ is absent.

Otherwise, we must use

$$\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\left\{\frac{\partial \mathbb{L}}{\partial u_\alpha} + \frac{1}{c^2} u^\alpha \left(\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right)\right\} = 0$$

or

$$\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\left\{\frac{\partial \mathbb{L}}{\partial u_\alpha} + u^\alpha \lambda\right\} = 0$$

with

$$\lambda = \frac{1}{c^2}\left(\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right)_{u_\alpha u^\alpha = c^2}$$

The momentum p^α conjugate to x_α is defined as

$$p^\alpha = \frac{\partial \mathbb{L}'}{\partial u_\alpha} \qquad \text{evaluated at} \qquad u_\alpha u^\alpha = c^2$$

$$= \frac{\partial \mathbb{L}}{\partial u_\alpha} + \lambda u^\alpha$$

$$= \frac{\partial \mathbb{L}}{\partial u_\alpha} + \frac{1}{c^2} u^\alpha \left(\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right)$$

which, for linear $\mathbb{L}$, reduces to

$$p^\alpha = \frac{\partial \mathbb{L}}{\partial u_\alpha}$$

- **Free Particles**

For a free particle, $\mathbb{L}$ doesn't depend on x. In order to be a scalar, it can only be a function of $u_\alpha u^\alpha = c^2$.

If we further demand $\mathbb{L}$ to be linear in u, we have

$$\mathbb{L} = \zeta \sqrt{u_\alpha u^\alpha} = \zeta \gamma \sqrt{c^2 - \boldsymbol{v}^2} = \zeta \gamma c \sqrt{1 - \beta^2} = \zeta c$$

where ζ is a constant and $\boldsymbol{\beta} c = \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$.

Using

$$\frac{\partial}{\partial u_\alpha}\left(u_\beta u^\beta\right) = \frac{\partial}{\partial u_\alpha}\left(g^{\beta\gamma} u_\beta u_\gamma\right)$$

$$= g^{\beta\gamma}\left(\delta^\alpha{}_\beta u_\gamma + \delta^\alpha{}_\gamma u_\beta\right)$$

$$= g^{\alpha\gamma} u_\gamma + g^{\beta\alpha} u_\beta$$

$$= 2 u^\alpha$$

we have

$$p^\alpha = \frac{\partial \mathbb{L}}{\partial u_\alpha} = \zeta \frac{u^\alpha}{\sqrt{u_\beta u^\beta}} = \frac{\zeta}{c} u^\alpha$$

which, when compared with the definition of 4-momentum, gives

$$\zeta = m c$$

ie

$$\mathbb{L} = m c \sqrt{u_\alpha u^\alpha}$$

Since $\mathbb{L}$ is linear in u, the Euler- Lagrange equation is simply

$$\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\left(\frac{\partial \mathbb{L}}{\partial u_\alpha}\right) = 0$$

so that

$$\frac{d}{d\tau} p^\alpha = m \frac{d}{d\tau} u^\alpha = 0$$

as expected.

Another choice of $\mathbb{L}$ is

$$\mathbb{L} = \xi u_\alpha u^\alpha = \xi c^2$$

$$\mathbb{L}' = \mathbb{L} + \frac{\lambda}{2}\left(u_\alpha u^\alpha - c^2\right)$$

with

$$\frac{\partial \mathbb{L}}{\partial x_\alpha} - \frac{d}{d\tau}\left\{\frac{\partial \mathbb{L}}{\partial u_\alpha} + u^\alpha \lambda\right\} = 0$$

and

$$\lambda = \frac{1}{c^2}\left(\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta}\right)$$

Now

$$\frac{\partial \mathbb{L}}{\partial u_\alpha} = 2\xi u^\alpha$$

$$\mathbb{L} - u_\beta \frac{\partial \mathbb{L}}{\partial u_\beta} = -\xi u_\alpha u^\alpha$$

$$\lambda = -\frac{\xi}{c^2} u_\alpha u^\alpha = -\xi$$

We iterate the fact that the constraint $u_\alpha u^\alpha = c^2$ must be used in the last expression since it denotes the actual evolution of the dynamical variable λ.

Hence

$$p^\alpha = \frac{\partial \mathbb{L}'}{\partial u_\alpha} = \frac{\partial \mathbb{L}}{\partial u_\alpha} + u^\alpha \lambda = \xi \left(2 - \frac{1}{c^2} u_\beta u^\beta \right) u^\alpha = \xi u^\alpha$$

so that

$$\xi = m \qquad \lambda = -m$$

and

$$\mathbb{L} = m u_\alpha u^\alpha$$

$$\mathbb{L}' = \frac{m}{2} \left(u_\alpha u^\alpha + c^2 \right)$$

$$\frac{d}{d\tau} p^\alpha = m \gamma \frac{d u^\alpha}{d t} = 0$$

Hence

$$m \gamma \frac{d}{d t} (\gamma c) = 0$$

$$m \gamma \frac{d}{d t} (\gamma \boldsymbol{v}) = 0$$

The 1st eq means that γ and hence $v = |\boldsymbol{v}|$ are constants. The 2nd eq then requires $\boldsymbol{v}$ to be constant, as expected of a free particle.

Since the momentum conjugate to λ is identically zero, the hamiltonian cannot contain the variable λ. Hence, the hamiltonian is defined by the Legendre transform

$$\mathbb{H} = \frac{\partial \mathbb{L}'}{\partial u_\alpha} u_\alpha - \mathbb{L}' = p^\alpha u_\alpha - \mathbb{L}'$$

where λ is assumed to be already eliminated from $\mathbb{L}'$. Note also that the constraint is used when calculating $\frac{\partial \mathbb{L}'}{\partial u_\alpha}$ but not used in $\mathbb{L}'$.

Obviously, $\mathbb{H}$ is a scalar, as it should be if the Hamilton equations

$$u^\alpha = \frac{\partial \mathbb{H}}{\partial p_\alpha}$$

$$\frac{d p^\alpha}{d\tau} = -\frac{\partial \mathbb{H}}{\partial x_\alpha}$$

are to be covariant.

However, this means $\mathbb{H}$ cannot be interpreted as the energy of the system since the later transforms as the time component of a 4-vector.

For the linear $\mathbb{L}$, ie

$$\mathbb{L}' = \mathbb{L} = m c \sqrt{u_\alpha u^\alpha}$$

$$p^\alpha = \frac{\partial\, \mathbb{L}'}{\partial\, u_\alpha} = m\, c\, \frac{u^\alpha}{\sqrt{u_\beta\, u^\beta}} = m\, u^\alpha$$

$$\frac{\partial\, \mathbb{L}'}{\partial\, u_\alpha}\, u_\alpha = m\, u^\alpha\, u_\alpha$$

so that

$$\mathbb{H} = \frac{\partial\, \mathbb{L}'}{\partial\, u_\alpha}\, u_\alpha - \mathbb{L}' = m\, u^\alpha\, u_\alpha - m\, c\, \sqrt{u_\alpha\, u^\alpha}$$

$$= \frac{1}{m}\, p^\alpha\, p_\alpha - c\, \sqrt{p_\alpha\, p^\alpha}$$

The equations of motion are

$$u^\alpha = \frac{\partial\, \mathbb{H}}{\partial\, p_\alpha} = \frac{2}{m}\, p^\alpha - \frac{c}{\sqrt{p_\alpha\, p^\alpha}}\, p^\alpha = \frac{1}{m}\, p^\alpha$$

$$\frac{d\, p^\alpha}{d\, \tau} = -\frac{\partial\, \mathbb{H}}{\partial\, x_\alpha} = 0$$

where the constraint $p_\alpha\, p^\alpha = m^2\, c^2$ is imposed, as required, after all derivatives are performed.

For the quadratic $\mathbb{L}$,

$$\mathbb{L} = m\, u_\alpha\, u^\alpha$$

$$\lambda = \frac{1}{c^2}\left(\mathbb{L} - u_\beta\, \frac{\partial\, \mathbb{L}}{\partial\, u_\beta}\right) = -\frac{m}{c^2}\, u_\alpha\, u^\alpha = -m$$

$$\mathbb{L}' = \mathbb{L} + \frac{\lambda}{2}\left(u_\alpha\, u^\alpha - c^2\right)$$

$$= m\, u_\alpha\, u^\alpha - \frac{m}{2}\left(u_\alpha\, u^\alpha - c^2\right)$$

$$= \frac{1}{2}\, m\left(u_\alpha\, u^\alpha + c^2\right)$$

$$p^\alpha = \frac{\partial\, \mathbb{L}'}{\partial\, u_\alpha} = m\, u^\alpha$$

$$\mathbb{H} = m\, u^\alpha\, u_\alpha - \frac{1}{2}\, m\left(u_\alpha\, u^\alpha + c^2\right)$$

$$= \frac{1}{2}\, m\left(u_\alpha\, u^\alpha - c^2\right)$$

$$= \frac{1}{2\, m}\, p_\alpha\, p^\alpha - \frac{1}{2}\, m\, c^2$$

The Hamilton's eqs are

$$u^\alpha = \frac{\partial\, \mathbb{H}}{\partial\, p_\alpha} = \frac{1}{m}\, p^\alpha$$

$$\frac{d\, p^\alpha}{d\, \tau} = -\frac{\partial\, \mathbb{H}}{\partial\, x_\alpha} = 0$$

- **EM Forces**

The interaction Lagrangian is

$$\mathbb{L}_{\text{int}} = \eta\, u \cdot A = \eta\, u_\alpha\, A^\alpha$$

where η is a constant to be determined.

Now

$$\frac{\partial \mathbb{L}_{\text{int}}}{\partial x_\alpha} = \eta\, u_\beta\, \partial^\alpha A^\beta$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathbb{L}_{\text{int}}}{\partial u_\alpha}\right) = \eta \frac{d}{d\tau} A^\alpha = \eta\, u_\beta\, \partial^\beta A^\alpha$$

so that

$$\frac{\partial \mathbb{L}_{\text{int}}}{\partial x_\alpha} - \frac{d}{d\tau}\left(\frac{\partial \mathbb{L}_{\text{int}}}{\partial u_\alpha}\right) = \eta\, u_\beta\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = \eta\, u_\beta\, F^{\alpha\beta}$$

Note that $\mathbb{L}_{\text{int}}$ is linear in u so that

$$\mathbb{L}_{\text{int}} - u_\alpha \frac{\partial \mathbb{L}_{\text{int}}}{\partial u_\alpha} = 0$$

- **Linear Form**

Choosing the linear form of $\mathbb{L}$ for the particle, ie

$$\mathbb{L} = m\, c \sqrt{u_\alpha u^\alpha} + \eta\, u_\alpha A^\alpha$$

the eq of motion is

$$-m \frac{d}{d\tau} u^\alpha + \eta\, u_\beta\, F^{\alpha\beta} = 0$$

Comparing with the covariant Lorentz force eq

$$m \frac{d}{d\tau} u^\alpha = \frac{q}{c} F^{\alpha\beta} u_\beta$$

we have

$$\eta = \frac{q}{c}$$

$$\mathbb{L} = m\, c \sqrt{u_\alpha u^\alpha} + \frac{q}{c} u_\alpha A^\alpha$$

with the momentum p^α conjugate to x_α being

$$p^\alpha = \frac{\partial \mathbb{L}}{\partial u_\alpha} = m\, c \frac{u^\alpha}{\sqrt{u_\beta u^\beta}} + \frac{q}{c} A^\alpha = m\, u^\alpha + \frac{q}{c} A^\alpha$$

The linear momentum is therefore

$$m\, u^\alpha = p^\alpha - \frac{q}{c} A^\alpha$$

The hamiltonian is

$$\mathbb{H} = \left(m\, u^\alpha + \frac{q}{c} A^\alpha\right) u_\alpha - m\, c \sqrt{u_\alpha u^\alpha} - \frac{q}{c} u_\alpha A^\alpha$$

$$= m\, u^\alpha u_\alpha - m\, c \sqrt{u_\alpha u^\alpha}$$

$$= \frac{1}{m}\left(p^\alpha - \frac{q}{c} A^\alpha\right)\left(p_\alpha - \frac{q}{c} A_\alpha\right) - c \sqrt{\left(p^\alpha - \frac{q}{c} A^\alpha\right)\left(p_\alpha - \frac{q}{c} A_\alpha\right)}$$

The eqs of motion are

$$u^\alpha = \frac{\partial \mathbb{H}}{\partial p_\alpha}$$

$$= \frac{2}{m}\left(p^\alpha - \frac{q}{c}A^\alpha\right) - \frac{c}{\sqrt{\left(p^\beta - \frac{q}{c}A^\beta\right)\left(p_\beta - \frac{q}{c}A_\beta\right)}}\left(p^\alpha - \frac{q}{c}A^\alpha\right)$$

$$= \frac{1}{m}\left(p^\alpha - \frac{q}{c}A^\alpha\right)$$

$$\frac{d\,p^\alpha}{d\,\tau} = -\frac{\partial \mathbb{H}}{\partial x_\alpha}$$

$$= -\left(p_\beta - \frac{q}{c}A_\beta\right)\left(-\frac{q}{c}\partial^\alpha A^\beta\right)\left[\frac{2}{m} - \frac{c}{\sqrt{\left(p^\gamma - \frac{q}{c}A^\gamma\right)\left(p_\gamma - \frac{q}{c}A_\gamma\right)}}\right]$$

$$= \frac{q}{m\,c}\left(p_\beta - \frac{q}{c}A_\beta\right)\partial^\alpha A^\beta$$

$$= \frac{q}{c}u_\beta\,\partial^\alpha A^\beta$$

The last eq can be written as

$$m\frac{d\,u^\alpha}{d\,\tau} = \frac{q}{c}\left[u_\beta\,\partial^\alpha A^\beta - \frac{d\,A^\alpha}{d\,\tau}\right]$$

$$= \frac{q}{c}\left[u_\beta\,\partial^\alpha A^\beta - u_\beta\,\partial^\beta A^\alpha\right]$$

$$= \frac{q}{c}u_\beta\,F^{\alpha\beta}$$

namely, the Lorentz force eq.

- **Quadratic Form**

If we use the quadratic form of $\mathbb{L}$ for the particle, ie

$$\mathbb{L} = m\,u_\alpha\,u^\alpha + \eta\,u_\alpha\,A^\alpha$$

$$\mathbb{L}' = m\,u_\alpha\,u^\alpha + \eta\,u_\alpha\,A^\alpha + \frac{\lambda}{2}\left(u_\alpha\,u^\alpha - c^2\right)$$

with

$$\lambda = \frac{1}{c^2}\left(\mathbb{L} - u_\beta\frac{\partial\,\mathbb{L}}{\partial\,u_\beta}\right)$$

Now

$$\frac{\partial\,\mathbb{L}}{\partial\,u_\alpha} = 2\,m\,u^\alpha + \eta\,A^\alpha$$

$$\mathbb{L} - u_\beta\frac{\partial\,\mathbb{L}}{\partial\,u_\beta} = -m\,u_\alpha\,u^\alpha$$

$$\lambda = -\frac{1}{c^2}m\,u_\alpha\,u^\alpha = -m$$

$$\mathbb{L}' = \frac{m}{2}\left(u_\alpha\,u^\alpha + c^2\right) + \eta\,u_\alpha\,A^\alpha$$

The eq of motion is again

$$-m \frac{d}{d\tau} u^\alpha + \eta\, u_\beta F^{\alpha\beta} = 0$$

so that $\eta = \frac{q}{c}$ as in the linear $\mathbb{L}$ case.

Thus

$$\mathbb{L} = m\, u_\alpha u^\alpha + \frac{q}{c} u_\alpha A^\alpha$$

$$\mathbb{L}' = \frac{m}{2} \left(u_\alpha u^\alpha + c^2 \right) + \frac{q}{c} u_\alpha A^\alpha$$

The conjugate momentum is defined as

$$p^\alpha = \frac{\partial\, \mathbb{L}'}{\partial\, u_\alpha} = m\, u^\alpha + \frac{q}{c} A^\alpha$$

so that the linear momentum is again

$$m\, u^\alpha = p^\alpha - \frac{q}{c} A^\alpha$$

The hamiltonian is

$$\begin{aligned}
\mathbb{H} &= \left(m\, u^\alpha + \frac{q}{c} A^\alpha \right) u_\alpha - \frac{m}{2} \left(u_\alpha u^\alpha + c^2 \right) - \frac{q}{c} u_\alpha A^\alpha \\
&= \frac{m}{2} \left(u_\alpha u^\alpha - c^2 \right) \\
&= \frac{1}{2m} \left(p^\alpha - \frac{q}{c} A^\alpha \right) \left(p_\alpha - \frac{q}{c} A_\alpha \right) - \frac{1}{2} m\, c^2 \qquad (12.29)
\end{aligned}$$

The eqs of motion are

$$\begin{aligned}
u^\alpha &= \frac{\partial\, \mathbb{H}}{\partial\, p_\alpha} \\
&= \frac{1}{m} \left(p^\alpha - \frac{q}{c} A^\alpha \right)
\end{aligned}$$

$$\begin{aligned}
\frac{d\, p^\alpha}{d\tau} &= -\frac{\partial\, \mathbb{H}}{\partial\, x_\alpha} \\
&= -\frac{1}{m} \left(p_\beta - \frac{q}{c} A_\beta \right) \left(-\frac{q}{c} \partial^\alpha A^\beta \right) \\
&= \frac{q}{m\, c} \left(p_\beta - \frac{q}{c} A_\beta \right) \partial^\alpha A^\beta \\
&= \frac{q}{c} u_\beta\, \partial^\alpha A^\beta \qquad (12.30)
\end{aligned}$$

which are the same as the linear $\mathbb{L}$ case.

2. Forces

Let frame K' be moving with velocity $\boldsymbol{v}$ with respect to K.

Let the velocity of a particle in these 2 frames is $\boldsymbol{u}'$ and $\boldsymbol{u}$, respectively. (ie., K' & K moves with velocity $-\boldsymbol{u}'$ and $-\boldsymbol{u}$ with respect to the rest frame K'' of the particle)

We shall consider the time rate of change of a 4-vector, $A^\alpha = \left(A^0,\ \boldsymbol{A} \right)$, attached to the particle.

From

$$\boldsymbol{A}_{||} = \gamma_v \left(\boldsymbol{A}_{||}' + \frac{\boldsymbol{v}}{c} A^{0'} \right)$$

$$A^0 = \gamma_v \left(A^{0'} + \frac{\boldsymbol{v}}{c} \cdot \boldsymbol{A}' \right)$$

$$\boldsymbol{A}_{\perp} = \boldsymbol{A}_{\perp}'$$

we have

$$\frac{d}{d\tau} \boldsymbol{A}_{||} = \gamma_v \left(\frac{d}{d\tau} \boldsymbol{A}_{||}' + \frac{\boldsymbol{v}}{c} \frac{d}{d\tau} A^{0'} \right)$$

$$\frac{d}{d\tau} A^0 = \gamma_v \left(\frac{d}{d\tau} A^{0'} + \frac{\boldsymbol{v}}{c} \cdot \frac{d}{d\tau} \boldsymbol{A}' \right)$$

$$\frac{d}{d\tau} \boldsymbol{A}_{\perp} = \frac{d}{d\tau} \boldsymbol{A}_{\perp}'$$

or

$$\gamma_u \frac{d}{dt} \boldsymbol{A}_{||} = \gamma_v \gamma_{u'} \left(\frac{d}{dt'} \boldsymbol{A}_{||}' + \frac{\boldsymbol{v}}{c} \frac{d}{dt'} A^{0'} \right)$$

$$\gamma_u \frac{d}{dt} A^0 = \gamma_v \gamma_{u'} \left(\frac{d}{dt'} A^{0'} + \frac{\boldsymbol{v}}{c} \cdot \frac{d}{dt'} \boldsymbol{A}' \right)$$

$$\gamma_u \frac{d}{dt} \boldsymbol{A}_{\perp} = \gamma_{u'} \frac{d}{dt'} \boldsymbol{A}_{\perp}'$$

Note that for A being the 4-position, ie, $A^\alpha = (c\,t, \ \boldsymbol{x})$, the 2nd eq becomes

$$\gamma_u\, c = \gamma_v \gamma_{u'} \left(c + \frac{\boldsymbol{v}}{c} \cdot \boldsymbol{u}' \right)$$

or

$$\gamma_u = \gamma_v \gamma_{u'} \left(1 + \frac{\boldsymbol{v} \cdot \boldsymbol{u}'}{c^2} \right)$$

with inverse ($\boldsymbol{v} \to -\boldsymbol{v}$, primed $\leftrightarrow$ unprimed)

$$\gamma_{u'} = \gamma_v \gamma_u \left(1 - \frac{\boldsymbol{v} \cdot \boldsymbol{u}}{c^2} \right)$$

If $A_\alpha A^\alpha$ is independent of the coordinates,

$$\frac{d}{d\tau} \left(A^{0\times 2} - \boldsymbol{A}^2 \right) = 2 \left(A^0 \frac{d}{d\tau} A^0 - \boldsymbol{A} \cdot \frac{d}{d\tau} \boldsymbol{A} \right) = 0$$

or

$$\frac{d}{d\tau} A^0 = \frac{1}{A^0} \boldsymbol{A} \cdot \frac{d}{d\tau} \boldsymbol{A}$$

For A being the 4-momentum, $A^\alpha = \left(p^0, \ \boldsymbol{p} \right) = \gamma\, m\, (c, \ \boldsymbol{u})$ we have

$$\frac{d}{d\tau} p^0 = \frac{\boldsymbol{u}}{c} \cdot \frac{d}{d\tau} \boldsymbol{p}$$

Hence

$$\frac{d}{dt}\boldsymbol{p}_{\|} = \frac{1}{1+\frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}}\left[\frac{d}{dt'}\boldsymbol{p}_{\|}' + \frac{\boldsymbol{v}}{c}\left(\boldsymbol{u}'\cdot\frac{d}{dt'}\boldsymbol{A}'\right)\right]$$

$$= \frac{d}{dt'}\boldsymbol{p}_{\|}' + \frac{1}{1+\frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}}\left[-\frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\frac{d}{dt'}\boldsymbol{p}_{\|}' + \frac{\boldsymbol{v}}{c}\left(\boldsymbol{u}'\cdot\frac{d}{dt'}\boldsymbol{p}'\right)\right]$$

$$= \frac{d}{dt'}\boldsymbol{p}_{\|}' + \frac{1}{1+\frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}}\frac{1}{c^2}\left[\boldsymbol{u}'\times\left(\boldsymbol{v}\times\frac{d}{dt'}\boldsymbol{p}'\right)\right]_{\|}$$

$$= \frac{d}{dt'}\boldsymbol{p}_{\|}' + \frac{1}{1+\frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}}\frac{1}{c^2}\left[\boldsymbol{u}_{\perp}'\times\left(\boldsymbol{v}\times\frac{d}{dt'}\boldsymbol{p}'\right)\right]_{\|}$$

$$= \frac{d}{dt'}\boldsymbol{p}_{\|}' + \frac{\gamma_v}{c^2}\left[\boldsymbol{u}_{\perp}\times\left(\boldsymbol{v}\times\frac{d}{dt'}\boldsymbol{p}'\right)\right]_{\|}$$

$$= \frac{d}{dt'}\boldsymbol{p}_{\|}' + \frac{\gamma_v}{c^2}\left[\boldsymbol{u}\times\left(\boldsymbol{v}\times\frac{d}{dt'}\boldsymbol{p}'\right)\right]_{\|} \qquad (12.32)$$

where we've used

$$\boldsymbol{u}_{\perp} = \frac{\boldsymbol{u}_{\perp}'}{\gamma_v\left(1+\frac{\boldsymbol{v}\cdot\boldsymbol{u}'}{c^2}\right)} \qquad (11.31)$$

Similarly

$$\frac{d}{dt}\boldsymbol{p}_{\perp} = \gamma_v\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{u}}{c^2}\right)\frac{d}{dt'}\boldsymbol{p}_{\perp}'$$

$$= \gamma_v\left\{\frac{d}{dt'}\boldsymbol{p}_{\perp}' - \frac{1}{c^2}\boldsymbol{v}\cdot\boldsymbol{u}\frac{d}{dt'}\boldsymbol{p}_{\perp}'\right\}$$

$$= \gamma_v\left\{\frac{d}{dt'}\boldsymbol{p}_{\perp}' + \frac{1}{c^2}\left[\boldsymbol{u}\times\left(\boldsymbol{v}\times\frac{d}{dt'}\boldsymbol{p}'\right)\right]_{\perp}\right\} \qquad (12.32)$$

Scalar Potential

Linear Case

Consider

$$\mathbb{L} = m\,c\,\sqrt{u_\alpha u^\alpha}\; e^{g\,\phi(x)/m\,c^2}$$

where $\phi(x)$ is a scalar potential and g a coupling constant.

Thus

$$\frac{\partial\mathbb{L}}{\partial x_\alpha} = m\,c\,\sqrt{u_\alpha u^\alpha}\;\frac{g}{m\,c^2}\frac{\partial\phi}{\partial x_\alpha}\,e^{g\,\phi(x)/m\,c^2}$$

$$= g\,\frac{\partial\phi}{\partial x_\alpha}\,e^{g\,\phi(x)/m\,c^2}$$

$$p^\alpha = \frac{\partial\mathbb{L}}{\partial u_\alpha} = m\,u^\alpha\,e^{g\,\phi(x)/m\,c^2}$$

$$\frac{d}{d\tau}\frac{\partial\mathbb{L}}{\partial u_\alpha} = m\left[\frac{d u^\alpha}{d\tau} + u^\alpha\frac{g}{m\,c^2}\frac{d\phi}{d\tau}\right]e^{g\,\phi(x)/m\,c^2}$$

The eq of motion is

$$m\frac{d u^\alpha}{d\tau} = g\frac{\partial \phi}{\partial x_\alpha} - u^\alpha \frac{g}{c^2}\frac{d\phi}{d\tau}$$
$$= g\left(\frac{\partial \phi}{\partial x_\alpha} - u^\alpha \frac{1}{c^2} u^\beta \frac{\partial \phi}{\partial x^\beta}\right)$$
$$= g\left(\partial^\alpha \phi - \frac{1}{c^2} u^\alpha u^\beta \partial_\beta \phi\right) \qquad (12.34)$$

For $\alpha = i$

$$m\frac{d u^i}{d\tau} = g\left[\partial^i \phi - \frac{1}{c^2} u^i u^\beta \partial_\beta \phi\right]$$
$$= g\left[\partial^i \phi - \frac{1}{c^2} u^i \gamma\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\phi\right]$$

where

$$u^\beta \partial_\beta = \gamma c\frac{\partial}{c\,\partial t} + \gamma\boldsymbol{v}\cdot\nabla = \gamma\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)$$

Hence

$$m\frac{d\boldsymbol{u}}{d\tau} = -g\left[\nabla\phi + \frac{1}{c^2}\boldsymbol{u}\gamma\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\phi\right]$$
$$m\frac{d\boldsymbol{u}}{dt} = -g\left[\frac{1}{\gamma}\nabla\phi + \frac{1}{c^2}\boldsymbol{u}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\phi\right] \qquad (12.36)$$

For $\alpha = 0$,

$$m\frac{d u^0}{d\tau} = g\left[\frac{\partial}{c\,\partial t}\phi - \frac{1}{c}\gamma^2\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\phi\right]$$
$$m\frac{d u^0}{dt} = \frac{g}{c}\left[\frac{1}{\gamma}\frac{\partial}{\partial t}\phi - \gamma\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\phi\right]$$
$$= \frac{g}{c}\gamma\left[\left(\frac{1}{\gamma^2} - 1\right)\frac{\partial}{\partial t}\phi - \boldsymbol{v}\cdot\nabla\phi\right]$$
$$= -\frac{g}{c}\gamma\left[\frac{v^2}{c^2}\frac{\partial}{\partial t}\phi + \boldsymbol{v}\cdot\nabla\phi\right]$$

Now

$$\boldsymbol{v}\cdot m\frac{d\boldsymbol{u}}{dt} = -g\left[\frac{1}{\gamma}\boldsymbol{v}\cdot\nabla\phi + \frac{1}{c^2}\gamma\boldsymbol{v}^2\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\phi\right]$$
$$= -g\gamma\left[\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right)\boldsymbol{v}\cdot\nabla\phi + \frac{v^2}{c^2}\frac{\partial}{\partial t}\phi\right]$$
$$= -g\gamma\left[\boldsymbol{v}\cdot\nabla\phi + \frac{v^2}{c^2}\frac{\partial}{\partial t}\phi\right]$$
$$= mc\frac{d u^0}{dt}$$

ie.

$$\boldsymbol{v}\cdot\frac{d}{dt}\boldsymbol{p} = \frac{dE}{dt} \qquad \text{where} \quad p^\alpha = \left(\frac{E}{c}, \boldsymbol{p}\right)$$

---

3. Uniform *B*

In a uniform, static, magnetic induction $\boldsymbol{B}$,

$$\frac{d\,\boldsymbol{p}}{d\,t} = \frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}$$

$$\frac{d\,E}{d\,t} = \boldsymbol{v}\cdot\frac{d\,\boldsymbol{p}}{d\,t} = \frac{q}{c}\boldsymbol{v}\cdot(\boldsymbol{v}\times\boldsymbol{B}) = 0 \qquad (12.37)$$

Now

$$\begin{aligned}\boldsymbol{v}\cdot\frac{d\,\boldsymbol{p}}{d\,t} &= m\,\boldsymbol{v}\cdot\frac{d\,\gamma\,\boldsymbol{v}}{d\,t} = m\left(v^2\frac{d\,\gamma}{d\,t}+\frac{1}{2}\gamma\frac{d\,v^2}{d\,t}\right)\\ &= m\left(v^2\frac{d\,\gamma}{d\,v^2}+\frac{1}{2}\gamma\right)\frac{d\,v^2}{d\,t}\\ &= m\left(\frac{1}{2}\frac{v^2}{c^2}\gamma^3+\frac{1}{2}\gamma\right)\frac{d\,v^2}{d\,t}\\ &= \frac{m}{2}\gamma^3\left(\frac{v^2}{c^2}+\frac{1}{\gamma^2}\right)\frac{d\,v^2}{d\,t}\\ &= \frac{m}{2}\gamma^3\frac{d\,v^2}{d\,t}\\ &= 0\end{aligned}$$

implies

$$\frac{d\,v^2}{d\,t} = 0 \qquad \text{or} \qquad \frac{d\,v}{d\,t} = 0 = \frac{d\,\gamma}{d\,t}$$

Hence, the speed of the particle remains constant.

Thus

$$\frac{d\,\boldsymbol{p}}{d\,t} = \gamma\,m\frac{d\,\boldsymbol{v}}{d\,t} = \frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}$$

$$\begin{aligned}\frac{d\,\boldsymbol{v}}{d\,t} &= \frac{q}{\gamma\,m\,c}\boldsymbol{v}\times\boldsymbol{B}\\ &= \boldsymbol{v}\times\boldsymbol{\omega}_B \qquad (12.38)\end{aligned}$$

where

$$\boldsymbol{\omega}_B = \frac{q}{\gamma\,m\,c}\boldsymbol{B} = \frac{q\,c}{E}\boldsymbol{B} \qquad \frac{E}{c} = \gamma\,m\,c \qquad (12.39)$$

We set $\boldsymbol{B}$ to be the || direction & label it as the ϵ_3 axis.
Writing

$$\boldsymbol{v} = \boldsymbol{v}_{||} + \boldsymbol{v}_{\perp}$$

we have

$$\frac{d\,\boldsymbol{v}}{d\,t} = \frac{d\,\boldsymbol{v}_{||}}{d\,t} + \frac{d\,\boldsymbol{v}_{\perp}}{d\,t} = \boldsymbol{v}_{\perp}\times\boldsymbol{\omega}_B$$

which means

$$\frac{d\,\boldsymbol{v}_{||}}{d\,t} = 0$$

$$\frac{d\,\boldsymbol{v}_{\perp}}{d\,t} = \boldsymbol{v}_{\perp}\times\boldsymbol{\omega}_B$$

Writing

$$\boldsymbol{v}_\perp = v_1\,\boldsymbol{\epsilon}_1 + v_2\,\epsilon_2$$

$$\boldsymbol{v}_\perp \times \boldsymbol{\omega}_B = \begin{vmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 \\ v_1 & v_2 & 0 \\ 0 & 0 & \omega_B \end{vmatrix} = v_2\,\omega_B\,\epsilon_1 - v_1\,\omega_B\,\epsilon_2$$

the eq of motion becomes

$$v_{||} = \text{const}$$

$$\frac{d\,v_1}{d\,t} = v_2\,\omega_B$$

$$\frac{d\,v_2}{d\,t} = -v_1\,\omega_B$$

Eliminating v_2 we have

$$\frac{d^2\,v_1}{d\,t^2} = \frac{d\,v_2}{d\,t}\,\omega_B = -\,\omega_B{}^2\,v_1$$

so that

$$v_1 = A\cos\omega_B\,t + B\sin\omega_B\,t$$

$$v_2 = \frac{1}{\omega_B}\,\frac{d\,v_1}{d\,t} = -A\sin\omega_B\,t + B\cos\omega_B\,t$$

Since $v_\perp{}^2 = v_1{}^2 + v_2{}^2 = \text{const}$, we have

$$v_\perp{}^2 = A^2 + B^2 = \text{const}$$

and

$$\boldsymbol{v}_\perp = \epsilon_1\,(\,A\cos\omega_B\,t + B\sin\omega_B\,t\,) + \epsilon_2\,(\,-A\sin\omega_B\,t + B\cos\omega_B\,t\,)$$

$$\begin{aligned}\frac{\boldsymbol{v}_\perp}{v_\perp} &= \epsilon_1\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega_B\,t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega_B\,t\right) \\ &\quad +\epsilon_2\left(-\frac{A}{\sqrt{A^2+B^2}}\sin\omega_B\,t + \frac{B}{\sqrt{A^2+B^2}}\cos\omega_B\,t\right) \\ &= \epsilon_1\,(\,\cos\phi\cos\omega_B\,t + \sin\phi\sin\omega_B\,t\,) + \epsilon_2\,(\,-\cos\phi\sin\omega_B\,t + \sin\phi\cos\omega_B\,t\,) \\ &= \epsilon_1\cos(\,\omega_B\,t - \phi\,) - \epsilon_2\sin(\,\omega_B\,t - \phi\,)\end{aligned}$$

where

$$\cos\phi = \frac{A}{\sqrt{A^2+B^2}} \qquad \sin\phi = \frac{B}{\sqrt{A^2+B^2}}$$

Agreeing to take the real part, we can write

$$\begin{aligned}\frac{\boldsymbol{v}_\perp}{v_\perp} &= (\,\epsilon_1 + i\,\epsilon_2\,)\,e^{i\,(\,\omega_B\,t - \phi\,)} \\ &= (\,\epsilon_1 - i\,\epsilon_2\,)\,e^{-i\,(\,\omega_B\,t - \phi\,)}\end{aligned}$$

Shifting the origin of t to eliminate ϕ, we can write

$$\begin{aligned}\boldsymbol{v}(t) &= v_{||}\,\epsilon_3 + v_\perp(\,\epsilon_1 - i\,\epsilon_2\,)\,e^{-i\,\omega_B\,t} \\ &= v_{||}\,\epsilon_3 + \omega_B\,a\,(\,\epsilon_1 - i\,\epsilon_2\,)\,e^{-i\,\omega_B\,t} \qquad (12.40)\end{aligned}$$

where $v_\perp = \omega_B\,a$ with a being the radius of the orbit in the $\epsilon_1 - \epsilon_2$ plane.

Integrating gives

$$\boldsymbol{x}(t) = \boldsymbol{X}_0 + v_{||}\,t\,\epsilon_3 + i\,a\,(\,\epsilon_1 - i\,\epsilon_2\,)\,e^{-i\,\omega_B\,t} \qquad (12.41)$$

where $\boldsymbol{X}_0 = \boldsymbol{x}(0) - i\,a\,(\,\epsilon_1 - i\,\epsilon_2\,)$.

Taking the real part gives

$$\boldsymbol{v}(t) = v_{\|}\, \epsilon_3 + \omega_B\, a(\, \epsilon_1 \cos \omega_B\, t - \epsilon_2 \sin \omega_B\, t\,)$$

$$\boldsymbol{x}(t) = \boldsymbol{X}_0 + v_{\|}\, t\, \epsilon_3 + a\,(\, \epsilon_1 \sin \omega_B\, t + \epsilon_2 \cos \omega_B\, t\,)$$

$$\boldsymbol{x}(0) - \boldsymbol{X}_0 = -\, a\, \epsilon_2$$

If $v_{\|} = 0$, we have

$$\boldsymbol{v}(t) = \boldsymbol{v}_\perp(t) = \omega_B\, a(\, \epsilon_1 \cos \omega_B\, t - \epsilon_2 \sin \omega_B\, t\,)$$

$$\boldsymbol{x}(t) - \boldsymbol{X}_0 = \boldsymbol{x}_\perp(t) - \boldsymbol{X}_0 = a\,(\, \epsilon_1 \sin \omega_B\, t + \epsilon_2 \cos \omega_B\, t\,)$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_\perp(0) = \omega_B\, a\, \boldsymbol{\epsilon}_1$$

$$\boldsymbol{x}(0) - \boldsymbol{X}_0 = a\, \epsilon_2$$

so that the particle is circling, in the $\epsilon_1 - \epsilon_2$ plane, around the point $\boldsymbol{X}_0$, with radius a, in a clockwise direction. ($\boldsymbol{B}$ is along $\epsilon_3 = \epsilon_1 \times \epsilon_2$)

For $v_{\|} \neq 0$, there'll also be uniform motion along ϵ_3 so that the particle spirals. (path is a helix)

Since $v_\perp$ = const, we see that

$$v_\perp = v_\perp(0) = \omega_B\, a = p_\perp / m\, \gamma$$

or

$$p_\perp = m\, \gamma\, \omega_B\, a = m\, \gamma\, \frac{q\, B}{\gamma\, m\, c}\, a = \frac{q\, B\, a}{c}$$

where $p_\perp$ is the magnitude of the space part of the 4-momentum in the transverse direction.

In Gaussian units

$$[p_\perp] = [\ \mathrm{erg}/(\mathrm{cm}/s)] = \left[\frac{q\, B\, a}{c}\right] = [\ \mathrm{esu} \cdot \mathrm{Gauss} \cdot \mathrm{cm}\ /\ (\ \mathrm{cm}/s)\ ]$$

For an electron charge,

$$q = 4.8 \times 10^{-10}\ \mathrm{esu}$$

Using $c = 3 \times 10^{10}$ cm/s, we have

$$p_\perp = \frac{4.8 \times 10^{-10}}{3 \times 10^{10}}\, B\, a = 1.6 \times 10^{-20}\, B\, a$$

where $p_\perp$ is in unit erg/(cm/s), B in Gauss, and a in cm.

Now

$$1\ \mathrm{eV} = 1.6 \times 10^{-19}\, J = 1.6 \times 10^{-12}\ \mathrm{erg}$$

$$1\ \mathrm{MeV} = 1.6 \times 10^{-6}\ \mathrm{erg}$$

$$1\ \mathrm{MeV}/c = 1.6 \times 10^{-6} \big/\, 3 \times 10^{10}\ \mathrm{erg}/(\mathrm{cm}/s)$$

$$= \frac{1.6}{3} \times 10^{-16}\ \mathrm{erg}/(\mathrm{cm}/s)$$

Hence

$$p_\perp = 1.6 \times 10^{-20}\, B\, a \Big/ \left(\frac{1.6}{3} \times 10^{-16}\right) = 3 \times 10^{-4}\, B\, a \qquad (12.42)$$

where $p_\perp$ is in MeV/c, B in Gauss, and a in cm.

A quicker way to get (12.42) is to note that

$$1\ \mathrm{MeV}/c = 10^6\ \frac{e}{c}\ \mathrm{Volt} = \frac{1}{3}\times 10^4\ \frac{e}{c}\ \mathrm{statvolt}$$

Dividing both sides of

$$p_\perp = \frac{q\,B\,a}{c}$$

with it then gives, when $q = e$,

$$\tilde{p}_\perp = 3\times 10^{-4}\ B\,a \qquad (12.42)$$

where $\tilde{p}_\perp$ means $p_\perp$ measured in units MeV/c.

Another way to derive the solutions (12.40) & (12.41) is as follows. From

$$\frac{d\,\boldsymbol{v}}{d\,t} = \boldsymbol{v}\times\boldsymbol{\omega}_B$$

we get

$$\begin{aligned}\frac{d^2\,\boldsymbol{v}}{d\,t^2} &= \frac{d\,\boldsymbol{v}}{d\,t}\times\boldsymbol{\omega}_B \\ &= (\boldsymbol{v}\times\boldsymbol{\omega}_B)\times\boldsymbol{\omega}_B \\ &= \boldsymbol{\omega}_B\,(\,\boldsymbol{v}\cdot\boldsymbol{\omega}_B\,) - \boldsymbol{v}\,\omega_B{}^2\end{aligned}$$

Hence

$$\frac{d^2\,\boldsymbol{v}_{||}}{d\,t^2} = \hat{\boldsymbol{\omega}}_B\cdot\frac{d^2\,\boldsymbol{v}}{d\,t^2} = 0 \qquad\qquad \hat{\boldsymbol{\omega}}_B = \boldsymbol{\omega}_B/\omega_B$$

$$\frac{d^2\,\boldsymbol{v}_\perp}{d\,t^2} = -\boldsymbol{v}_\perp\ \omega_B{}^2$$

so that $\boldsymbol{v}_\perp$ oscillates with frequency ω_B as indicated by (12.40).

---

4. $\boldsymbol{E}\times\boldsymbol{B}$ Drift

Let the fields be $\boldsymbol{E}$ and $\boldsymbol{B}$ in frame K.

The Lorentz force on a particle moving with velocity $\boldsymbol{v}$ in this frame is,

$$\frac{d\,\boldsymbol{p}}{d\,t} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right) \qquad\qquad \boldsymbol{p} = \gamma\,m\,\boldsymbol{v}$$

Consider a transformation to a frame $K\,'$ moving with velocity $\boldsymbol{u}$ with respect to K.

$$\boldsymbol{E}_{||}\,' = \boldsymbol{E}_{||}$$

$$\boldsymbol{E}_\perp\,' = \gamma_u\left(\boldsymbol{E}_\perp + \frac{\boldsymbol{u}}{c}\times\boldsymbol{B}\right)$$

$$\boldsymbol{B}_{||}\,' = \boldsymbol{B}_{||}$$

$$\boldsymbol{B}_\perp\,' = \gamma_u\left(\boldsymbol{B}_\perp - \frac{\boldsymbol{u}}{c}\times\boldsymbol{E}\right)$$

First, we note that if $\boldsymbol{u}$ is perpendicular to both $\boldsymbol{E}$ & $\boldsymbol{B}$, we have

$$\boldsymbol{u} = \alpha\,\boldsymbol{E}\times\boldsymbol{B}$$

$$\boldsymbol{E}_{||}\,' = \boldsymbol{E}_{||} = 0$$

$$\boldsymbol{B}_{||}\,' = \boldsymbol{B}_{||} = 0$$

where α is an arbitrary constant and $\boldsymbol{E}$ & $\boldsymbol{B}$ is assumed not to be collinear. Thus, $\boldsymbol{u}$ is also perpendicular to both $\boldsymbol{E}\,'$ & $\boldsymbol{B}\,'$.

In other words

$$\boldsymbol{E} = \boldsymbol{E}_\perp \qquad \boldsymbol{E}\,' = \boldsymbol{E}_\perp\,'$$

$$\boldsymbol{B} = \boldsymbol{B}_\perp \qquad \boldsymbol{B}\,' = \boldsymbol{B}_\perp\,'$$

so that we can drop the subscript $\perp$.

Let's see if we can adjust α to make $\boldsymbol{E}\,' = 0$. Thus, we require

$$\boldsymbol{E} + \frac{\alpha}{c}(\boldsymbol{E}\times\boldsymbol{B})\times\boldsymbol{B} = 0$$

$$= \boldsymbol{E} + \frac{\alpha}{c}\left[(\boldsymbol{E}\cdot\boldsymbol{B})\,\boldsymbol{B} - B^2\,\boldsymbol{E}\right]$$

$$= \left(1 - \frac{\alpha}{c}B^2\right)\boldsymbol{E} + \frac{\alpha}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\,\boldsymbol{B}$$

Since $\boldsymbol{E}$ & $\boldsymbol{B}$ are not collinear, we have 2 equations (2 vector components) but only one variable α. There is therefore in general no solutions.

However, if $\boldsymbol{E}$ happens to be perpendicular to $\boldsymbol{B}$, ie

$$\boldsymbol{E}\cdot\boldsymbol{B} = 0$$

a solution is possible with

$$\alpha = \frac{c}{B^2}$$

so that

$$\boldsymbol{u} = \alpha\,\boldsymbol{E}\times\boldsymbol{B} = \frac{c}{B^2}\,\boldsymbol{E}\times\boldsymbol{B} = c\,\frac{E}{B}\,\boldsymbol{n}$$

where $\boldsymbol{n}$ is the unit vector along direction $\boldsymbol{E}\times\boldsymbol{B}$.
and

$$\boldsymbol{E}\,' = 0$$

$$\boldsymbol{B}\,' = \gamma_u\left[\boldsymbol{B} - \frac{1}{B^2}(\boldsymbol{E}\times\boldsymbol{B})\times\boldsymbol{E}\right]$$

$$= \gamma_u\left[\boldsymbol{B} - \frac{1}{B^2}E^2\,\boldsymbol{B}\right]$$

$$= \gamma_u\left[1 - \frac{E^2}{B^2}\right]\boldsymbol{B}$$

Now

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{E^2}{B^2}}}$$

Hence

$$\boldsymbol{B}\,' = \sqrt{1 - \frac{E^2}{B^2}}\;\boldsymbol{B} = \frac{1}{\gamma_u}\,\boldsymbol{B} \qquad (12.44)$$

Note that the condition $u < c$ implies the fore-going transformation is possible only if $E < B$.

Since the particle sees only a static uniform magnetic field $\boldsymbol{B}\,'$ in $K\,'$, its motion is as described in section 12.3.

For $E > B$, one can transform to a frame in which $\boldsymbol{B}\,' = 0$. (see Prob 12.4)

5. Nonuniform *B*

Gradient Field

For a static, slightly non-uniform magnetic field

$$\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{B}_0 + (\,\boldsymbol{x}\cdot\nabla\,)\,\boldsymbol{B}\,\big|_{\boldsymbol{x}=0} + \dots$$
$$= \boldsymbol{B}_0 + \Delta\,\boldsymbol{B}$$

where $\boldsymbol{B}_0 = \boldsymbol{B}(\boldsymbol{0})$ is the field at the origin. The Lorentz force is

$$\frac{d}{d\,t}\,\boldsymbol{p} = q\,\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}(\boldsymbol{x}) \qquad (\,\boldsymbol{p} = \gamma_v\,m\,\boldsymbol{v}\,)$$
$$= q\,\frac{\boldsymbol{v}}{c}\times(\boldsymbol{B}_0 + \Delta\,\boldsymbol{B}\Big)$$

From

$$\boldsymbol{v}\cdot\frac{d}{d\,t}\,\boldsymbol{p} = q\,\boldsymbol{v}\cdot\Big[\,\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\,(\boldsymbol{x})\,\Big] = 0$$

we see that v = const and we can always write

$$\frac{d\,\boldsymbol{v}}{d\,t} = \frac{q}{\gamma_v\,m\,c}\,\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{x})$$

A simple, precession type solution is possible only if $v_{||}$ = const, which requires $\Delta\,\boldsymbol{B} = \boldsymbol{B}_0\,f(\boldsymbol{x})$ with $\boldsymbol{B}_0$ = const, so that

$$\frac{d\,\boldsymbol{v}}{d\,t} = \frac{q}{\gamma_v\,m\,c}\,\boldsymbol{v}\times\boldsymbol{B}_0\,[\,1 + f\,(\boldsymbol{x})\,\Big]$$

and hence

$$\hat{\boldsymbol{B}}_0\cdot\frac{d\,\boldsymbol{v}}{d\,t} = \frac{d\,\boldsymbol{v}_{||}}{d\,t} = \frac{q}{\gamma_v\,m\,c}\,\hat{\boldsymbol{B}}_{\boldsymbol{0}}\cdot\Big\{\,\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}_0\,[\,1 + f(\boldsymbol{x})\,]\,\Big\} = 0$$

and

$$\frac{d\,\boldsymbol{v}}{d\,t} = \boldsymbol{v}\times\boldsymbol{\omega}_0\,[\,1 + f(\boldsymbol{x})\,]$$
$$= \boldsymbol{v}\times\boldsymbol{\omega}_B(\boldsymbol{x})$$

where

$$\boldsymbol{\omega}_0 = \frac{q}{\gamma\,m\,c}\,\boldsymbol{B}_0 \qquad \boldsymbol{\omega}_B(\boldsymbol{x}) = \boldsymbol{\omega}_0\,[\,1 + f(\boldsymbol{x})\,]$$

Note that $\boldsymbol{\omega}_B(\boldsymbol{x})$ is a function of t through $\boldsymbol{x} = \boldsymbol{x}(t)$.

Since both v and $v_{||}$ are constant, so is $v_\perp = \sqrt{v^2 - {v_{||}}^2}$.

As an example, if the spatial variation can be described by a gradient in a direction $\boldsymbol{n}$ perpendicular to $\boldsymbol{B}_0$, then

$$\boldsymbol{B}(\boldsymbol{x}) = \hat{\boldsymbol{B}}_0\, B(\boldsymbol{x}) = \frac{1}{B_0}\, \boldsymbol{B}_0\, B(\boldsymbol{x})$$

$$\nabla\, B(\boldsymbol{x}) = \boldsymbol{n}\, \frac{\partial B}{\partial \xi}$$

$$(\boldsymbol{x}\cdot\nabla)\, \boldsymbol{B}(\boldsymbol{x}) = \hat{\boldsymbol{B}}_0\, (\boldsymbol{x}\cdot\nabla)\, B(\boldsymbol{x}) = \frac{1}{B_0}\, \boldsymbol{B}_0\, (\boldsymbol{x}\cdot\boldsymbol{n})\, \frac{\partial B}{\partial \xi}$$

$$\boldsymbol{B}(\boldsymbol{x}) \simeq \boldsymbol{B}_0 \left[1 + \frac{1}{B_0}\, (\boldsymbol{x}_\perp\cdot\boldsymbol{n}) \left(\frac{\partial B}{\partial \xi} \right)_{\xi=0} \right]$$

where ξ is the coordinate along the $\boldsymbol{n}$ direction. We have

$$f(\boldsymbol{x}) = (\boldsymbol{x}_\perp\cdot\boldsymbol{n}) \left(\frac{\partial B}{B\, \partial \xi} \right)_{\xi=0}$$

The eq of motion is therefore

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{v}\times\boldsymbol{\omega}_0 \left[1 + (\boldsymbol{x}_\perp\cdot\boldsymbol{n}) \left(\frac{\partial B}{B\, \partial \xi} \right)_{\xi=0} \right]$$

Since $\boldsymbol{v}_{||}$ =const, we can concentrate on $\boldsymbol{v}_\perp$.

$$\frac{d\boldsymbol{v}_\perp}{dt} = \boldsymbol{v}_\perp\times\boldsymbol{\omega}_0 \left[1 + (\boldsymbol{x}_\perp\cdot\boldsymbol{n}) \left(\frac{\partial B}{B\, \partial \xi} \right)_{\xi=0} \right]$$

Setting

$$\boldsymbol{v}_\perp = \boldsymbol{v}_{\perp 0} + \boldsymbol{v}_1$$

where $\boldsymbol{v}_{\perp 0}$ satisfies

$$\frac{d\boldsymbol{v}_{\perp 0}}{dt} = \boldsymbol{v}_{\perp 0}\times\boldsymbol{\omega}_0$$

we have

$$\frac{d\boldsymbol{v}_{\perp 0}}{dt} + \frac{d\boldsymbol{v}_1}{dt} = (\boldsymbol{v}_{\perp 0} + \boldsymbol{v}_1)\times\boldsymbol{\omega}_0\, [1 + f(\boldsymbol{x})]$$

$$= \boldsymbol{v}_{\perp 0}\times\boldsymbol{\omega}_0 + \boldsymbol{v}_{\perp 0}\times\boldsymbol{\omega}_0\, f(\boldsymbol{x}) + \boldsymbol{v}_1\times\boldsymbol{\omega}_0\, [1 + f(\boldsymbol{x})]$$

or

$$\frac{d\boldsymbol{v}_1}{dt} = \{ \boldsymbol{v}_{\perp 0}\, f(\boldsymbol{x}) + \boldsymbol{v}_1\, [1 + f(\boldsymbol{x})] \}\times\boldsymbol{\omega}_0$$

Now, if some component of the motion of the particle is periodic (in particular, we have in mind the trajectory depicted in fig 12.3), we expect from $\frac{d v_\perp}{dt} = 0$ that

$$\left\langle \frac{d\boldsymbol{v}_\perp}{dt} \right\rangle = 0$$

where $\langle \ldots \rangle$ denotes time average over a period.

Since $\left\langle \frac{d\boldsymbol{v}_{\perp 0}}{dt} \right\rangle = 0$, this means $\left\langle \frac{d\boldsymbol{v}_1}{dt} \right\rangle = 0$. Hence

$$\langle \boldsymbol{v}_1\, [1 + f(\boldsymbol{x})] \rangle = -\langle \boldsymbol{v}_{\perp 0}\, f(\boldsymbol{x}) \rangle$$

For $|\, f(\boldsymbol{x})\, | \ll 1$, the drift velocity becomes

$$\boldsymbol{v}_G = \langle \boldsymbol{v}_1 \rangle \simeq -\langle \boldsymbol{v}_{\perp 0}\, f(\boldsymbol{x}) \rangle$$

For gradient fields,

$$\boldsymbol{v}_G = \langle \boldsymbol{v}_1 \rangle \simeq -\left\langle \boldsymbol{v}_{\perp 0} \, (\boldsymbol{x}_\perp \cdot \boldsymbol{n}) \left(\frac{\partial B}{B \, \partial \xi} \right)_{\xi=0} \right\rangle$$

$$\simeq -\left\langle \boldsymbol{v}_{\perp 0} \, (\boldsymbol{x}_{\perp 0} \cdot \boldsymbol{n}) \left(\frac{\partial B}{B \, \partial \xi} \right)_{\xi=0} \right\rangle$$

Using

$$\langle a \, b \rangle = \frac{1}{2} a^* \, b$$

for quantities with exponential time dependence, we have

$$\langle \boldsymbol{v}_{\perp 0} \, (\boldsymbol{x}_{\perp 0} \cdot \boldsymbol{n}) \rangle = \frac{1}{2} \omega_0 \, a^2 \, (\epsilon_1 - i \, \epsilon_2) \, [i \, (\epsilon_1 - i \, \epsilon_2) \cdot \boldsymbol{n}]$$

$$\boldsymbol{v}_G = -\frac{\omega_0 \, a^2}{2} \left(\frac{\partial B}{B \, \partial \xi} \right)_{\xi=0} i \, (\epsilon_1 + i \, \epsilon_2) \, [\, (\epsilon_1 - i \, \epsilon_2) \cdot \boldsymbol{n} \,]$$

The real part of which is

$$\boldsymbol{v}_G = -\frac{\omega_0 \, a^2}{2} \left(\frac{\partial B}{B \, \partial \xi} \right)_{\xi=0} [\, \epsilon_1 (\epsilon_2 \cdot \boldsymbol{n}) - \epsilon_2 (\epsilon_1 \cdot \boldsymbol{n})]$$

If $\boldsymbol{n} = \epsilon_1$,

$$\boldsymbol{v}_G = \frac{\omega_0 \, a^2}{2} \left(\frac{\partial B}{B \, \partial \xi} \right)_{\xi=0} \epsilon_2$$

If $\boldsymbol{n} = \epsilon_2$,

$$\boldsymbol{v}_G = -\frac{\omega_0 \, a^2}{2} \left(\frac{\partial B}{B \, \partial \xi} \right)_{\xi=0} \epsilon_1$$

This means we can write

$$\boldsymbol{v}_G = \frac{\omega_0 \, a^2}{2} \left(\frac{\partial B}{B \, \partial \xi} \right)_{\xi=0} \epsilon_3 \times \boldsymbol{n}$$

$$= \frac{a^2}{2} \left(\frac{\partial B}{B \, \partial \xi} \right)_{\xi=0} \boldsymbol{\omega}_0 \times \boldsymbol{n}$$

■ Curved Field

Assuming the particle still spirals around the lines of $\boldsymbol{B}$, it'll experience a centrifugal force $m \, {v_{||}}^2 / R$, where R is the radius of curvature of the line of forces at the position of the particle. This is equivalent to the existence of a radial effective electric field

$$q \, \boldsymbol{E}_{\text{eff}} = \gamma \, m \, \frac{{v_{||}}^2}{R^2} \, \boldsymbol{R} \qquad (12.56)$$

where $\boldsymbol{R}$ points from the local center of curvature to the particle.

The particle is thus under a $E\times B$ field and the center of its spiral will drift with velocity

$$\boldsymbol{v}_c = \frac{c}{B^2}\,\boldsymbol{E}_{\text{eff}}\times\boldsymbol{B}$$

$$= \frac{c}{B^2}\,\gamma\, m\,\frac{{v_{||}}^2}{R^2}\,\boldsymbol{R}\times\boldsymbol{B} \qquad (12.57)$$

$$= \frac{q\,{v_{||}}^2}{\omega_B}\,\frac{\boldsymbol{R}\times\boldsymbol{B}}{B\,R} \qquad (12.58)$$

where

$$\omega_B = \frac{q\,B}{\gamma\, m\, c}$$

Note that

$$v_C = \frac{q\,{v_{||}}^2}{\omega_B}$$

In cylindrical coordinates (ρ, ϕ, z) so that $\boldsymbol{B} = B_\phi\,\hat{\boldsymbol{\phi}}$. (see Fig.12.4)

We have (see Arfken)

$$\ddot{\boldsymbol{r}} = \hat{\boldsymbol{\rho}}\left(\ddot{\rho} - \rho\,\dot{\phi}^2\right) + \hat{\boldsymbol{\phi}}\left(2\,\dot{\rho}\,\dot{\phi} + \rho\,\ddot{\phi}\right) + \hat{\boldsymbol{z}}\,\ddot{z}$$

$$\boldsymbol{v}\times\boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{\rho}} & \hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \dot{\rho} & \rho\,\dot{\phi} & \dot{z} \\ 0 & B_\phi & 0 \end{vmatrix} = -\hat{\boldsymbol{\rho}}\,\dot{z}\,B_\phi + \hat{\boldsymbol{z}}\,\dot{\rho}\,B_\phi$$

Hence the Lorentz force equation is

$$\gamma\, m\left[\hat{\boldsymbol{\rho}}\left(\ddot{\rho} - \rho\,\dot{\phi}^2\right) + \hat{\boldsymbol{\phi}}\left(2\,\dot{\rho}\,\dot{\phi} + \rho\,\ddot{\phi}\right) + \hat{\boldsymbol{z}}\,\ddot{z}\right] = \frac{q}{c}\left[-\hat{\boldsymbol{\rho}}\,\dot{z}\,B_\phi + \hat{\boldsymbol{z}}\,\dot{\rho}\,B_\phi\right]$$

or

$$\gamma\, m\left(\ddot{\rho} - \rho\,\dot{\phi}^2\right) = -\frac{q}{c}\,\dot{z}\,B_\phi$$

$$\gamma\, m\left(2\,\dot{\rho}\,\dot{\phi} + \rho\,\ddot{\phi}\right) = 0$$

$$\gamma\, m\,\ddot{z} = \frac{q}{c}\,\dot{\rho}\,B_\phi$$

or

$$\ddot{\rho} - \rho\,\dot{\phi}^2 = -\omega_B\,\dot{z}$$

$$2\,\dot{\rho}\,\dot{\phi} + \rho\,\ddot{\phi} = 0$$

$$\ddot{z} = \omega_B\,\dot{\rho} \qquad (12.59)$$

If the zeroth order trajectory is a helix with radius $a \ll R$, we have

$$\rho \simeq R$$

$$\dot{\phi} \simeq \frac{v_{||}}{R}$$

so that the 1st of (12.59) becomes

$$R\left(\frac{v_{||}}{R}\right)^2 = \omega_B\,\dot{z}$$

or

$$\dot{z} = \frac{{v_{||}}^2}{R\,\omega_B} \qquad (12.60)$$

which is just the drift velocity v_C.

For a current free region,

$$\nabla \times \boldsymbol{B} = 0$$

For a $2 - D$ field shown in Fig.12.4, $\frac{\partial \boldsymbol{B}}{\partial z} = 0$, $B_z = 0$ so that

$$\nabla \times \boldsymbol{B} = \hat{\boldsymbol{z}} \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho B_\phi \right) - \frac{\partial}{\partial \phi} B_\rho \right] = 0$$

or

$$\frac{\partial}{\partial \rho} \left(\rho B_\phi \right) = \frac{\partial}{\partial \phi} B_\rho$$

Near the origin in Fig.12.4(b), $B = B_\phi$, we have

$$\frac{\partial}{\partial \rho} (\rho B) = 0$$

$$= B + \rho \frac{\partial B}{\partial \rho}$$

or

$$\frac{1}{B} \frac{\partial B}{\partial \rho} = -\frac{1}{\rho}$$

Using $\rho = R$, we can write this as

$$\frac{\nabla_\perp B}{B} = -\frac{1}{R} \hat{\boldsymbol{R}} = -\frac{1}{R^2} \boldsymbol{R}$$

which will be valid away from the origin.

Thus $\boldsymbol{v}_G$ and $\boldsymbol{v}_C$ can be combined to give

$$v_D = \frac{1}{\omega_B R} \left(v_{||}{}^2 + \frac{1}{2} v_\perp{}^2 \right) \frac{\boldsymbol{R} \times \boldsymbol{B}}{B R}$$

---

6. Adiabatic Invariance

For a periodic coordinate q_i, its action integral is defined as

$$J_i = \oint d q_i \, p_i \qquad (12.63)$$

where p_i is its conjugate momentum.

For adiabatic changes of the system, the action is invariant. By adiabatic changes, we mean changes that are slow compared to the periods of the motion.

For motion in a uniform $\boldsymbol{B}$, we set

$$J = \oint \boldsymbol{P}_\perp \cdot d\,\boldsymbol{l} \qquad (12.64)$$

where $\boldsymbol{P}_\perp$ is the transverse component of the conjugate momentum and $d\,\boldsymbol{l}$ is the line element of the (circular) orbit.

Now,

$$\boldsymbol{P}_\perp = \gamma m \boldsymbol{v}_\perp + \frac{q}{c} \boldsymbol{A}$$

so that

$$J = \oint \gamma m \boldsymbol{v}_\perp \cdot d\,\boldsymbol{l} + \frac{q}{c} \oint \boldsymbol{A} \cdot d\,\boldsymbol{l}$$

Now, $\boldsymbol{v}_\perp$ is parallel to $d\,\boldsymbol{l}$ with a magnitude $v_\perp = \omega_B\, a$. Writing $d\,\boldsymbol{l} = a\, d\,\boldsymbol{\theta}$, we have

$$\oint \gamma\, m\, \boldsymbol{v}_\perp \cdot d\,\boldsymbol{l} = \int_0^{2\pi} d\,\theta\, \gamma\, m\, \omega_B\, a^2 = 2\,\pi\, \gamma\, m\, \omega_B\, a^2 = 2\,\pi\, \frac{q}{c}\, B\, a^2$$

where

$$\omega_B = \frac{q\, B}{\gamma\, m\, c}$$

Writing

$$\boldsymbol{p}_\perp = \gamma\, m\, \boldsymbol{v}_\perp$$

we have

$$p_\perp = \gamma\, m\, v_\perp = \gamma\, m\, \omega_B\, a = \frac{q}{c}\, B\, a$$

$$\frac{p_\perp{}^2}{B} = \left(\frac{q}{c}\right)^2 \Phi$$

Also

$$\oint \boldsymbol{A} \cdot d\,\boldsymbol{l} = \int_S d\,\boldsymbol{\sigma} \cdot (\nabla \times \boldsymbol{A}) = \int_S d\,\boldsymbol{\sigma} \cdot \boldsymbol{B} = -B\,\pi\, a^2 = -\,\Phi$$

where S is the surface enclosed by the orbital and $\Phi = B\,\pi\, a^2$ is the magnetic flux through S. The negative sign arises because the particle goes clockwise around $\boldsymbol{B}$ while $\boldsymbol{\sigma}$ points in the direction around which $d\,\boldsymbol{l}$ advances in the counter- clockwise direction.

Hence

$$J = \pi\, \frac{q}{c}\, B\, a^2 = \frac{q}{c}\, \Phi \qquad (12.68)$$

Consider $\boldsymbol{B}$ as shown in fig. 12.5. Since $\boldsymbol{B}$ does not work v remains unchanged:

$$v^2 = v_{||}{}^2 + v_\perp{}^2 = v_0{}^2 \qquad (12.70)$$

where v_0 is v at time $t = 0$.

Assuming adiabatic invariance, we have

$$\frac{p_\perp{}^2}{B} = \frac{p_{\perp 0}{}^2}{B_0}$$

Since v and hence γ is not changed, we have

$$\frac{v_\perp{}^2}{B} = \frac{v_{\perp 0}{}^2}{B_0} \qquad (12.71)$$

Hence

$$\begin{aligned} v_{||}{}^2 &= v^2 - v_\perp{}^2 \\ &= v_0{}^2 - v_\perp{}^2 \\ &= v_0{}^2 - \frac{v_{\perp 0}{}^2}{B_0}\, B \qquad (12.72) \end{aligned}$$

For a particle in $1 - D$ motion

$$E = \frac{1}{2}\, m\, v^2 + V(z)$$

or

$$v^2 = \frac{2}{m}\,[\, E - V(z)\,]$$

(12.72) can therefore be interpreted as due to a potential

$$V(z) = \frac{m}{2} \frac{v_{\perp 0}{}^2}{B_0} B(z)$$

Now, if $\boldsymbol{B}$ has a gradient along the z direction, ie.

$$\frac{\partial B(z)}{\partial z} \neq 0$$

The condition

$$\nabla \cdot \boldsymbol{B} = 0$$

will demand corresponding gradients along other directions.

Assuming axial symmetry, we have

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho B_\rho\right) + \frac{\partial}{\partial z} B_z = 0 \qquad \left(\frac{\partial}{\partial \phi} B_\phi = 0 \right)$$

Hence

$$\frac{\partial}{\partial \rho}\left(\rho B_\rho\right) = -\rho \frac{\partial B_z(z)}{\partial z}$$

$$\rho B_\rho = -\frac{\rho^2}{2} \frac{\partial B_z(z)}{\partial z}$$

$$B_\rho = -\frac{\rho}{2} \frac{\partial B_z(z)}{\partial z} \qquad (12.73)$$

where B_z is assumed to be a function of z only.

The z component of $\boldsymbol{v}\times\boldsymbol{B}$ is

$$v_x B_y - v_y B_x$$

Assuming circular motion in the $x - y$ plane, we have

$$v_x = -\rho \dot{\phi} \sin\phi \qquad v_y = \rho \dot{\phi} \cos\phi \qquad (v_\rho = 0)$$

$$B_x = B_\rho \cos\phi \qquad B_y = B_\rho \sin\phi \qquad (B_\phi = 0)$$

so that

$$(\boldsymbol{v}\times\boldsymbol{B})_z = \rho \dot{\phi} B_\rho \left(-\sin^2\phi - \cos^2\phi\right) = -\rho \dot{\phi} B_\rho$$

The z component of the Lorentz force eq is therefore

$$\ddot{z} = \frac{q}{\gamma m c}\left(-\rho \dot{\phi} B_\rho\right)$$

$$= \frac{q}{2 \gamma m c} \rho^2 \dot{\phi} \frac{\partial B_z(z)}{\partial z} \qquad (12.74)$$

Using

$$\rho^2 \dot{\phi} \simeq -\left(a^2 \omega_B\right)_0 = -\left(\frac{v_{\perp 0}{}^2}{\omega_{B0}}\right)$$

we have

$$\ddot{z} = -\frac{v_{\perp 0}{}^2}{2 B_0} \frac{\partial B_z(z)}{\partial z}$$
$$= \frac{1}{2} \frac{d}{d z} v_{\|}{}^2$$

so that

$$v_{\|}{}^2 = C - \frac{v_{\perp 0}{}^2}{B_0} B(z)$$

where C is some constant. Eq (12.72) is thus recovered.

7. Darwin Lagrangian

Interaction between moving charges cannot be described by the Lagrangian approach since the instantaneous nature of the latter does not allow for retardation effects.

However, approximation correct to $O\left(\frac{v^2}{c^2}\right)$ is still possible. Such Lagrangians are named after Darwin.

Consider the case of 2 particles.

Let the fields experienced by particle 1 due to particle 2 be

$$A_{12}{}^{\alpha} = (\Phi_{12},\ \boldsymbol{A}_{12})$$

In the Coulomb gauge

$$\Phi_{12} = \frac{q_2}{r}$$

where $r = |\boldsymbol{x}_1 - \boldsymbol{x}_2|$ is the instantaneous distance between the particles, is exact.

Neglecting retardation effects, we have

$$\boldsymbol{A}_{12} \simeq \frac{1}{c} \int d^3 x' \boldsymbol{J}_t(\boldsymbol{x}') \frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}'|} \qquad (12.78)$$

where $\boldsymbol{J}_t$ is the transverse part of the current density due to particle 2,

$$\boldsymbol{J}(x') = q_2 \boldsymbol{v}_2\ \delta(\boldsymbol{x}' - \boldsymbol{x}_2)$$

Now, the longitudinal current density is

$$\boldsymbol{J}_l(\boldsymbol{x}') = \frac{1}{4\pi} \nabla' \frac{\partial \Phi}{\partial t} \qquad (6.51)$$

Using

$$\Phi(\boldsymbol{x}') = \frac{q_2}{|\boldsymbol{x}' - \boldsymbol{x}_2|}$$

and

$$|\boldsymbol{x}' - \boldsymbol{x}_2| = \sqrt{(\boldsymbol{x}' - \boldsymbol{x}_2) \cdot (\boldsymbol{x}' - \boldsymbol{x}_2)}$$

we have

$$\frac{\partial \Phi}{\partial t} = -\frac{q_2}{|\boldsymbol{x}' - \boldsymbol{x}_2|^3} (\boldsymbol{x}' - \boldsymbol{x}_2) \cdot \left(-\frac{d \boldsymbol{x}_2}{d t}\right)$$
$$= \frac{q_2}{|\boldsymbol{x}' - \boldsymbol{x}_2|^3} (\boldsymbol{x}' - \boldsymbol{x}_2) \cdot \boldsymbol{v}_2$$

Hence

$$\boldsymbol{J}_l(\boldsymbol{x}') = \frac{q_2}{4\pi} \nabla' \left(\frac{\boldsymbol{v}_2 \cdot (\boldsymbol{x}' - \boldsymbol{x}_2)}{|\ \boldsymbol{x}' - \boldsymbol{x}_2\ |^3} \right)$$

$$\boldsymbol{J}_t(\boldsymbol{x}') = \boldsymbol{J}(\boldsymbol{x}') - \boldsymbol{J}_l(\boldsymbol{x}')$$

$$= q_2 \boldsymbol{v}_2 \ \delta(\boldsymbol{x}' - \boldsymbol{x}_2) - \frac{q_2}{4\pi} \nabla' \left(\frac{\boldsymbol{v}_2 \cdot (\boldsymbol{x}' - \boldsymbol{x}_2)}{|\boldsymbol{x}' - \boldsymbol{x}_2|^3} \right) \qquad (12.79)$$

$$\boldsymbol{A}_{12} \simeq \frac{1}{c} \int d^3 x' \ \frac{1}{|\boldsymbol{x}' - \boldsymbol{x}_1||} \left[q_2 \boldsymbol{v}_2 \ \delta(\boldsymbol{x}' - \boldsymbol{x}_2) - \frac{q_2}{4\pi} \nabla' \left(\frac{\boldsymbol{v}_2 \cdot (\boldsymbol{x}' - \boldsymbol{x}_2)}{|\boldsymbol{x}' - \boldsymbol{x}_2|^3} \right) \right]$$

$$= \frac{q_2 \boldsymbol{v}_2}{c\, r} - \frac{q_2}{4\pi c} \int d^3 x' \ \frac{1}{|\boldsymbol{x}' - \boldsymbol{x}_1|} \nabla' \left(\frac{\boldsymbol{v}_2 \cdot (\boldsymbol{x}' - \boldsymbol{x}_2)}{|\boldsymbol{x}' - \boldsymbol{x}_2|^3} \right)$$

Consider now the integral

$$I = \int d^3 x' \ \frac{1}{|\boldsymbol{x}' - \boldsymbol{x}_1|} \nabla' \left(\frac{\boldsymbol{v}_2 \cdot (\boldsymbol{x}' - \boldsymbol{x}_2)}{|\boldsymbol{x}' - \boldsymbol{x}_2|^3} \right)$$

Setting

$$\boldsymbol{y} = \boldsymbol{x}' - \boldsymbol{x}_2$$

$$\boldsymbol{x}' - \boldsymbol{x}_1 = \boldsymbol{x}' - \boldsymbol{x}_2 + \boldsymbol{x}_2 - \boldsymbol{x}_1 = \boldsymbol{y} - \boldsymbol{r}$$

we have

$$I = \int d^3 y \ \frac{1}{|\ \boldsymbol{y} - \boldsymbol{r}\ |} \nabla_y \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{y}}{y^3} \right)$$

$$= -\int d^3 y \ \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{y}}{y^3} \right) \nabla_y \left(\frac{1}{|\ \boldsymbol{y} - \boldsymbol{r}\ |} \right)$$

$$= \int d^3 y \ \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{y}}{y^3} \right) \nabla_r \left(\frac{1}{|\ \boldsymbol{y} - \boldsymbol{r}\ |} \right)$$

$$= \nabla_r \int d^3 y \ \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{y}}{y^3} \right) \left(\frac{1}{|\ \boldsymbol{y} - \boldsymbol{r}\ |} \right)$$

$$= \nabla_r \boldsymbol{v}_2 \cdot \int d^3 y \ \left(\frac{\boldsymbol{y}}{y^3} \right) \left(\frac{1}{|\ \boldsymbol{y} - \boldsymbol{r}\ |} \right)$$

Consider the integral

$$K^i = \int d^3 y \ \left(\frac{y^i}{y^3} \right) \left(\frac{1}{|\ \boldsymbol{y} - \boldsymbol{r}\ |} \right)$$

In terms of spherical coordinates $\boldsymbol{y} = (y, \theta, \phi)$ with the z axis along $\boldsymbol{r}$, we have

$$K^1 = \int_0^\infty y^2\, d\, y \int_{-1}^1 d \cos\theta \int_0^{2\pi} d\, \phi \, \frac{y \sin\theta \cos\phi}{y^3} \ \frac{1}{\sqrt{y^2 + r^2 - 2\, y\, r \cos\theta}}$$

$$= 0$$

since

$$\int_0^{2\pi} d\, \phi \cos\phi = 0$$

Similarly,

$$K^2 = 0 \qquad \text{since} \quad \int_0^{2\pi} d\, \phi \sin\phi = 0$$

and

$$K^3 = 2\pi \int_0^\infty d\,y \int_{-1}^{1} d\cos\theta \;\; \frac{\cos\theta}{\sqrt{y^2 + r^2 - 2\,y\,r\cos\theta}}$$

$$= \;-2\pi \int_0^\infty d\,y\, \frac{2}{12\,y^2\,r^2} \left\{ \left[2\left(y^2 + r^2\right) + 2\,y\,r\cos\theta \right] \sqrt{y^2 + r^2 - 2\,y\,r\cos\theta} \;\right\}_{-1}^{1}$$

$$= \;-\frac{2\pi}{3\,r^2} \int_0^\infty d\,y\, \frac{1}{y^2} \left\{ \left[y^2 + r^2 + \;y\,r\cos\theta \right] \sqrt{y^2 + r^2 - 2\,y\,r\cos\theta} \;\right\}_{-1}^{1}$$

$$= \;-\frac{2\pi}{3\,r^2} \int_0^\infty d\,y\, \frac{1}{y^2} \left\{ \left(y^2 + r^2 + \;y\,r \right) \sqrt{y^2 + r^2 - 2\,y\,r} \right.$$

$$\left. -\left(y^2 + r^2 - \;y\,r \right) \sqrt{y^2 + r^2 + 2\,y\,r} \;\right\}$$

$$= \;-\frac{2\pi}{3\,r^2} \left\{ \int_0^r d\,y\, \frac{1}{y^2} \left[\left(y^2 + r^2 + \;y\,r \right)(r - y) - \left(y^2 + r^2 - \;y\,r \right)(y + r) \right] \right.$$

$$\left. + \int_r^\infty d\,y\, \frac{1}{y^2} \left[\left(y^2 + r^2 + \;y\,r \right)(y - r) - \left(y^2 + r^2 - \;y\,r \right)(y + r) \right] \right\}$$

The necessity for breaking up the integral comes from that of choosing the correct branch for

$$\sqrt{y^2 + r^2 - 2\,y\,r} \; = \begin{cases} y - r & \text{for} \quad \text{y > r} \\ r - y & \qquad\;\; \text{r > y} \end{cases}$$

Hence

$$K^3 = -\frac{2\pi}{3\,r^2} \left\{ \int_0^r d\,y\, \frac{2}{y^2} \left[-\left(y^2 + r^2 \right) y + y\,r^2 \right] + \int_r^\infty d\,y\, \frac{2}{y^2} \left[-\left(y^2 + r^2 \right) r + y^2\,r \right] \right\}$$

$$= \;-\frac{2\pi}{3\,r^2} \left\{ -2 \int_0^r d\,y\,y - 2\,r^3 \int_r^\infty d\,y\, \frac{1}{y^2} \right\}$$

$$= \;\frac{4\pi}{3\,r^2} \left[\frac{1}{2} r^2 + r^2 \right]$$

$$= \;2\pi$$

Since z is in the $\boldsymbol{r}$ direction, we can write

$$\boldsymbol{K} = \int d^3\,y\, \left(\frac{\boldsymbol{y}}{y^3} \right) \left(\frac{1}{|\;\boldsymbol{y} - \boldsymbol{r}\;|} \right) = 2\pi\,\hat{\boldsymbol{r}} = 2\pi\,\frac{\boldsymbol{r}}{r}$$

$$I = \nabla_r (\boldsymbol{v}_2 \cdot \boldsymbol{K}) = 2\pi\,\nabla_r \left(\boldsymbol{v}_2 \cdot \frac{\boldsymbol{r}}{r} \right)$$

$$\boldsymbol{A}_{12} \simeq \frac{q_2\,\boldsymbol{v}_2}{c\,r} - \frac{q_2}{4\pi\,c}\,I$$

$$= \;\frac{q_2\,\boldsymbol{v}_2}{c\,r} - \frac{q_2}{2\,c}\,\nabla_r \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{r}}{r} \right)$$

$$= \;\frac{q_2}{c} \left[\frac{\boldsymbol{v}_2}{r} - \frac{1}{2}\,\nabla_r \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{r}}{r} \right) \right]$$

Using

$$\frac{\partial}{\partial x_i}\left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{r}}{r}\right) = \frac{\partial}{\partial x_i}\left(\frac{v_{2\,j}\, x_j}{r}\right)$$

$$= v_{2\,j}\left[\frac{\delta_{i\,j}}{r} - \frac{x_j}{r^2}\frac{\partial r}{\partial x_i}\right]$$

$$= \frac{v_{2\,i}}{r} - v_{2\,j}\frac{x_j}{r^3}x_i$$

$$= \frac{v_{2\,i}}{r} - (\boldsymbol{v}_2 \cdot \boldsymbol{r})\frac{1}{r^3}x_i$$

or

$$\nabla_r\left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{r}}{r}\right) = \frac{1}{r}\left[\boldsymbol{v}_2 - \hat{\boldsymbol{r}}\left(\boldsymbol{v}_2 \cdot \hat{\boldsymbol{r}}\right)\right]$$

we have

$$\boldsymbol{A}_{12} = \frac{q_2}{2\,c\,r}\left[\boldsymbol{v}_2 + \hat{\boldsymbol{r}}\left(\boldsymbol{v}_2 \cdot \hat{\boldsymbol{r}}\right)\right]$$

$$= \frac{q_2}{2\,c\,r}\left[\boldsymbol{v}_2 + \frac{\boldsymbol{r}\,(\boldsymbol{v}_2 \cdot \boldsymbol{r})}{r^2}\right] \qquad (12.80)$$

The interaction Lagrangian is

$$L_{\text{int}} = \frac{q_1}{c}\,u_{1\,\alpha}\,A_{12}^{\alpha}$$

$$= q_1\left[\Phi_{12} - \frac{1}{c}\boldsymbol{v}_1 \cdot \boldsymbol{A}_{12}\right] \qquad (\gamma \approx 1)$$

$$= q_1\left\{\frac{q_2}{r} - \frac{q_2}{2\,c^2\,r}\boldsymbol{v}_1 \cdot \left[\boldsymbol{v}_2 + \frac{\boldsymbol{r}\,(\boldsymbol{v}_2 \cdot \boldsymbol{r})}{r^2}\right]\right\}$$

$$= \frac{q_1\,q_2}{r}\left\{1 - \frac{1}{2\,c^2}\left[\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \frac{(\boldsymbol{v}_1 \cdot \boldsymbol{r})\,(\boldsymbol{v}_2 \cdot \boldsymbol{r})}{r^2}\right]\right\} \qquad (12.81)$$

which is commonly known as the Darwin Lagrangian.

8. Lagrangian for Fields

A field is a system with a dynamical variable at each point in space.

For a scalar field $\phi(x)$, the action is defined as

$$S = \int d^4\,x\,\mathcal{L}$$

where, for our purposes, the Lagrangian density

$$\mathcal{L} = \mathcal{L}[\,\phi(x),\ \partial^\alpha\,\phi(x)\,]$$

is assumed to be a function of the field ϕ & its 4-gradient $\partial^\alpha\,\phi$ only.

The equations of motion are obtained through a variation in the 4-dimensional space x,

$$\frac{\partial\,\mathcal{L}}{\partial\,\phi} - \partial^\alpha\,\frac{\partial\,\mathcal{L}}{\partial\,\partial^\alpha\,\phi} = 0$$

Obviously, Lorentz covariance is assured if $\mathcal{L}$ is a scalar.

For a vector field $A^\alpha(x)$, the obvious generalization is

$$\frac{\partial\,\mathcal{L}}{\partial\,A^\beta} - \partial^\alpha\,\frac{\partial\,\mathcal{L}}{\partial\,\partial^\alpha\,A^\beta} = 0$$

For the source- free electromagnetic field, the simplest choice is

$$\begin{aligned}\mathcal{L} &= \xi\, F_{\alpha\beta}\, F^{\alpha\beta} \\ &= \xi \left[\partial_\alpha A_\beta - \partial_\beta A_\alpha \right] \left[\partial^\alpha A^\beta - \partial^\beta A^\alpha \right] \\ &= \xi\, g_{\mu\sigma}\, g_{\nu\tau} \left[\partial^\mu A^\nu - \partial^\nu A^\mu \right] \left[\partial^\sigma A^\tau - \partial^\tau A^\sigma \right]\end{aligned}$$

where ξ is a dimensionless constant. (Note that the use of F means that the homogeneous Maxwell eqs are automatically satisfied so that we need only deal with the inhomogeneous ones),

Hence

$$\frac{\partial \mathcal{L}}{\partial A^\beta} = 0$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial\, \partial^\alpha A^\beta} &= \xi\, g_{\mu\sigma}\, g_{\nu\tau} \left\{ \left[\delta_\alpha{}^\mu\, \delta_\beta{}^\nu - \delta_\alpha{}^\nu\, \delta_\beta{}^\mu \right] \left[\partial^\sigma A^\tau - \partial^\tau A^\sigma \right] \right. \\ &\qquad \left. + \left[\partial^\mu A^\nu - \partial^\nu A^\mu \right] \left[\delta_\alpha{}^\sigma\, \delta_\beta{}^\tau - \delta_\alpha{}^\tau\, \delta_\beta{}^\sigma \right] \right\} \\ &= \xi \left\{ \left[g_{\alpha\sigma}\, g_{\beta\tau} - g_{\beta\sigma}\, g_{\alpha\tau} \right] \left[\partial^\sigma A^\tau - \partial^\tau A^\sigma \right] \right. \\ &\qquad \left. + \left[\partial^\mu A^\nu - \partial^\nu A^\mu \right] \left[g_{\mu\alpha}\, g_{\nu\beta} - g_{\mu\beta}\, g_{\nu\alpha} \right] \right\} \\ &= \xi \left\{ \left[g_{\alpha\sigma}\, g_{\beta\tau} - g_{\beta\sigma}\, g_{\alpha\tau} \right] F^{\sigma\tau} + F^{\mu\nu} \left[g_{\mu\alpha}\, g_{\nu\beta} - g_{\mu\beta}\, g_{\nu\alpha} \right] \right\} \\ &= \xi \left\{ F_{\alpha\beta} - F_{\beta\alpha} + F_{\alpha\beta} - F_{\beta\alpha} \right\} \\ &= 4\, \xi\, F_{\alpha\beta}\end{aligned}$$

The factor 4 in the last expression can be obtained at once since there are 4 ∂A factors in $\mathcal{L}$ and each of them gives the same contribution.

The source free eqs are therefore

$$\partial^\alpha F_{\alpha\beta} = 0$$

irregardless of the value of ξ.

In the presence of source, the interaction Lagrangian density is obviously

$$\mathcal{L}_{\text{int}} = \eta\, \frac{1}{c}\, J_\alpha\, A^\alpha$$

where η is a dimensionless constant.

With

$$\frac{\partial \mathcal{L}_{\text{int}}}{\partial A^\beta} = \eta\, \frac{1}{c}\, J_\beta$$

$$\frac{\partial \mathcal{L}_{\text{int}}}{\partial\, \partial^\alpha A^\beta} = 0$$

the eq of motion of

$$\mathcal{L} = \xi\, F_{\alpha\beta}\, F^{\alpha\beta} + \eta\, \frac{1}{c}\, J_\alpha\, A^\alpha$$

are

$$\eta\, \frac{1}{c}\, J_\beta - 4\, \xi\, \partial^\alpha F_{\alpha\beta} = 0$$

Comparing with the Maxwell eqs

$$\partial^\alpha F_{\alpha\beta} = \frac{4\pi}{c} J_\beta$$

we have

$$\frac{\xi}{\eta} = \frac{1}{16\pi}$$

so that

$$\mathcal{L} = \eta \left[\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{c} J_\alpha A^\alpha \right]$$

η can be determined by comparing $\int d^3 \mathcal{L}_{\text{int}}$ with $\mathbb{L}_{\text{int}}$ between a charged particle and an EM field discussed in section 12.1. This gives $\eta = 1$ so that

$$\mathcal{L} = \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{c} J_\alpha A^\alpha$$

---

9. Proca Lagrangian

In relativistic quantum mechanics, the Klein- Gordon eq

$$\left(\Box + \mu^2 \right) \psi = 0$$

describes a boson with mass $m = \frac{\mu \hbar}{c}$. Hence, the wave eq

$$\Box A^\alpha = 0$$

may be interpreted as describing a particle of zero mass.

Conversely, the eqs

$$\left(\Box + \mu^2 \right) A^\alpha = \frac{4\pi}{c} J^\alpha \qquad (12.93)$$

should describe, in the Lorentz gauge, an electrodynamics with finite photon mass.

Eq (12.93) can be derived from the Proca Lagrangian

$$\mathcal{L}_{\text{Proca}} = \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{c} J_\alpha A^\alpha - \frac{\mu^2}{8\pi} A_\alpha A^\alpha \qquad (12.91)$$

since

$$\frac{\partial \mathcal{L}_{\text{mass}}}{\partial A_\alpha} = -\frac{\mu^2}{4\pi} A^\alpha$$

If the source is a point charge at rest at the origin,

$$J^\alpha(x) = (q\, \delta(\boldsymbol{x}),\ \mathbf{0})$$

$$\left(\Box + \mu^2\right)\phi = \frac{4\pi}{c}\, q\, \delta(\boldsymbol{x})$$
$$= \left(-\nabla^2 + \mu^2\right)\phi$$

Using

$$\nabla^2 \frac{1}{r} = -4\pi\, \delta(\boldsymbol{x})$$

we see that

$$\phi(x) = q\, \frac{e^{-\mu r}}{r} \qquad (12.94)$$

which falls off exponentially with effective range $\frac{1}{\mu}$. Thus, testing the $\frac{1}{r^2}$ law for coulomb field can be used to estimated the value the mass of photon.

The free field wave eq is

$$\left(\Box + \mu^2\right) A^\alpha = 0$$

with allows plane wave solutions $e^{\pm i k \cdot r}$ with the constraint

$$-k^{0\,2} + \boldsymbol{k}^2 + \mu^2 = 0$$

or

$$\left(\frac{\omega}{c}\right)^2 = \boldsymbol{k}^2 + \mu^2 \qquad (12.95)$$

Since the minimum value of $\boldsymbol{k}^2$ is 0, the lowest allowed frequency is $\omega_{\min} = \mu\, c$.

Upper limit of μ can be estimated from the lowest Schumann mode.

---

10. Stress Tensors

■ Canonical Stress Tensor (Free Fields)

The canonical stress tensor is defined as (note that our definition is the negative of that used by Jackson)

$$T^{\alpha\beta} = -\frac{\partial \mathcal{L}}{\partial\, \partial_\alpha\, \phi}\, \partial^\beta \phi + g^{\alpha\beta}\, \mathcal{L} \qquad \text{(scalar field)}$$
$$= -\frac{\partial \mathcal{L}}{\partial\, \partial_\alpha\, \phi^\mu}\, \partial^\beta \phi^\mu + g^{\alpha\beta}\, \mathcal{L} \qquad \text{(vector field)} \qquad (12.102)$$

For the free electromagnetic field

$$\mathcal{L} = \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta}$$
$$\frac{\partial \mathcal{L}}{\partial\, \partial_\alpha A^\beta} = \frac{1}{4\pi} F^\alpha{}_\beta = \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\beta}$$
$$T^{\alpha\beta} = -\frac{1}{4\pi} F^\alpha{}_\mu\, \partial^\beta A^\mu + \frac{1}{16\pi} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}$$
$$= -\frac{1}{4\pi} g^{\alpha\nu} F_{\nu\mu}\, \partial^\beta A^\mu + \frac{1}{16\pi} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \qquad (12.104)$$

Using

$$F^{i0} = -F^{0i} = E_i = F_{0i} = -F_{i0}$$
$$F^{ij} = -\epsilon_{ijk} B_k = F_{ij}$$

we have

$$F_{\mu\nu} F^{\mu\nu} = F_{0i} F^{0i} + F_{ij} F^{ij} + F_{i0} F^{i0}$$
$$= \; -2 E_i E_i + \epsilon_{ijk} B_k \epsilon_{ijl} B_l$$

Now

$$\epsilon_{ijk} \epsilon_{ijl} B_k B_l = \left(\delta_{jj} \delta_{kl} - \delta_{jl} \delta_{kj}\right) B_k B_l$$
$$= \; 3 B_k B_k - B_j B_j$$
$$= \; 2 \boldsymbol{B}^2$$

Hence

$$F_{\mu\nu} F^{\mu\nu} = 2\left(- \boldsymbol{E}^2 + \boldsymbol{B}^2\right)$$
$$\mathcal{L} = -\frac{1}{8\pi}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right)$$

$$F_{0\mu} \partial^0 A^\mu = \; F_{0i} \partial^0 A^i = E_i \frac{\partial}{c\,\partial t} A^i = \boldsymbol{E} \cdot \frac{\partial}{c\,\partial t} \boldsymbol{A}$$

$$F_{0\mu} \partial^j A^\mu = F_{0i} \partial^j A^i = F_{0i}\left(F^{ji} + \partial^i A^j\right)$$
$$= \; E_i\left(-\epsilon_{jik} B_k + \partial^i A^j\right)$$
$$= \; [-\boldsymbol{E} \times \boldsymbol{B} - (\boldsymbol{E} \cdot \nabla)\, \boldsymbol{A}\,]_j$$

$$F_{j\mu} \partial^0 A^\mu = F_{j0} \partial^0 A^0 + F_{ji} \partial^0 A^i$$
$$= \; -E_j \frac{\partial}{c\,\partial t} \phi - \epsilon_{jik} B_k \frac{\partial}{c\,\partial t} A^i$$
$$= \; \left(-\boldsymbol{E} \frac{\partial \phi}{c\,\partial t} - \frac{\partial \boldsymbol{A}}{c\,\partial t} \times \boldsymbol{B}\right)_j$$

Using

$$\boldsymbol{E} = \; -\nabla \phi - \frac{\partial \boldsymbol{A}}{c\,\partial t} \qquad\qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

we have

$$F_{0\mu} \partial^0 A^\mu = -\boldsymbol{E} \cdot (\,\boldsymbol{E} + \nabla \phi\,) = -\boldsymbol{E}^2 - \boldsymbol{E} \cdot \nabla \phi$$

For free fields,

$$\nabla \cdot \boldsymbol{E} = 0$$

so that

$$\nabla \cdot (\phi\, \boldsymbol{E}) = \phi\, \nabla \cdot \boldsymbol{E} + \boldsymbol{E} \cdot \nabla \phi = \boldsymbol{E} \cdot \nabla \phi$$

and

$$F_{0\mu} \partial^0 A^\mu = -\boldsymbol{E}^2 - \nabla \cdot (\phi\, \boldsymbol{E}) = F^0{}_\mu \partial^0 A^\mu$$

Thus

$$T^{00} = \; -\frac{1}{4\pi} F^0{}_\mu \partial^0 A^\mu = -\frac{1}{4\pi} F_{0\mu} \partial^0 A^\mu$$
$$= \; \frac{1}{4\pi}\left[\boldsymbol{E}^2 + \nabla \cdot (\phi\, \boldsymbol{E})\right] - \frac{1}{8\pi}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right)$$
$$= \; \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \frac{1}{4\pi} \nabla \cdot (\phi\, \boldsymbol{E}) \qquad (12.105)$$

$$T^{0j} = -\frac{1}{4\pi} F^0{}_\mu \partial^j A^\mu = -\frac{1}{4\pi} F_{0\mu} \partial^j A^\mu$$
$$= \frac{1}{4\pi} [\boldsymbol{E}\times\boldsymbol{B} + (\boldsymbol{E}\cdot\nabla)\boldsymbol{A}]_j$$
$$= \frac{1}{4\pi}\left[(\boldsymbol{E}\times\boldsymbol{B})_j + \nabla\cdot\left(A_j \boldsymbol{E}\right)\right] \qquad (12.105)$$

where we've used

$$\nabla\cdot\left(A_j \boldsymbol{E}\right) = A_j \nabla\cdot\boldsymbol{E} + (\boldsymbol{E}\cdot\nabla) A_j = (\boldsymbol{E}\cdot\nabla) A_j$$

$$\frac{\partial \boldsymbol{A}}{c\,\partial t}\times\boldsymbol{B} = -(\boldsymbol{E} + \nabla\phi)\times\boldsymbol{B}$$
$$= -\boldsymbol{E}\times\boldsymbol{B} - \nabla\phi\times\boldsymbol{B}$$
$$= -\boldsymbol{E}\times\boldsymbol{B} - \nabla\times(\phi\boldsymbol{B}) + \phi\nabla\times\boldsymbol{B}$$
$$= -\boldsymbol{E}\times\boldsymbol{B} - \nabla\times(\phi\boldsymbol{B}) + \phi\frac{\partial \boldsymbol{E}}{c\,\partial t}$$

$$-\boldsymbol{E}\frac{\partial\phi}{c\,\partial t} - \frac{\partial \boldsymbol{A}}{c\,\partial t}\times\boldsymbol{B} = -\boldsymbol{E}\frac{\partial\phi}{c\,\partial t} + \boldsymbol{E}\times\boldsymbol{B} + \nabla\times(\phi\boldsymbol{B}) - \phi\frac{\partial \boldsymbol{E}}{c\,\partial t}$$
$$= \boldsymbol{E}\times\boldsymbol{B} + \nabla\times(\phi\boldsymbol{B}) - \frac{\partial}{c\,\partial t}(\phi\boldsymbol{E})$$

Therefore

$$F_{j\mu}\partial^0 A^\mu = \left[\boldsymbol{E}\times\boldsymbol{B} + \nabla\times(\phi\boldsymbol{B}) - \frac{\partial}{c\,\partial t}(\phi\boldsymbol{E})\right]_j$$

$$T^{j0} = -\frac{1}{4\pi} F^j{}_\mu \partial^0 A^\mu = \frac{1}{4\pi} F_{j\mu}\partial^0 A^\mu$$
$$= \frac{1}{4\pi}\left[\boldsymbol{E}\times\boldsymbol{B} + \nabla\times(\phi\boldsymbol{B}) - \frac{\partial}{c\,\partial t}(\phi\boldsymbol{E})\right]_j \qquad (12.105)$$

$$\int d^3x\, T^{00} = \int d^3x \left[\frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \frac{1}{4\pi}\nabla\cdot(\phi\boldsymbol{E})\right]$$
$$= \frac{1}{8\pi}\int d^3x\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)$$
$$= E_{\text{field}}$$

where the term involving $\nabla\cdot(\phi\boldsymbol{E})$ can be transformed to a surface integral by the Green's theorem and hence vanishes provided the fields vanish at infinity.

Thus T^{00} can be interpreted as the energy density of the EM field as long as it is always used inside integrals.

Similarly,

$$\int d^3x\, T^{0j} = \frac{1}{4\pi}\int d^3x\, [\boldsymbol{E}\times\boldsymbol{B}]_j = c\, P_{\text{field}}{}^j$$

where $\boldsymbol{P}_{\text{field}}$ is the Poynting vector so that T^{j0}/c can be interpreted as the field momentum density.

Now

$$\partial_\alpha T^{\alpha\beta} = \partial_\alpha \left\{ -\frac{\partial \mathcal{L}}{\partial\, \partial_\alpha\, \phi^\mu} \partial^\beta \phi^\mu + g^{\alpha\beta} \mathcal{L} \right\}$$

$$= \quad -\left(\partial_\alpha \frac{\partial \mathcal{L}}{\partial\, \partial_\alpha\, \phi^\mu} \right) \partial^\beta \phi^\mu - \frac{\partial \mathcal{L}}{\partial\, \partial_\alpha\, \phi^\mu} \partial_\alpha \partial^\beta \phi^\mu + \partial^\beta \mathcal{L}$$

$$= \quad -\frac{\partial \mathcal{L}}{\partial\, \phi^\mu} \partial^\beta \phi^\mu - \frac{\partial \mathcal{L}}{\partial\, \partial_\alpha\, \phi^\mu} \partial^\beta \partial_\alpha \phi^\mu + \partial^\beta \mathcal{L}$$

If $\mathcal{L} = \mathcal{L}\left(\phi^\alpha, \partial^\beta \phi^\alpha \right)$,

$$\partial^\beta \mathcal{L} = \quad \frac{\partial \mathcal{L}}{\partial\, \phi^\mu} \partial^\beta \phi^\mu + \frac{\partial \mathcal{L}}{\partial\, (\partial_\alpha\, \phi^\mu)} \partial^\beta (\partial_\alpha \phi^\mu)$$

Hence

$$\partial_\alpha T^{\alpha\beta} = -\partial^\beta \mathcal{L} + \partial^\beta \mathcal{L} = 0 \qquad (12.107)$$

$$0 = \int d^3 x\, \partial_\alpha T^{\alpha\beta} = \int d^3 x\, \partial_0 T^{0\,\beta} + \int d^3 x\, \partial_i T^{i\,\beta}$$

$$= \quad \partial_0 \int d^3 x\, T^{0\,\beta} + \int d^3 x\, \partial_i T^{i\,\beta}$$

$$= \quad \partial_0 \int d^3 x\, T^{0\,\beta} \qquad \text{for localized sources}$$

$$= \quad \frac{d}{c\, d\, t} E_{\text{field}} \qquad (12.108)$$

$$\frac{d}{c\, d\, t} {P_{\text{field}}}^j = \int d^3 x\;\, \partial_0 T^{0\,j}$$

$$= \quad -\int d^3 x\;\, \partial_i T^{i\,j} = 0 \qquad (12.108)$$

Thus, the equation

$$\partial_\alpha T^{\alpha\beta} = 0$$

implies conservation of both of the field energy & field momentum.

■ Symmetric Stress Tensor (Free Fields)

$T^{\alpha\beta}$ is not satisfactory since

1. T^{00} & $T^{0\,j}$ are not exactly the same as the energy & momentum densities, respectively.
2. $T^{0\,j} \neq T^{j\,0}$ (not symmetric)

A proper stress tensor should be symmetric so that conservation of energy & linear momentum implies that of the angular momentum.

In non-relativistic treatment, the field angular momentum density is defined as

$$\boldsymbol{l}_{\text{field}} = \boldsymbol{x} \times \boldsymbol{p}_{\text{field}} = \frac{1}{4\,\pi\, c}\, \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B})$$

so that

$$\boldsymbol{L}_{\text{field}} = \int d^3 x\, \boldsymbol{l}_{\text{field}} = \frac{1}{4\,\pi\, c} \int d^3 x\;\, \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B})$$

Covariant generalization leads to the 3rd rank tensor

$$M^{\alpha\beta\gamma} = T^{\alpha\beta} x^\gamma - T^{\alpha\gamma} x^\beta$$

with conservation of angular momentum equivalent to

$$\partial_\alpha M^{\alpha\beta\gamma} = 0$$

Now

$$\partial_\alpha M^{\alpha\beta\gamma} = \left(\partial_\alpha T^{\alpha\beta}\right) x^\gamma + T^{\alpha\beta}\,\partial_\alpha x^\gamma - (\partial_\alpha T^{\alpha\gamma})\, x^\beta - T^{\alpha\gamma}\,\partial_\alpha x^\beta$$
$$= \left(\partial_\alpha T^{\alpha\beta}\right) x^\gamma + T^{\gamma\beta} - (\partial_\alpha T^{\alpha\gamma})\, x^\beta - T^{\beta\gamma}$$

In case of conservation of energy & linear momentum

$$\partial_\alpha T^{\alpha\beta} = 0$$

we have

$$\partial_\alpha M^{\alpha\beta\gamma} = T^{\gamma\beta} - T^{\beta\gamma}$$

automatic conservation of angular momentum thus require T to be symmetric.

Now, the asymmetric part of

$$T^{\alpha\beta} = -\frac{1}{4\pi}\, g^{\alpha\nu}\, F_{\nu\mu}\,\partial^\beta A^\mu + \frac{1}{16\pi}\, g^{\alpha\beta}\, F_{\mu\nu}\, F^{\mu\nu}$$

is contained in the 1st term. Using

$$\partial^\beta A^\mu = -F^{\mu\beta} + \partial^\mu A^\beta$$

we have

$$T^{\alpha\beta} = \frac{1}{4\pi}\, g^{\alpha\nu}\, F_{\nu\mu}\, F^{\mu\beta} + \frac{1}{16\pi}\, g^{\alpha\beta}\, F_{\mu\nu}\, F^{\mu\nu} - \frac{1}{4\pi}\, g^{\alpha\nu}\, F_{\nu\mu}\,\partial^\mu A^\beta$$
$$= \Theta^{\alpha\beta} + T_D{}^{\alpha\beta}$$

where

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\, g^{\alpha\nu}\, F_{\nu\mu}\, F^{\mu\beta} + \frac{1}{16\pi}\, g^{\alpha\beta}\, F_{\mu\nu}\, F^{\mu\nu}$$
$$= \frac{1}{4\pi}\, F^\alpha{}_\mu\, F^{\mu\beta} + \frac{1}{16\pi}\, g^{\alpha\beta}\, F_{\mu\nu}\, F^{\mu\nu}$$
$$T_D{}^{\alpha\beta} = -\frac{1}{4\pi}\, g^{\alpha\nu}\, F_{\nu\mu}\,\partial^\mu A^\beta$$

Now

$$F^\beta{}_\mu\, F^{\mu\alpha} = F^{\beta\mu}\, F_\mu{}^\alpha = (-)^2\, F^{\mu\beta}\, F^\alpha{}_\mu = F^\alpha{}_\mu\, F^{\mu\beta}$$

so that Θ is symmetric.

Furthermore

$$T_D{}^{\alpha\beta} = -\frac{1}{4\pi}\, F^\alpha{}_\mu\,\partial^\mu A^\beta$$
$$= -\frac{1}{4\pi}\, F^{\alpha\mu}\,\partial_\mu A^\beta$$
$$= -\frac{1}{4\pi}\left[\partial_\mu\left(F^{\alpha\mu} A^\beta\right) - \left(\partial_\mu F^{\alpha\mu}\right) A^\beta\right]$$
$$= -\frac{1}{4\pi}\,\partial_\mu\left(F^{\alpha\mu} A^\beta\right) \qquad \text{for free fields}$$
$$= \frac{1}{4\pi}\,\partial_\mu\left(F^{\mu\alpha} A^\beta\right) \qquad (12.112)$$

Thus

$$\begin{aligned} T_D{}^{0\,\beta} = & \frac{1}{4\pi}\,\partial_\mu\left(F^{\mu 0}\,A^\beta\right) \\ = & \frac{1}{4\pi}\,\partial_i\left(F^{i\,0}\,A^\beta\right) \\ = & \frac{1}{4\pi}\,\partial_i\left(E_i\,A^\beta\right) \\ = & \frac{1}{4\pi}\,\nabla\cdot\left(\boldsymbol{E}\,A^\beta\right) \end{aligned}$$

so that

$$\begin{aligned} \int d^3x\,T_D{}^{0\,\beta} = & \frac{1}{4\pi}\int d^3x\,\nabla\cdot\left(\boldsymbol{E}\,A^\beta\right) \\ = & \frac{1}{4\pi}\oint d\,\boldsymbol{S}\cdot\left(\boldsymbol{E}\,A^\beta\right) \\ = & 0 \end{aligned}$$

provided the fields are localized.

Also

$$\begin{aligned} \partial_\alpha T_D{}^{\alpha\beta} = & -\frac{1}{4\pi}\,\partial_\alpha\left(F^\alpha{}_\mu\,\partial^\mu A^\beta\right) \\ = & -\frac{1}{4\pi}\,F^\alpha{}_\mu\,\partial_\alpha\partial^\mu A^\beta \qquad (\text{ since } \partial_\alpha F^\alpha{}_\mu = 0\ ) \end{aligned}$$

Now, $F^\alpha{}_\mu$ is antisymmetric while $\partial_\alpha\partial^\mu$ is symmetric. Their contraction vanishes,

$$\partial_\alpha T_D{}^{\alpha\beta} = 0$$

For those who need more concrete proof,

$$\begin{aligned} F^\alpha{}_\mu\,\partial_\alpha\partial^\mu = & -F_\mu{}^\alpha\,\partial_\alpha\partial^\mu \\ = & -F_\alpha{}^\mu\,\partial_\mu\partial^\alpha \qquad (\ \alpha \leftrightarrow \mu\ ) \\ = & -F^\alpha{}_\mu\,\partial^\mu\partial_\alpha \qquad (\ a^\mu\,b_\mu = a_\mu\,b^\mu\ ) \\ = & -F^\alpha{}_\mu\,\partial_\alpha\partial^\mu \\ = & 0 \qquad (\ a = -a = 0\ ) \end{aligned}$$

Thus, the symmetric part

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\,F^\alpha{}_\mu\,F^{\mu\beta} + \frac{1}{16\pi}\,g^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu}$$

also satisfies

$$\partial_\alpha\Theta^{\alpha\beta} = 0 \qquad (12.116)$$

$$\partial_0\int d^3x\,\Theta^{0\,\beta} = 0$$

It therefore embodies the energy- momentum conservation relations. In fact, it is the symmetric stress tensor that we were looking for.

Explicit calculations of $\Theta^{\alpha\beta}$ is similar to those for $T^{\alpha\beta}$ and will be left as an exercise. The result is

$$\Theta^{\alpha\beta} = \left(\begin{array}{c|c} u & c\,\boldsymbol{g} \\ \hline c\,\boldsymbol{g} & -T^{(M)}_{i\,j} \end{array}\right) \qquad (12.115)$$

where

$$u = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \qquad \text{(Energy density)}$$

$$\boldsymbol{g} = \frac{1}{4\pi c}\,\boldsymbol{E}\times\boldsymbol{B} \qquad \text{(Poynting vector)}$$

$$T^{(M)}_{i\,j} = \frac{1}{4\pi}\left[E_i\,E_j + B_i\,B_j - \frac{1}{2}\,u\,\delta_{i\,j}\right] \qquad \text{(Maxwell stress tensor)}$$

For convenience, other forms of Θ are listed below

$$\Theta_{\alpha\beta} = \left(\begin{array}{c|c} u & -c\,\boldsymbol{g} \\ \hline -c\,\boldsymbol{g} & -T^{(M)}_{i\,j} \end{array}\right) \Theta^{\alpha}{}_{\beta} = \left(\begin{array}{c|c} u & -c\,\boldsymbol{g} \\ \hline c\,\boldsymbol{g} & T^{(M)}_{i\,j} \end{array}\right)$$

$$\Theta_{\alpha}{}^{\beta} = \left(\begin{array}{c|c} u & c\,\boldsymbol{g} \\ \hline -c\,\boldsymbol{g} & T^{(M)}_{i\,j} \end{array}\right)$$

■ Source

In the presence of source,

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}\,J^\beta$$

The deivergence of

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\,F^{\alpha}{}_{\mu}\,F^{\mu\beta} + \frac{1}{16\pi}\,g^{\alpha\beta}\,F_{\mu\nu}\,F^{\mu\nu}$$

no longer vanishes. Instead, we have

$$\begin{aligned}\partial_\alpha \Theta^{\alpha\beta} &= \frac{1}{4\pi}\,\partial_\alpha\left(F^{\alpha}{}_{\mu}\,F^{\mu\beta}\right) + \frac{1}{16\pi}\,\partial^\beta\left(F_{\mu\nu}\,F^{\mu\nu}\right) \\ &= \frac{1}{4\pi}\left[\left(\partial_\alpha F^{\alpha}{}_{\mu}\right)F^{\mu\beta} + F^{\alpha}{}_{\mu}\,\partial_\alpha F^{\mu\beta} + \frac{1}{4}\,\partial^\beta\left(F_{\mu\nu}\,F^{\mu\nu}\right)\right]\end{aligned}$$

Hence

$$\begin{aligned}\partial_\alpha \Theta^{\alpha\beta} - \frac{1}{c}\,J_\mu\,F^{\mu\beta} &= \frac{1}{4\pi}\left[F^{\alpha}{}_{\mu}\,\partial_\alpha F^{\mu\beta} + \frac{1}{4}\,\partial^\beta\left(F_{\mu\nu}\,F^{\mu\nu}\right)\right] \\ &= \frac{1}{4\pi}\left[F_{\alpha\mu}\,\partial^\alpha F^{\mu\beta} + \frac{1}{4}\left(\partial^\beta F_{\mu\nu}\right)F^{\mu\nu} + \frac{1}{4}\,F_{\mu\nu}\,\partial^\beta F^{\mu\nu}\right]\end{aligned}$$

$$= \frac{1}{4\pi}\left[F_{\alpha\mu}\,\partial^\alpha F^{\mu\beta} + \frac{1}{2} F_{\mu\nu}\,\partial^\beta F^{\mu\nu}\right]$$
$$= \frac{1}{8\pi}\left[2 F_{\alpha\mu}\,\partial^\alpha F^{\mu\beta} + F_{\mu\nu}\,\partial^\beta F^{\mu\nu}\right]$$
$$= \frac{1}{8\pi}\left[2 F_{\nu\mu}\,\partial^\nu F^{\mu\beta} + F_{\mu\nu}\,\partial^\beta F^{\mu\nu}\right]$$
$$= \frac{1}{8\pi}\left[F_{\nu\mu}\,\partial^\nu F^{\mu\beta} + F_{\mu\nu}\,\partial^\mu F^{\nu\beta} + F_{\mu\nu}\,\partial^\beta F^{\mu\nu}\right]$$
$$= \frac{1}{8\pi} F_{\mu\nu}\left[\partial^\nu F^{\beta\mu} + \partial^\mu F^{\nu\beta} + \partial^\beta F^{\mu\nu}\right]$$
$$= \frac{1}{8\pi} F_{\mu\nu}\left[\partial^\mu F^{\nu\beta} + \left(\partial^\nu F^{\beta\mu} + \partial^\beta F^{\mu\nu}\right)\right]$$
$$= \frac{1}{8\pi} F_{\mu\nu}\left[\partial^\mu F^{\nu\beta} - \partial^\mu F^{\nu\beta}\right]$$
$$= 0$$

where we've used the homogeneous Maxwell eqs

$$\partial^\nu F^{\beta\mu} + \partial^\beta F^{\mu\nu} + \partial^\mu F^{\nu\beta} = 0$$

Hence

$$\partial_\alpha \Theta^{\alpha\beta} = \frac{1}{c} J_\mu F^{\mu\beta}$$
$$= -\frac{1}{c} F^{\beta\mu} J_\mu \qquad (12.118)$$

For $\beta = 0$

$$\partial_\alpha \Theta^{\alpha 0} = \partial_0 \Theta^{00} + \partial_i \Theta^{i\,0} = \frac{\partial}{c\,\partial t} u + c\,\nabla\cdot\boldsymbol{g}$$
$$F^{0\,\mu} J_\mu = F^{0\,i} J_i = (-E_i)\,(-\boldsymbol{J})_i = \boldsymbol{E}\cdot\boldsymbol{J}$$

so that

$$\frac{\partial}{c\,\partial t} u + c\,\nabla\cdot\boldsymbol{g} = -\frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{J} \qquad (12.119)$$

or

$$\frac{\partial}{\partial t} u + \nabla\cdot\boldsymbol{S} = -\boldsymbol{E}\cdot\boldsymbol{J} \qquad \boldsymbol{S} = c^2\,\boldsymbol{g} = \text{energy flow}$$

Note: the left side of the relation $J_i = (-\boldsymbol{J})_i$ denotes the covariant component i of the 4-vector $J^\alpha = \left(J^0, \boldsymbol{J}\right)$.

For $\beta = j$

$$\partial_\alpha \Theta^{\alpha\,j} = \partial_0 \Theta^{0\,j} + \partial_i \Theta^{i\,j}$$
$$= \frac{\partial}{\partial t} g^j - \partial_i T^{(M)}_{i\,j}$$
$$F^{j\,\mu} J_\mu = F^{j\,0} J_0 + F^{j\,i} J_i$$
$$= E_j\,\rho\,c - \epsilon_{j\,i\,k} B_k\,(-\boldsymbol{J})_i$$
$$= c\left[\rho\,\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}\right]_j$$

Hence

$$\frac{\partial}{\partial t}\boldsymbol{g} - \nabla \cdot T^{(M)} = -\left[\rho\,\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}\right] \qquad (12.120)$$

The Lorentz 4-force density is defined as

$$f^{\alpha} = \frac{1}{c} F^{\beta\mu} J_{\mu} = \left(\frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{J},\ \rho\,\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}\right) \qquad (12.121)$$

so that

$$\partial_{\alpha}\Theta^{\alpha\beta} = -f^{\beta}$$

or

$$\partial\cdot\Theta = -f$$

If the sources are charged particles in motion, we have

$$\int d^3x\, f^{\alpha} = \frac{d}{d\,t} P^{\alpha}_{\text{particles}}$$

so that

$$0 = \int d^3x\left(\partial_{\alpha}\Theta^{\alpha\beta} + f^{\beta}\right) = \frac{d}{d\,t}\left(P^{\beta}_{\text{field}} + P^{\beta}_{\text{particles}}\right)$$

---

11. Green Functions

The inhomogeneous Maxwell eqs are

$$\begin{aligned}\partial_{\alpha}F^{\alpha\beta} &= \frac{4\pi}{c} J^{\beta}\\ &= \partial_{\alpha}\left(\partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}\right)\\ &= \square A^{\beta} - \partial_{\alpha}\partial^{\beta}A^{\alpha} \qquad \square = \partial_{\alpha}\partial^{\alpha}\\ &= \square A^{\beta} - \partial^{\beta}\partial_{\alpha}A^{\alpha}\end{aligned}$$

In the Lorentz gauge

$$\partial_{\alpha}A^{\alpha} = 0$$

we have

$$\square A^{\beta} = \frac{4\pi}{c} J^{\beta} \qquad (12.123)$$

Define

$$\square_x D(x, x') = \delta^{(4)}(x - x') \qquad (12.124)$$

where

$$\delta^{(4)}(x - x') = \delta(x_0 - x_0{}')\,\delta(\boldsymbol{x} - \boldsymbol{x}')$$

In the absence of boundary conditions

$$D(x, x') = D(x - x') = D(z)$$

where

$$z = x - x'$$

so that (12.124) becomes

$$\square_z D(z) = \delta^{(4)}(z)$$

The Fourier transform $\tilde{D}(k)$ of $D(z)$ is defined by

$$\tilde{D}(k) = \int d^4z\, D(z)\, e^{i\,k\cdot z} \qquad k\cdot z = k^0 z^0 - \boldsymbol{k}\cdot\boldsymbol{z}$$

Using

$$\delta^{(4)}(z) = \frac{1}{(2\pi)^4} \int d^4 k\, e^{-i k \cdot z} \qquad (12.126)$$

the inverse transform is found to be

$$D(z) = \frac{1}{(2\pi)^4} \int d^4 k\, \tilde{D}(k)\, e^{-i k \cdot z} \qquad (12.125)$$

Using

$$\begin{aligned} \Box_z e^{-i k \cdot z} &= \partial_\alpha \partial^\alpha e^{-i k \cdot z} \\ &= (-i k^\alpha)(-i k_\alpha)\, e^{-i k \cdot z} \\ &= -k \cdot k\, e^{-i k \cdot z} \end{aligned}$$

we have

$$\begin{aligned} \Box_z D(z) &= \frac{1}{(2\pi)^4} \int d^4 k\, \tilde{D}(k)\, \Box_z e^{-i k \cdot z} \\ &= -\frac{1}{(2\pi)^4} \int d^4 k\, \tilde{D}(k)\, k \cdot k\, e^{-i k \cdot z} \\ &= \delta^{(4)}(z) \\ &= \frac{1}{(2\pi)^4} \int d^4 k\, e^{-i k \cdot z} \end{aligned}$$

which requires

$$-\tilde{D}(k)\, k \cdot k = 1$$

or

$$\begin{aligned} \tilde{D}(k) &= -\frac{1}{k \cdot k} \qquad (12.127) \\ &= \frac{1}{\boldsymbol{k}^2 - \left(\frac{\omega}{c}\right)^2} \qquad k^\alpha = \left(\frac{\omega}{c}, \boldsymbol{k}\right) \end{aligned}$$

Hence

$$\begin{aligned} D(z) &= -\frac{1}{(2\pi)^4} \int d^4 k\, \frac{1}{k \cdot k}\, e^{-i k \cdot z} \\ &= -\frac{1}{(2\pi)^4} \int d^3 k \int d k_0\, \frac{1}{k_0^2 - \boldsymbol{k}^2}\, e^{-i k_0 z_0 + i \boldsymbol{k} \cdot \boldsymbol{z}} \end{aligned}$$

The integral

$$I_k = \int_{-\infty}^{\infty} d k_0\, \frac{1}{k_0^2 - \boldsymbol{k}^2}\, e^{-i k_0 z_0}$$

is improper due to the poles at $k_0 = \pm |\boldsymbol{k}| = \pm\kappa$. However, finite value can still be obtained if the path of integration is distorted (into the complex k_0 plane) to avoid these poles. Obviously, the value thus obtained depends on the particular distortion chosen.

For example, the paths for the retarded, D_r, and advanced, D_a, Green functions are shown in Fig 12.8.

Now

$$-i\,k_0\,z_0 = -i\,(\,\text{Re }\,k_0\,)\,z_0 + (\,\text{Im}\,k_0\,)\,z_0$$

Thus, for $z_0 > 0$,

$$e^{-i\,k_0\,z_0} \xrightarrow[|k_0|\to\infty]{} 0 \qquad\qquad \text{if} \qquad \text{Im }\,k_0 < 0$$

Hence, the contour must be closed in the lower k_0 plane. (the contour is thus traveled in the clockwise, or negative, sense) Conversely, for $z_0 < 0$, the contour must be closed in the upper plane. (the contour is traveled in the counter- clockwise, or positive sense)

Consider D_r.
For $z_0 > 0$, both poles are enclosed so that

$$\begin{aligned} I_k &= -2\,\pi\,i \sum_{k_0=\pm\kappa} \text{Res}\left(\frac{1}{k_0{}^2 - \kappa^2}\, e^{-i\,k_0\,z_0} \right) \\ &= -2\,\pi\,i\left[\,\frac{1}{2\,\kappa}\,e^{-i\,\kappa\,z_0} - \frac{1}{2\,\kappa}\,e^{i\,\kappa\,z_0}\,\right] \\ &= -\frac{2\,\pi}{\kappa}\,\sin\,(\kappa\,z_0) \end{aligned}$$

For $z_0 < 0$, no pole is enclosed so that

$$I_k = 0.$$

For D_a,

$$I_k = 0 \qquad\qquad\qquad \text{for} \qquad z_0 > 0$$

$$\begin{aligned} I_k &= 2\,\pi\,i \sum_{k_0=\pm\kappa} \text{Res}\left(\frac{1}{k_0{}^2 - \kappa^2}\, e^{-i\,k_0\,z_0} \right) \\ &= \frac{2\,\pi}{\kappa}\,\sin\,(\kappa\,z_0) \qquad\qquad \text{for} \qquad z_0 < 0 \end{aligned}$$

Hence

$$D_r(z) = \frac{\theta(z_0)}{(2\,\pi)^3} \int d^3\,k\;\frac{\sin\,(\kappa\,z_0)}{\kappa}\,e^{i\,\mathbf{k}\cdot\mathbf{z}}$$

Using spherical coordinates so that $\mathbf{z} = R\,\hat{z}$ is along the z-axis, and $\mathbf{k} = (\,\kappa,\,\theta,\,\phi\,)$, we have

$$\mathbf{k}\cdot\mathbf{z} = \kappa\,R\cos\theta$$
$$d^3\,k = \kappa^2\,d\,\kappa\,d\,\cos\theta\,d\,\phi$$

so that

$$\begin{aligned} D_r(z) &= \frac{\theta(z_0)}{(2\,\pi)^2} \int_0^\infty \kappa^2\,d\,\kappa \int_{-1}^{1} d\,\cos\theta\;\frac{\sin\,(\kappa\,z_0)}{\kappa}\,e^{i\,\kappa\,R\cos\theta} \\ &= \frac{\theta(z_0)}{(2\,\pi)^2} \int_0^\infty \kappa^2\,d\,\kappa\;\frac{\sin\,(\kappa\,z_0)}{i\,\kappa^2\,R}\left(e^{i\,\kappa\,R} - e^{-i\,\kappa\,R}\right) \\ &= \frac{\theta(z_0)}{2\,\pi^2\,R} \int_0^\infty d\,\kappa\,\sin\,(\,\kappa\,z_0)\,\sin(\,\kappa\,R\,) \qquad\qquad (12.130) \end{aligned}$$

Using

$$\sin(\kappa z_0)\sin(\kappa R) = -\frac{1}{4}\left(e^{i\kappa z_0} - e^{-i\kappa z_0}\right)\left(e^{i\kappa R} - e^{-i\kappa R}\right)$$

$$= -\frac{1}{4}\left[e^{i\kappa(z_0+R)} + e^{-i\kappa(z_0+R)} - e^{i\kappa(z_0-R)} - e^{-i\kappa(z_0-R)}\right]$$

$$= -\frac{1}{4}\left\{\left[e^{i\kappa(z_0+R)} - e^{i\kappa(z_0-R)}\right] + [\kappa \to -\kappa]\right\}$$

we have

$$\int_0^\infty d\kappa \sin(\kappa z_0)\sin(\kappa R) = -\frac{1}{4}\int_0^\infty d\kappa\left\{\left[e^{i\kappa(z_0+R)} - e^{i\kappa(z_0-R)}\right] + [\kappa \to -\kappa]\right\}$$

$$= -\frac{1}{4}\int_{-\infty}^{\infty} d\kappa\left[e^{i\kappa(z_0+R)} - e^{i\kappa(z_0-R)}\right]$$

Hence

$$D_r(z) = \frac{\theta(z_0)}{8\pi^2 R}\int_{-\infty}^{\infty} d\kappa\left[-e^{i\kappa(z_0+R)} + e^{i\kappa(z_0-R)}\right]$$

$$= \frac{\theta(z_0)}{4\pi R}[-\delta(z_0+R) + \delta(z_0-R)]$$

$$= \frac{\theta(z_0)}{4\pi R}\delta(z_0-R)$$

or

$$D_r(x-x') = \frac{\theta(x_0 - x_0')}{4\pi R}\delta(x_0 - x_0' - R) \qquad (12.131)$$

$$= \frac{\theta(x_0 - x_0')}{4\pi\,|\mathbf{x}-\mathbf{x}'|}\delta(x_0 - x_0' - |\mathbf{x}-\mathbf{x}'|)$$

Similarly, we have

$$D_a(x-x') = \frac{\theta(x_0' - x_0)}{4\pi R}\delta(x_0' - x_0 - R) \qquad (12.132)$$

Using

$$\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|}\delta(x-x_i)$$

we have

$$\delta\left[(x-x')^2\right] = \delta\left[(x_0-x_0')^2 - R^2\right]$$

$$= \frac{1}{2R}\delta(x_0 - x_0' - R) + \frac{1}{|-2R|}\delta(x_0 - x_0' + R)$$

$$= \frac{1}{2R}[\delta(x_0 - x_0' - R) + \delta(x_0 - x_0' + R)]$$

Since $R \geq 0$, we have

$$\delta(x_0 - x_0' \mp R) = 2R\,\theta[\pm(x_0 - x_0')]\,\delta\left[(x-x')^2\right]$$

The Green's functions can thus be written as:

$$D_r(x-x') = \frac{\theta(x_0 - x_0')}{2\pi}\,\delta\left[(x-x')^2\right]$$

$$D_a(x-x') = \frac{\theta(x_0' - x_0)}{2\pi}\,\delta\left[(x-x')^2\right] \qquad (12.133)$$

Now, $\theta(\pm z_0)\,\delta(z\cdot z)$ denotes the forward / backward light cone centered at $z=0$. Since the Lorentz transformation, by definition, leaves light cones invariant, eqs (12.133) are covariant despite the fact that step functions $\theta(\pm z_0)$ are not.

The solution to the inhomogeneous eq

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha$$

is simply

$$A^\alpha(x) = A_0^\alpha(x) + \frac{4\pi}{c}\int d^4x'\,D(x-x')\,J^\alpha(x')$$

where A_0 is a solution of the homogeneous ($J=0$) equation. The use of different Green functions corresponds to different boundary conditions.

For D_r, we have $x_0 > x_0'$ so that

$$\int d^4x'\,D_r(x-x')\,J^\alpha(x') = \int_{-\infty}^{x_0} d\,x_0' \int d^3x\,D_r(x-x')\,J^\alpha(x')$$

which vanishes at the distant past $x_0 = -\infty$. This means

$$A^\alpha(x) = A_0^\alpha(x) \qquad \text{at} \qquad x_0 = -\infty$$

In a scattering problem, the source J is localized in time and space. $A_0^\alpha(x)$ is then the incoming wave and we write

$$A^\alpha(x) = A_{\text{in}}^\alpha(x) + \frac{4\pi}{c}\int d^4x'\,D_r(x-x')\,J^\alpha(x') \qquad (12.134)$$

Similarly, we have

$$A^\alpha(x) = A_{\text{out}}^\alpha(x) + \frac{4\pi}{c}\int d^4x'\,D_a(x-x')\,J^\alpha(x') \qquad (12.135)$$

The radiation field $A_{\text{rad}}^\alpha(x)$ is defined as

$$A_{\text{rad}}^\alpha(x) = A_{\text{out}}^\alpha(x) - A_{\text{in}}^\alpha(x)$$

Hence

$$A_{\text{rad}}^\alpha(x) = \frac{4\pi}{c}\int d^4x'\,[D_r(x-x') - D_a(x-x')]\,J^\alpha(x')$$

$$= \frac{4\pi}{c}\int d^4x'\,\tilde{D}(x-x')\,J^\alpha(x') \qquad (12.136)$$

where

$$\tilde{D}(x-x') = D_r(x-x') - D_a(x-x') \qquad (12.137)$$

satisfies the homogeous ($J=0$) equation.

For a charged particle in motion. Let its charge be q, its position $\boldsymbol{r}(t)$, its velocity $\boldsymbol{v}(t) = \dfrac{d\,\boldsymbol{r}(t)}{d\,t}$, when seen in frame K. We have

$$\rho(x) = \rho(\,\boldsymbol{x},\ t\,) = q\,\delta[\,\boldsymbol{x} - \boldsymbol{r}(t)\,]$$

$$\boldsymbol{J}(x) = \boldsymbol{J}(\,\boldsymbol{x},\ t\,) = \boldsymbol{v}(t)\,\rho(\,\boldsymbol{x},\ t\,) = q\,\boldsymbol{v}(t)\,\delta[\,\boldsymbol{x} - \boldsymbol{r}(t)\,]$$

so that

$$J^\alpha(x) = J^\alpha(\,\boldsymbol{x},\ t\,) = (\,\rho\,c,\ \boldsymbol{J}\,)$$

$$= \quad q\,\delta[\,\boldsymbol{x} - \boldsymbol{r}(t)\,]\,[\,c,\ \boldsymbol{v}(t)\,]$$

$$= \quad \frac{q}{\gamma_v(t)}\,\delta[\,\boldsymbol{x} - \boldsymbol{r}(t)\,]\,u^\alpha(t)$$

where $u^\alpha = \gamma_v\,(t)\,[\,c,\ \boldsymbol{v}(t)\,]$ is the 4-velocity of the particle.

In this form, the 4-vector character of J^α is not obvious. We now proceed to make this characteristic explicit.

To begin, let us examine the 4-vector counterparts of the 3-space quantities involved.

$x^\alpha = (\,c\,t\,,\ \ \boldsymbol{x}\,)$ is the 4-position in frame K

$r^\alpha(t) = \left[\,r^0(t),\ \ \boldsymbol{r}(t)\,\right] = [\,c\,t,\ \ \boldsymbol{r}(t)\,]$ is that for the particle

In terms of the proper time τ, we have

$d\,t = \gamma_v[\tau(t)]\,d\,\tau$

where $\tau(t)$ is the proper time corresponding to t in frame K and $\gamma_v[\tau(t)]$ is γ_v at time t or proper time $\tau(t)$.

$$t = \int_0^{\tau(t)} d\,\tau'\,\gamma_v[\tau'\,(t)]$$

where for convenience, we've set $\tau(0) = 0$, ie., $\tau = 0$ at $t = 0$.

Hence

$$r^\alpha(\tau) = \left[\,c\int_0^\tau d\,\tau'\,\gamma_v(\tau'),\ \ \ \boldsymbol{r}(\tau)\,\right]$$

so that

$$u^\alpha(\tau) = \frac{d\,r^\alpha(\tau)}{d\,\tau} = \left[c\,\gamma_v(\tau),\ \ \frac{d\,\boldsymbol{r}(\boldsymbol{\tau})}{d\,\tau}\right] = \gamma_v\,(\,c,\ \boldsymbol{v}\,)$$

as it should be.

The 4-delta function generalized from $\delta^3[\,\boldsymbol{x} - \boldsymbol{r}(t)\,]$ is

$$\delta^4[\,x - r(\tau)\,] = \delta\left[\,c\,t - r^0(\tau)\,\right]\ \delta^3[\,\boldsymbol{x} - \boldsymbol{r}(\tau)\,]$$

$$= \delta\left[c\,t - c\int_0^{\tau} d\,\tau'\,\gamma_v(\tau') \right] \delta^3[\,\boldsymbol{x} - \boldsymbol{r}(\tau)\,]$$

$$= \frac{1}{c\,\gamma_v(\tau)}\,\delta[\,\tau - \tau(t)\,]\,\delta^3[\,\boldsymbol{x} - \boldsymbol{r}(\tau)\,]$$

$$= \frac{1}{c\,\gamma_v(t)}\,\delta[\,\tau - \tau(t)\,]\,\delta^3[\,\boldsymbol{x} - \boldsymbol{r}(t)\,]$$

where we have used the general functional relation

$$g[\,\tau(t)\,] \equiv g(t)$$

and $\tau(t)$ is the proper time that corresponds to t:

$$c\,t - c\int_0^{\tau(t)} d\,\tau'\,\gamma_v(\tau') = 0$$

Hence

$$c\int d\,\tau\;\delta^4[\,x - r(\tau)\,]\,f(\tau) = \frac{1}{\gamma_v(t)}\,\delta^3\,\{\,\boldsymbol{x} - \boldsymbol{r}(t)\,]\,f(t)$$

Setting $f = u^\alpha$, we have

$$J^\alpha(x) = \frac{q}{\gamma_v(t)}\,\delta^3[\,\boldsymbol{x} - \boldsymbol{r}(t)\,]\,u^\alpha(t)$$

$$= q\,c\int d\,\tau\;\delta^4[\,x - r(\tau)\,]\,u^\alpha(\tau) \qquad (12.139)$$