13
Collisions Between Charged Particles, Eneregy Loss and Scattering

1. Coulomb Collision

A particle of charge $z\,e$ and mass M passes by an electron of charge $-e$ and mass m in an atom (see Fig.13.1). We are interested in the energy transfer of such an encounter.

■ $\Delta\,\boldsymbol{E}(\boldsymbol{b})$

Approximations:
1. If $M \gg m$, the path of the particle will not be appreciably deflected.
2. If the velocity v of the particle is large enough, the duration of energy exchange will be short enough such that the recoil of the electron is negligible.

Assuming the particle travels with uniform speed v along the line $x^2 = b$. The $\boldsymbol{E}$ fields at the origin are

$$E^1 = -\frac{\gamma\, z\, e\, v\, t}{\left[(\gamma\, v\, t)^2 + b^2\right]^{3/2}}$$

$$E^2 = -\frac{\gamma\, z\, e\, b}{\left[(\gamma\, v\, t)^2 + b^2\right]^{3/2}}$$

$$E^3 = 0 \qquad\qquad [\text{ see (11.152) }]$$

Only the transverse component E_2 give rise to finite time integral or energy transfer. The impulse delivered to a charge $-e$ at $\boldsymbol{x}_0$ is

$$\begin{aligned}
\Delta\, p &= \int_{-\infty}^{\infty} d\,t\, \frac{d\,p_2}{d\,t} \\
&= \int_{-\infty}^{\infty} d\,t\,(-e)\,E_2(t) \\
&= \int_{-\infty}^{\infty} d\,t\, \frac{\gamma\, z\, e^2\, b}{\left[(\gamma\, v\, t)^2 + b^2\right]^{3/2}} \\
&= \int_{0}^{\infty} d\,t\, \frac{2\,\gamma\, z\, e^2\, b}{\left[(\gamma\, v\, t)^2 + b^2\right]^{3/2}}
\end{aligned}$$

Using

$$\int d x \frac{1}{\left(x^2 + a^2\right)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}} = \frac{1}{a^2 \sqrt{1 + \left(\frac{a}{x}\right)^2}}$$

we have

$$\Delta p = \frac{2 z e^2 b}{\gamma^2 v^3} \int_0^\infty d t \frac{1}{\left[t^2 + \left(\frac{b}{\gamma v}\right)^2\right]^{3/2}}$$

$$= \frac{2 z e^2 b}{\gamma^2 v^3} \cdot \left(\frac{\gamma v}{b}\right)^2$$

$$= \frac{2 z e^2}{b v} \qquad (13.1)$$

Assuming the deviation of the position of the electron away from the origin is negligible, the energy transfer is therefore

$$\Delta E(b) = \frac{(\Delta p)^2}{2 m} = \frac{2 z^2 e^4}{m v^2} \left(\frac{1}{b^2}\right) \qquad (13.2)$$

The singularity at $b = 0$ emphasizes the fact that (13.2) is valid only for large b.

■ θ

The angular deflection of the incident particle is

$$\theta \simeq \frac{\Delta P}{P}$$

provided $\Delta P \ll P$, where $P = \gamma M v$ is the momentum of the incident particle. From conservation of momentum, we have $\Delta P = -\Delta p$.

Hence, for small deflections

$$\theta \simeq \frac{2 z e^2}{b v P} \qquad (13.3)$$

For non-relativistic particles, the exact result is given by the Rutherford scattering:

$$2 \tan \frac{\theta}{2} = \frac{2 z e^2}{P v b} \qquad (13.4)$$

Since $2 \tan \frac{\theta}{2} \simeq \theta$ for small θ, (13.3) agrees with (13.4).

■ $b_{\min}$

The maximum allowable energy transfer occurs at head-on collisions [see Prob 11.23(b)],

$$\Delta E_{\max} \simeq 2 m \gamma^2 v^2 \qquad (13.5)$$

which may be used to define $b_{\min}$ below which (13.2) is invalid. Thus

$$\frac{2 z^2 e^4}{m v^2} \left(\frac{1}{b_{\min}{}^2}\right) = 2 m \gamma^2 v^2$$

or

$$b_{\min} = \frac{z e^2}{\gamma m v^2} \qquad (13.6)$$

As discussed in Prob 13.1, an improvement over (13.2) is

$$\Delta E(b) = \frac{2 z^2 e^4}{m v^2} \left(\frac{1}{b^2 + b_{min}^2} \right) \qquad (13.7)$$

Another way to estimate b_{min} is to set it to the distance d travelled by the electron during the collision.

$$b_{min} \simeq d \simeq \overline{v} \Delta t$$

where $\overline{v}$ is the average velocity of the electron and Δt the collision time.

Assuming the electron to be at rest initially, we have

$$\overline{v} \simeq \frac{1}{2} \left(\frac{\Delta p}{m} \right) = \frac{z e^2}{m b v}$$

The collision time is

$$\Delta t \simeq \frac{b}{\gamma v} \qquad (11.153)$$

Hence

$$b_{min} \simeq d \simeq \frac{z e^2}{\gamma m v^2} \qquad (13.6)$$

- **b_{max}**

Since the collision time Δt is directly proportional to the impact parameter b, it will become comparable with the period $\tau = \frac{1}{\omega}$ of the electron orbit around the atom for large enough b. Now, the direction of the energy transfer depends on the direction of the transverse motion of the electron. The net energy transfer will therefore be much smaller than that given by (13.2) if the electron can complete a full period of orbital motion.

We thus set

$$b_{max} \simeq \gamma v \tau = \frac{\gamma v}{\omega} \qquad (13.9)$$

above which $\Delta E \simeq 0$.

- **$\frac{dE}{dx}$**

Let

N = density of atom

Z = number of electrons per atom

The number of electrons in a thickness $d x$ located between b and $b + d b$ away from the path of the incident particle is

$$d n = 2 \pi N Z b \, d b \, d x$$

The energy loss per unit length is therefore

$$\frac{dE}{dx} = \int \Delta E \frac{dn}{dx} = 2\pi N Z \int \Delta E\, b\, db \qquad (13.11)$$

Using (13.2), we have

$$\frac{dE}{dx} = 4\pi N Z \frac{z^2 e^4}{m v^2} \int_{b_{\min}}^{b_{\max}} \frac{1}{b} db \qquad (13.12)$$

$$= 4\pi N Z \frac{z^2 e^4}{m v^2} \ln \frac{b_{\max}}{b_{\min}}$$

$$= 4\pi N Z \frac{z^2 e^4}{m v^2} \ln B \qquad (13.13)$$

where

$$B = \frac{b_{\max}}{b_{\min}} \simeq \frac{\gamma v}{\omega} \Big/ \frac{z e^2}{\gamma m v^2} = \frac{\gamma^2 m v^3}{z e^2 \omega} \qquad (13.14)$$

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2. Harmonically Bounded Electron

Purpose:
More accurate estimate of $b_{\max}$.
(for $b > b_{\max}$, $\Delta E \to 0$ much faster than $\frac{1}{b^2}$)

Simplification:

1. Electron harmonically bounded with amplitude $|\boldsymbol{x}|$.
2. ΔE small.
3. $|\boldsymbol{x}| \ll b$.
4. $\boldsymbol{E}$ spatially constant throughout electron orbit & hence replaced by that at nucleus (dipole approximation).

Equation of motion for electron:

$$\ddot{\boldsymbol{x}} + \Gamma \dot{\boldsymbol{x}} + \omega_0^2 \boldsymbol{x} = -\frac{e}{m} \boldsymbol{E} \qquad (13.15)$$

where Γ is a damping constant, ω_0 the binding frequency in the absence of $\boldsymbol{E}$.

Let

$$\boldsymbol{x}(t) = \frac{1}{2\pi} \int d\omega\, e^{-i\omega t}\, \boldsymbol{x}(\omega) \qquad (13.16)$$

$$\boldsymbol{E}(t) = \frac{1}{2\pi} \int d\omega\, e^{-i\omega t}\, \boldsymbol{E}(\omega) \qquad (13.17)$$

Note that Jackson used the factor $\frac{1}{\sqrt{2\pi}}$.

All $f(t)$ are of course the same in both conventions. However, $F_{\text{Jackson}}(\omega) = \frac{1}{\sqrt{2\pi}} F_{\text{our}}(\omega)$.

Reality of $\boldsymbol{x}(t)$ and $\boldsymbol{E}(t)$ requires

$$\boldsymbol{x}(-\omega) = \boldsymbol{x}^*(\omega) \qquad \boldsymbol{E}(-\omega) = \boldsymbol{E}^*(\omega) \qquad (13.18)$$

(13.15) becomes

$$\frac{1}{2\pi}\int d\omega\; e^{-i\omega t}\left[-\omega^2 - i\,\omega\,\Gamma + \omega_0^{\,2}\right]\boldsymbol{x}(\omega) = -\frac{e}{m}\,\frac{1}{2\pi}\int d\omega\; e^{-i\omega t}\,\boldsymbol{E}(\omega)$$

which implies

$$\left[-\omega^2 - i\,\omega\,\Gamma + \omega_0^{\,2}\right]\boldsymbol{x}(\omega) = -\frac{e}{m}\,\boldsymbol{E}(\omega)$$

$$\boldsymbol{x}(\omega) = -\frac{e}{m\left[-\omega^2 - i\,\omega\,\Gamma + \omega_0^{\,2}\right]}\,\boldsymbol{E}(\omega) \qquad (13.19)$$

$$= -\frac{e}{m}\,\frac{\omega_0^{\,2} - \omega^2 + i\,\omega\,\Gamma}{\left(\omega_0^{\,2} - \omega^2\right)^2 + (\omega\,\Gamma)^2}\,\boldsymbol{E}(\omega)$$

Rate of work done on electron:

$$\frac{dE}{dt} = \int d^3x'\,\boldsymbol{E}\cdot\boldsymbol{J}$$

Energy transfer:

$$\Delta E = \int_{-\infty}^{\infty} dt\,\frac{dE}{dt}$$

$$= \int_{-\infty}^{\infty} dt\int d^3x'\,\boldsymbol{E}\cdot\boldsymbol{J}$$

$$= \int_{-\infty}^{\infty} dt\int d^3x'\,\boldsymbol{E}\cdot\{-e\,\boldsymbol{v}\,\delta[\,\boldsymbol{x}'-\boldsymbol{x}(t)\,]\}$$

$$= -e\int_{-\infty}^{\infty} dt\;\boldsymbol{E}\cdot\boldsymbol{v} \qquad [\,\boldsymbol{v} = \dot{\boldsymbol{x}}(t)\,]$$

Now

$$\boldsymbol{E}(t)\cdot\boldsymbol{v}(t) = \int\frac{d\omega}{2\pi}\,e^{-i\omega t}\,\boldsymbol{E}(\omega)\cdot\int\frac{d\omega'}{2\pi}\,e^{-i\omega' t}\,(-i\,\omega')\,\boldsymbol{x}(\omega')$$

$$= \int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}\,e^{-i(\omega+\omega')t}\,(-i\,\omega')\,\boldsymbol{E}(\omega)\cdot\boldsymbol{x}(\omega')$$

Hence

$$\Delta E = -e\int\frac{d\omega}{2\pi}\int\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty} dt\;e^{-i(\omega+\omega')t}\,(-i\,\omega')\,\boldsymbol{E}(\omega)\cdot\boldsymbol{x}(\omega')$$

$$= -e\int\frac{d\omega}{2\pi}\int d\omega'\,\delta(\omega+\omega')\,(-i\,\omega')\,\boldsymbol{E}(\omega)\cdot\boldsymbol{x}(\omega')$$

$$= \; -i\, e \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, \omega\, \boldsymbol{E}(\omega)\cdot \boldsymbol{x}(-\omega)$$

$$= \; -i\, e\left[\int_{-\infty}^{0} + \int_{0}^{\infty}\right] \frac{d\,\omega}{2\,\pi}\, \omega\, \boldsymbol{E}(\omega)\cdot \boldsymbol{x}(-\omega)$$

$$= \; -i\, e\left\{-\int_{\infty}^{0} \frac{d\,\omega}{2\,\pi}\,(-\omega)\, \boldsymbol{E}(-\omega)\cdot \boldsymbol{x}(\omega) \; + \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \omega\, \boldsymbol{E}(\omega)\cdot \boldsymbol{x}(-\omega)\right\}$$

$$= \; -i\, e \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \{-\omega\, \boldsymbol{E}(-\omega)\cdot \boldsymbol{x}(\omega) + \omega\, \boldsymbol{E}(\omega)\cdot \boldsymbol{x}(-\omega)\}$$

$$= \; e \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \omega\, \{i\, \boldsymbol{E}^{*}(\omega)\cdot \boldsymbol{x}(\omega) - i\, \boldsymbol{E}(\omega)\cdot \boldsymbol{x}^{*}(\omega)\}$$

Let

$$z = i\, \boldsymbol{E}^{*}(\omega)\cdot \boldsymbol{x}(\omega) = a + i\, b$$

we have

$$i\, \boldsymbol{E}^{*}(\omega)\cdot \boldsymbol{x}(\omega) - i\, \boldsymbol{E}(\omega)\cdot \boldsymbol{x}^{*}(\omega)\,\} = z + z^{*} = 2\, a = 2\,\mathrm{Re}\,[i\, \boldsymbol{E}^{*}(\omega)\cdot \boldsymbol{x}(\omega)\,]$$

Hence

$$\Delta E = 2\, e \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \omega\; \mathrm{Re}\,[i\, \boldsymbol{E}^{*}(\omega)\cdot \boldsymbol{x}(\omega)\,]$$

$$= \; 2\, e\; \mathrm{Re} \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, i\, \omega\; \boldsymbol{x}(\omega)\cdot \boldsymbol{E}^{*}(\omega) \qquad (13.23)$$

$$= \; -2\, e\; \mathrm{Im} \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \omega\; \boldsymbol{x}(\omega)\cdot \boldsymbol{E}^{*}(\omega)$$

Using (13.19), we have

$$\Delta E = -2\, e\; \mathrm{Im} \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \omega\; \frac{-e}{m\left[\,\omega_0^{2} - \omega^{2} \,-\, i\,\omega\,\Gamma\,\right]}\, \boldsymbol{E}(\omega)\cdot \boldsymbol{E}^{*}(\omega)$$

$$= \; 2\,\frac{e^{2}}{m} \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \omega\; \mathrm{Im}\left[\frac{1}{\omega_0^{2} - \omega^{2} \,-\, i\,\omega\,\Gamma}\right] \left|\, \boldsymbol{E}(\omega)\,\right|^{2}$$

$$= \; 2\,\frac{e^{2}}{m} \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \omega\; \left[\frac{\omega\,\Gamma}{\left(\omega_0^{2} - \omega^{2}\right)^{2} \,+\, (\omega\,\Gamma)^{2}}\right] \left|\, \boldsymbol{E}(\omega)\,\right|^{2}$$

The Lorentzian

$$\frac{1}{\left(\omega_0{}^2-\omega^2\right)^2 + (\omega\,\Gamma)^2}$$

is peaked at $\omega = \sqrt{\omega_0{}^2 - \frac{1}{2}\Gamma^2}$ with half-width Γ. Thus, for small damping Γ, we can replace $\boldsymbol{E}(\omega)$ with $\boldsymbol{E}(\omega_0)$ so that

$$\Delta E = 2\,\frac{e^2}{m}\;|\,\boldsymbol{E}(\omega_0)\,|^2\quad I \qquad (13.25)$$

where

$$I = \int_0^\infty \frac{d\,\omega}{2\,\pi}\;\frac{\omega^2\,\Gamma}{\left(\omega_0{}^2-\omega^2\right)^2 + (\omega\,\Gamma)^2}$$

Setting

$$x = \frac{\omega}{\Gamma}$$

we have

$$\begin{aligned} I &= \int_0^\infty \frac{d\,x}{2\,\pi}\,\Gamma\;\frac{x^2\,\Gamma^3}{\left[\omega_0{}^2-(x\,\Gamma)^2\right]^2 + \left(x\,\Gamma^2\right)^2} \\ &= \int_0^\infty \frac{d\,x}{2\,\pi}\;\frac{x^2}{\left[\left(\frac{\omega_0}{\Gamma}\right)^2 - x^2\right]^2 + x^2} \qquad (13.25) \\ &= \int_0^\infty \frac{d\,x}{2\,\pi}\;\frac{1}{\left(\frac{a}{x}-x\right)^2 + 1} \qquad a = \left(\frac{\omega_0}{\Gamma}\right)^2 \\ &= \int_{-\infty}^\infty \frac{d\,x}{4\,\pi}\;\frac{1}{\left(\frac{a}{x}-x\right)^2 + 1} \end{aligned}$$

The poles are at

$$\left(\frac{a}{x}-x\right)^2 = -\,1$$

or

$$\frac{a}{x} - x = \;\pm i$$

ie

$$x^2 \pm i\,x - a = 0$$

so that there're 4 poles at

$$x = \frac{1}{2}\left[\mp\, i \pm \sqrt{-1+4\,a}\,\right]$$

Closing the contour in the upper plane, the poles enclosed are, for $4\,a > 1$,

$$x_\pm = \frac{1}{2}\left[\, i \pm \sqrt{-1+4\,a}\,\right]$$

which obey

$$\frac{a}{x_\pm} - x_\pm = \; -i$$

so that

$$\begin{aligned}
I = \frac{1}{2}\,i \sum_{x=x_\pm} \text{Res}\, \frac{1}{\left(\frac{a}{x}-x\right)^2 + 1} \\
&= \frac{1}{2}\,i \sum_{x=x_\pm} \frac{1}{2\left(\frac{a}{x}-x\right)\left(-\frac{a}{x^2}-1\right)} \\
&= \frac{1}{4} \sum_{x=x_\pm} \frac{1}{\frac{a}{x^2}+1} \\
&= \frac{1}{4} \sum_{x=x_\pm} \frac{x}{\frac{a}{x}+x} \\
&= \frac{1}{4} \sum_{x=x_\pm} \frac{x}{2\,x - i} \\
&= \frac{1}{4} \sum \frac{x_\pm}{\pm\sqrt{-1+4\,a}} \\
&= \frac{1}{8}\left\{ \frac{i+\sqrt{-1+4\,a}}{\sqrt{-1+4\,a}} + \frac{i-\sqrt{-1+4\,a}}{-\sqrt{-1+4\,a}} \right\} \\
&= \frac{1}{4}
\end{aligned}$$

Note: the above manipulation is just repeated use of the two relations involving $x_\pm$ to avoid tedious arithmatics. If you got lost, plug in the values of $x_\pm$ from the start & simplify.

Hence

$$\Delta\,E = \frac{e^2}{2\,m}\,|\,\boldsymbol{E}(\omega_0)\,|^2 \qquad (13.26)$$

Given

$$E_2(t) = -\frac{\gamma\; z\,e\,b}{\left[\,(\gamma\,v\,t)^2 + b^2\,\right]^{3/2}}$$

we have

$$\begin{aligned}
E_2(\omega) = \int_{-\infty}^{\infty} d\,t\,e^{i\,\omega\,t}\,E_2(t) \\
&= -\gamma\; z\,e\,b \int_{-\infty}^{\infty} d\,t\; \frac{e^{i\,\omega\,t}}{\left[\,(\gamma\,v\,t)^2 + b^2\,\right]^{3/2}} \qquad (13.27) \\
&= -\frac{\gamma\; z\,e}{b^2} \int_{-\infty}^{\infty} d\,t\; \frac{e^{i\,\omega\,t}}{\left[\left(\frac{\gamma\,v\,t}{b}\right)^2 + 1\right]^{3/2}}
\end{aligned}$$

Setting

$$x = \gamma v t / b \qquad t = b x / \gamma v$$

we have

$$E_2(\omega) = -\frac{z e}{b v} \int_{-\infty}^{\infty} d x \, \frac{e^{i \omega b x / \gamma v}}{\left(x^2 + 1\right)^{3/2}} \qquad (13.28)$$

From Bateman Vol I, p.11 & p.66,

$$\int_0^{\infty} d x \, \frac{\cos \xi x}{\left(x^2 + a^2\right)^{\nu + \frac{1}{2}}} = \frac{\sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{\xi}{2 a}\right)^{\nu} K_\nu(a \xi)$$

$$\int_0^{\infty} d x \, \frac{x \sin \xi x}{\left(x^2 + a^2\right)^{3/2}} = \xi K_0(a \xi)$$

where

$$\xi > 0 \qquad \mathrm{Re}\, a > 0 \qquad \mathrm{Re}\, \nu > -\frac{1}{2}$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} d x \, \frac{e^{i \xi x}}{\left(x^2 + a^2\right)^{\nu + \frac{1}{2}}} &= \int_0^{\infty} d x \, \frac{e^{i \xi x} + e^{-i \xi x}}{\left(x^2 + a^2\right)^{\nu + \frac{1}{2}}} \\ &= 2 \int_0^{\infty} d x \, \frac{\cos \xi x}{\left(x^2 + a^2\right)^{\nu + \frac{1}{2}}} \\ &= 2 \frac{\sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{\xi}{2 a}\right)^{\nu} K_\nu(a \xi) \end{aligned}$$

In particular

$$\begin{aligned} \int_{-\infty}^{\infty} d x \, \frac{e^{i \xi x}}{\left(x^2 + 1\right)^{3/2}} &= \frac{2 \sqrt{\pi}}{\Gamma\left(\frac{3}{2}\right)} \left(\frac{\xi}{2}\right) K_1(\xi) \\ &= 2 \xi K_1(\xi) \end{aligned}$$

Similarly

$$\begin{aligned} \int_{-\infty}^{\infty} d x \, \frac{x e^{i \xi x}}{\left(x^2 + 1\right)^{3/2}} &= \int_0^{\infty} d x \, \frac{x\left(e^{i \xi x} - e^{-i \xi x}\right)}{\left(x^2 + 1\right)^{3/2}} \\ &= 2 i \int_0^{\infty} d x \, \frac{x \sin \xi x}{\left(x^2 + 1\right)^{3/2}} \\ &= 2 i \xi K_0(\xi) \end{aligned}$$

Hence

$$\begin{aligned} E_2(\omega) &= -\frac{z e}{b v} 2 \frac{\omega b}{\gamma v} K_1\left(\frac{\omega b}{\gamma v}\right) \qquad (13.29) \\ &= -\frac{z e}{b v} 2 \xi K_1(\xi) \end{aligned}$$

where

$$\xi = \frac{\omega b}{\gamma v} = \frac{b}{b_{\max}} \qquad (13.32)$$

Similarly, from

$$E_1(t) = -\frac{\gamma\, z\, e\, v\, t}{\left[(\gamma v t)^2 + b^2\right]^{3/2}}$$

we have

$$E_1(\omega) = \frac{z e}{b v \gamma} 2 i \frac{\omega b}{\gamma v} K_0\left(\frac{\omega b}{\gamma v}\right)$$

$$= \frac{z e}{b v \gamma} 2 i \xi K_0(\xi) \qquad (13.30)$$

Hence

$$|\boldsymbol{E}(\omega_0)|^2 = |E_1(\omega_0)|^2 + |E_2(\omega_0)|^2$$

$$= 4\left(\frac{z e}{b v}\right)^2 \xi^2 \left[\frac{1}{\gamma^2} K_0{}^2(\xi) + K_1{}^2(\xi)\right]$$

$$\Delta E(b) = \frac{2}{m}\left(\frac{z e^2}{b v}\right)^2 \xi^2 \left[\frac{1}{\gamma^2} K_0{}^2(\xi) + K_1{}^2(\xi)\right]$$

$$= \frac{2 z^2 e^4}{m v^2}\left(\frac{1}{b^{*2}}\right)$$

where

$$\frac{1}{b^{*2}} = \left(\frac{\xi}{b}\right)^2 \left[\frac{1}{\gamma^2} K_0{}^2(\xi) + K_1{}^2(\xi)\right]$$

$$= \frac{1}{b_{\max}{}^2}\left[\frac{1}{\gamma^2} K_0{}^2\left(\frac{b}{b_{\max}}\right) + K_1{}^2\left(\frac{b}{b_{\max}}\right)\right]$$

For b, $\xi \to 0$,

$$K_0(\xi) \to -\left[\ln\frac{\xi}{2} + 0.5772 + \ldots\right] \simeq \ln\left(\frac{1.123}{\xi}\right)$$

$$\xi K_0(\xi) \to -\xi \ln\frac{\xi}{2} \to 0$$

$$K_1(\xi) \to \frac{1}{\xi} \qquad \xi K_1(\xi) \to 1$$

Hence

$$\frac{1}{b^{*2}} \to \frac{1}{b^2}$$

$$\Delta E(b) \to \frac{2 z^2 e^4}{m v^2}\left(\frac{1}{b^2}\right)$$

For b, $\xi \to \infty$

$$K_0(\xi) \approx K_1(\xi) \approx \sqrt{\frac{\pi}{2\xi}}\, e^{-\xi}$$

$$\frac{1}{b^{*2}} \to \frac{\xi}{b^2}\frac{\pi}{2} e^{-2\xi}\left(\frac{1}{\gamma^2} + 1\right)$$

$$\Delta E(b) \to \frac{2 z^2 e^4}{m v^2}\left(\frac{1}{b^2}\right)\xi \frac{\pi}{2} e^{-2\xi}\left(\frac{1}{\gamma^2} + 1\right)$$

which justifies the assignment of $b_{\max}$ as discussed in the last section.

3. Energy Loss Formulae

Classical Formula

Consider an atom of Z electrons.
In a semi- classical description, its states are (effectively) described as having f_j electrons in orbits of frequency ω_j. To agree better with experiment, the ocillator strengths f_j are allowed to become fractional numbers. However, as may be justified by a quantum mechanical treatment, the sum rule

$$\sum_j f_j = \text{Z}$$

is still valid.

This means an energy loss of

$$\frac{d E}{d x} = 2 \pi N \sum_j f_j \int_{b_{\min}}^{\infty} d b \, b \, \Delta E_j \qquad (13.34)$$

with

$$\Delta E_j = \frac{2 z^2 e^4}{m v^2} \left(\frac{1}{b_j^{*2}} \right)$$

$$\frac{1}{b_j^{*2}} = \left(\frac{\xi_j}{b} \right)^2 \left[\frac{1}{\gamma^2} K_0^2(\xi_j) + K_1^2(\xi_j) \right]$$

$$= \left(\frac{1}{b_{\max}{}^j} \right)^2 \left[\frac{1}{\gamma^2} K_0^2(\xi_j) + K_1^2(\xi_j) \right]$$

$$\xi_j = \frac{\omega_j b}{\gamma v} = \frac{b}{b_{\max}{}^j}$$

Setting

$$\xi_{\min}{}^j = \frac{b_{\min}}{b_{\max}{}^j} \qquad \text{where} \quad b_{\min} = \frac{z e^2}{\gamma m v^2}$$

we have

$$\frac{d E}{d x} = 2 \pi N \frac{2 z^2 e^4}{m v^2} \sum_j f_j \left(b_{\max}{}^j \right)^2 \int_{\xi_{\min}{}^j}^{\infty} d \xi_j \; \xi_j \; \Delta E_j$$

$$= \; 4 \pi N \frac{z^2 e^4}{m v^2} \sum_j f_j \int_{\xi_{\min}{}^j}^{\infty} d \xi_j \; \xi_j \left[\frac{1}{\gamma^2} K_0^2(\xi_j) + K_1^2(\xi_j) \right]$$

Now (see Integrals of K)

$$I = \int_{\xi_{\min}}^{\infty} d\,\xi\,\xi\left(\frac{1}{\gamma^2}K_0{}^2 + K_1{}^2\right)$$

$$= -\frac{1}{2}\left\{\left(1-\beta^2\right)\xi^2\left(K_1{}^2 - K_0{}^2\right) + 2\,\xi\,K_0\,K_1 + \xi^2\left(K_0{}^2 - K_1{}^2\right)\right\}_{\xi_{\min}}^{\infty}$$

$$= -\left\{\xi\,K_0\,K_1 - \frac{1}{2}\beta^2\,\xi^2\left(K_1{}^2 - K_0{}^2\right)\right\}_{\xi_{\min}}^{\infty}$$

$$= \xi_{\min}\,K_0\,(\xi_{\min})\,K_1\,(\xi_{\min}) - \frac{1}{2}\beta^2\,\xi_{\min}{}^2\left[K_1\,(\xi_{\min})^2 - K_0\,(\xi_{\min})^2\right]$$

Using

$$K_2 = K_0 + \frac{2}{z}K_1$$

we have

$$\int^{z} d\,t\,t\,K_1(t)^2 = \frac{z}{2}\left[K_2(\,z\,K_2 - 2\,K_1\,) - z\,K_1{}^2\right]$$

$$= \frac{z}{2}\left[z\,K_2\,K_0 - z\,K_1{}^2\right]$$

$$= \frac{z^2}{2}\left[\left(K_0 + \frac{2}{z}K_1\right)K_0 - K_1{}^2\right]$$

$$= \frac{z^2}{2}\left[K_0{}^2 + \frac{2}{z}K_1\,K_0 - K_1{}^2\right]$$

$$= \frac{z}{2}\left[2\,K_0\,K_1 + z\left(K_0{}^2 - K_1{}^2\right)\right]$$

Therefore

$$\frac{d\,E}{d\,x} = 4\,\pi\,N\,\frac{z^2\,e^4}{m\,v^2}\sum_j f_j\left\{\xi\,K_0\,K_1 - \frac{v^2}{2\,c^2}\xi^2\left(K_1{}^2 - K_0{}^2\right)\right\}_{\xi=\xi_{\min}{}^j} \tag{13.35}$$

In general

$$\xi_{\min}{}^j = \frac{b_{\min}}{b_{\max}{}^j} \ll 1$$

Using the small ξ expansion

$$K_0(\xi) \to -\left[\ln\frac{\xi}{2} + 0.5772 + \ldots\right] \approx \ln\left(\frac{1.123}{\xi}\right)$$

$$K_1(\xi) \to \frac{1}{\xi}$$

where

$$e^{0.5772} = 1.781 = \frac{2}{1.123}$$

we have

$$\left\{ \xi K_0 K_1 - \frac{v^2}{2 c^2} \xi^2 \left(K_1{}^2 - K_0{}^2 \right) \right\}_{\xi = \xi_{\min}{}^j}$$

$$\approx \ln\left(\frac{1.123}{\xi_{\min}{}^j} \right) - \frac{v^2}{2 c^2}$$

so that

$$\frac{d E}{d x} \approx 4 \pi N \frac{z^2 e^4}{m v^2} \sum_j f_j \left[\ln\left(\frac{1.123}{\xi_{\min}{}^j} \right) - \frac{v^2}{2 c^2} \right]$$

$$= 4 \pi N \frac{z^2 e^4}{m v^2} \sum_j f_j \left[\ln\left(\frac{1.123 \gamma v}{\omega_j b_{\min}} \right) - \frac{v^2}{2 c^2} \right]$$

Setting

$$\sum_j f_j \ln \omega_j = Z \ln \langle \omega \rangle \qquad (13.38)$$

we have

$$\frac{d E}{d x} \approx 4 \pi N Z \frac{z^2 e^4}{m v^2} \left[\ln\left(\frac{1.123 \gamma v}{\langle \omega \rangle b_{\min}} \right) - \frac{v^2}{2 c^2} \right]$$

$$\approx 4 \pi N Z \frac{z^2 e^4}{m v^2} \left[\ln B_c - \frac{v^2}{2 c^2} \right] \qquad (13.36)$$

where

$$B_c = \frac{1.123 \gamma v}{\langle \omega \rangle b_{\min}} = \frac{1.123 \gamma^2 m v^3}{\langle \omega \rangle z e^2} \qquad (13.37)$$

Eq(13.36) is good for slow α particles or heavy nuclei, but overestimates considerably for lighter or fast particles.

■ Integrals of *K*

See Abramowitz & Stegun

Some useful relations are:

$$K_\nu(z)^* = K_\nu(z^*) K_{-\nu}(z) = -K_\nu(z)$$

$$-K_{\nu-1}(z) + K_{\nu+1}(z) = \frac{2 \nu}{z} K_\nu(z)$$

$$K_\nu{}'(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z)$$

$$= -K_{\nu+1}(z) + \frac{\nu}{z} K_\nu(z)$$

Let

$$\mathbb{L}_\nu = e^{\nu \pi i} K_\nu$$

we have

$$\mathbb{L}_{\nu-1}(z) - \mathbb{L}_{\nu+1}(z) = \frac{2 \nu}{z} \mathbb{L}_\nu(z)$$

$$\mathbb{L}_\nu{}'(z) = \mathbb{L}_{\nu-1}(z) - \frac{\nu}{z} \mathbb{L}_\nu(z)$$

Specializing to K_0 & K_1, we have

$$K_{0,1}(\xi)^* = K_{0,1}(\xi^*)$$

$$K_{-1}(\xi) = -K_1(\xi)$$

$$K_0 - K_2 = -\frac{2}{\xi} K_1 \qquad \text{or} \qquad K_2 = K_0 + \frac{2}{\xi} K_1$$

$$-K_1{}' = K_0 + \frac{1}{\xi} K_1$$

$$K_0{}' = -K_{-1} = -K_1$$

$$K_2{}' = -K_1 - \frac{2}{\xi} K_2$$

$$= -K_1 - \frac{2}{\xi} K_0 - \frac{4}{\xi^2} K_1$$

$$= -\left(1 + \frac{4}{\xi^2}\right) K_1 - \frac{2}{\xi} K_0$$

For any cylinder functions $\mathbb{C}$ and $\mathbb{D}$,

$$\int^z d\,t \left[\left(k^2 - l^2 \right) t - \left(\mu^2 - \nu^2 \right) \frac{1}{t} \right] \mathbb{C}_\mu\,(k\,t)\, \mathbb{D}_\nu(l\,t)$$

$$= z \left\{ k\, \mathbb{C}_{\mu+1}(k\,z)\, \mathbb{D}_\nu(l\,z) - l\, \mathbb{C}_\mu(k\,z)\, \mathbb{D}_{\nu+1}(l\,z) \right\} - (\mu - \nu)\, \mathbb{C}_\mu(k\,z)\, \mathbb{D}_\nu(l\,z)$$

For $\mu = \nu$,

$$\left(k^2 - l^2 \right) \int^z d\,t\;\; t\, \mathbb{C}_\mu(k\,t)\, \mathbb{D}_\mu(l\,t)$$

$$= z \left\{ k\, \mathbb{C}_{\mu+1}(k\,z)\, \mathbb{D}_\mu(l\,z) - l\, \mathbb{C}_\mu(k\,z)\, \mathbb{D}_{\mu+1}(l\,z) \right\}$$

or

$$\int^z d\,t\;\; t\, \mathbb{C}_\mu(k\,t)\, \mathbb{D}_\mu(l\,t) = \frac{z}{k^2 - l^2} \left\{ k\, \mathbb{C}_{\mu+1}(k\,z)\, \mathbb{D}_\mu(l\,z) - l\, \mathbb{C}_\mu(k\,z)\, \mathbb{D}_{\mu+1}\,(l\,z) \right\}$$

To evaluate

$$I = \int^z d\,t\,t\, K_\mu(t)^2$$

we start with

$$I\,' = \int^z d\,t\;\; t\, K_\mu(k\,t)\, K_\mu(t)$$

$$= \frac{z}{k^2 - 1} \left\{ k\, K_{\mu+1}(k\,z)\, K_\mu(z) - K_\mu(k\,z)\, K_{\mu+1}(z) \right\}$$

and set

$$k = 1 + \epsilon$$

so that

$$I = \lim_{\epsilon \to 0} I\,'$$

Thus

$$k^2 - 1 \to 2\epsilon$$

$$K_\mu(k\,z) = K_\mu(\,z + \epsilon\,z\,) \to K_\mu(z) + \epsilon\,z\,K_\mu{}'(z)$$

so that

$$I \underset{\epsilon\to 0}{=} \frac{z}{2\epsilon}\left\{ (1+\epsilon)\left[K_{\mu+1}(z) + \epsilon\,z\,K_{\mu+1}{}'(z) \right] K_\mu(z) - \left[K_\mu(z) + \epsilon\,z\,K_\mu{}'(z) \right] K_{\mu+1}(z) \right\}$$

$$\underset{\epsilon\to 0}{=} \frac{z}{2}\left[K_{\mu+1}\,K_\mu + z\left(K_{\mu+1}{}'\,K_\mu - K_\mu{}'\,K_{\mu+1}\right) \right] + O(\epsilon)$$

Using

$$K_\nu{}'(z) = -K_{\nu-1}(z) - \frac{\nu}{z}\,K_\nu(z)$$

$$= -K_{\nu+1}(z) + \frac{\nu}{z}\,K_\nu(z)$$

we have

$$K_{\mu+1}{}'\,K_\mu - K_\mu{}'\,K_{\mu+1} = -\left[K_\mu + \frac{\mu+1}{z}\,K_{\mu+1}\right] K_\mu + \left[K_{\mu+1} - \frac{\mu}{z}\,K_\mu\right] K_{\mu+1}$$

$$= -K_\mu{}^2 + K_{\mu+1}{}^2 - \frac{2\mu+1}{z}\,K_\mu\,K_{\mu+1}$$

Thus

$$\int^z d\,t\,t\,K_\mu(t)^2 = \frac{z}{2}\left[z\left(K_{\mu+1}{}^2 - K_\mu{}^2\right) - 2\,\mu\,K_{\mu+1}\,K_\mu \right]$$

so that

$$\int^z d\,t\,t\,K_0(t)^2 = \frac{z^2}{2}\left(K_1{}^2 - K_0{}^2\right)$$

$$\int^z d\,t\,t\,K_1(t)^2 = \frac{z}{2}\left[z\left(K_2{}^2 - K_1{}^2\right) - 2\,K_2\,K_1 \right]$$

■ Quantum Effects

■ Discreteness of atomic levels

Atomic energy levels are quantized, which means

$$\Delta E > \hbar\,\omega_0$$

where $\hbar\,\omega_0$ is the 1st excitation energy.

Using

$$v_0 = \frac{e^2}{\hbar} = \frac{c}{137} \qquad \alpha = \frac{e^2}{\hbar\,c} = \frac{1}{137}$$

$$I_H = \frac{m\,e^4}{2\,\hbar^2} \simeq 13.6\ \text{eV}$$

we have

$$e^2 = \hbar\,v_0$$

$$m = \frac{2\,\hbar^2\,I_H}{e^4} = \frac{2\,I_H}{v_0{}^2}$$

The minimum energy transfer given by (13.2) is

$$\Delta E(b_{\max}) = \frac{2 z^2 e^4}{m v^2}\left(\frac{1}{b_{\max}{}^2}\right) \qquad b_{\max} = \frac{\gamma v}{\omega_0}$$

$$= \frac{2 z^2 e^4 \omega_0{}^2}{m v^4 \gamma^2}$$

$$= \frac{2 z^2 \hbar^2 v_0{}^2 \omega_0{}^2}{\frac{2 I_H}{v_0{}^2} v^4 \gamma^2}$$

$$= \left(\frac{z}{\gamma}\right)^2 \left(\frac{v_0}{v}\right)^4 \hbar \omega_0 \left(\frac{\hbar \omega_0}{I_H}\right) \qquad (13.39)$$

Since $\hbar \omega_0$ is of the order of I_H, ΔE can be smaller than $\hbar \omega_0$ if v is large enough. However, (13.39) is still correct in a statistical sense.

- **Uncertainty principle**

Due to the wave- particle duality, the trajectory of a particle travelling with momentum p can be defined only with an uncertainty of $\Delta x \gtrsim \frac{\hbar}{p}$. Our discussions on ΔE are therefore valid only for

$$b \gtrsim \frac{\hbar}{p}$$

where p is the center of mass momentum.

For a heavy incident particle, p is close to the momentum of the electron

$$b_{\min}^{(q)} = \frac{\hbar}{\gamma m v} \qquad (13.40)$$

For $e - e$ collisions

$$b_{\min}^{(q)} = \frac{\hbar}{m c} \sqrt{\frac{2}{\gamma - 1}} \qquad (13.41)$$

Defining

$$\eta = \frac{b_{\min}}{b_{\min}^{(q)}} = \frac{z e^2}{\gamma m v^2} \gamma m \frac{v}{\hbar} = \frac{z e^2}{\hbar v} \qquad (13.42)$$

we see that

$b_{\min}$ is used for $\eta > 1$

$b_{\min}^{(q)}$ is used for $\eta < 1$

Furthermore

$$B_q = \frac{b_{\max}}{b_{\min}^{(q)}} = \frac{b_{\max}}{b_{\min}} \frac{b_{\min}}{b_{\min}^{(q)}}$$

$$= B\,\eta = \frac{\gamma^2 m v^3}{z e^2 \langle\omega\rangle} \frac{z e^2}{\hbar v}$$

$$= \frac{\gamma^2 m v^2}{\hbar \langle\omega\rangle} \qquad (13.43)$$

which implies

$$\frac{d E_q}{d x} = 4 \pi N Z \frac{z^2 e^4}{m v^2} \left[\ln\left(\frac{\gamma^2 m v^2}{\hbar \langle\omega\rangle} \right) - \frac{v^2}{2 c^2} \right]$$

which is to be compared with Bethe's quantum result

$$\frac{d E_q}{d x} = 4 \pi N Z \frac{z^2 e^4}{m v^2} \left[\ln\left(\frac{2 \gamma^2 m v^2}{\hbar \langle\omega\rangle} \right) - \frac{v^2}{c^2} \right] \qquad (13.44)$$

For $e - e$ scattering,

$$B_{\text{el}} \simeq (\gamma - 1) \sqrt{\frac{\gamma + 1}{2}} \frac{m c^2}{\hbar \langle\omega\rangle} \xrightarrow[\gamma \gg 1]{} \frac{\gamma^{3/2}}{\sqrt{2}} \frac{m c^2}{\hbar \langle\omega\rangle} \qquad (13.45)$$

To find the energy loss as a function of incident particle energy, we set

$$\max(\Delta E) = \epsilon = \frac{2 z^2 e^4}{m v^2} \frac{1}{b_{\min}{}^2(\epsilon)}$$

so that

$$b_{\min}(\epsilon) = \frac{2 z e^2}{v \sqrt{2 m \epsilon}}$$

Keeping

$$b_{\max} = 1.123 \frac{\gamma v}{\langle\omega\rangle}$$

we have

$$B_c(\epsilon) = \frac{b_{\max}}{b_{\min}(\epsilon)} = \frac{1.123 \gamma v^2 \sqrt{2 m \epsilon}}{2 z e^2 \langle \omega \rangle}$$

Using the rule

$$B_q(\epsilon) = \eta B_c(\epsilon) = \frac{z e^2}{\hbar v} B_c(\epsilon)$$

$$= \frac{1.123}{2} \frac{\gamma v \sqrt{2 m \epsilon}}{\hbar \langle \omega \rangle}$$

we have

$$\frac{d E_q}{d x}(\epsilon) = 4 \pi N Z \frac{z^2 e^4}{m v^2} \left[\ln B_q(\epsilon) - \frac{v^2}{2 c^2} \right] \qquad (13.46)$$

Better agreement with the full quantum result can be obtained by setting

$$B_q(\epsilon) = \lambda \frac{\gamma v \sqrt{2 m \epsilon}}{\hbar \langle \omega \rangle} \qquad (13.47)$$

with λ being some numerical constant of order unity. ($\lambda = 1$ for Bethe's result).

In analogy with B_c, we can write

$$B_q(\epsilon) = \frac{b_{\max}}{b_{\min}{}^{(q)}(\epsilon)} \tag{13.48}$$

with

$$b_{\min}{}^{(q)}(\epsilon) \simeq \frac{\hbar}{\Delta p} \simeq \frac{\hbar}{\sqrt{2 m \epsilon}} \tag{13.49}$$

- **Fluctuations**

According to standard quantum mechanical interpretattion, the energy loss calculated above denotes only the mean value of a large number of measurements. If the number of collisions is large, and the energy transfer small, the distribution of the measured values will, according to the central limit theorem, be Gaussian.

The mean square energy loss is defined as

$$\frac{d E^2}{d x} = 2 \pi N Z \int_{b_{\min}}^{b_{\max}} [\Delta E(b)]^2 \, b \, d b$$

Using

$$\Delta E(b) = \frac{2 z^2 e^4}{m v^2} \left(\frac{1}{b^2}\right) \tag{13.2}$$

and

$$\int_{b_{\min}}^{b_{\max}} \frac{1}{b^4} b \, d b = \frac{1}{2} \left(\frac{1}{b_{\min}{}^2} - \frac{1}{b_{\max}{}^2}\right)$$

$$\simeq \frac{\gamma^2 m^2 v^4}{2 z^2 e^4} \qquad \text{for } b_{\min} \ll b_{\max}$$

we have

$$\frac{d E^2}{d x} = 2 \pi N Z \left(\frac{2 z^2 e^4}{m v^2}\right)^2 \frac{\gamma^2 m^2 v^4}{2 z^2 e^4}$$

$$= 4 \pi N Z \frac{z^2 e^4}{m v^2} \gamma^2 m v^2$$

$$= 4 \pi N Z \frac{z^2 e^4}{m v^2} \gamma^2 m v^2 \left(1 - \frac{\beta^2}{2}\right) \tag{13.50}$$

where the term $-\beta^2/2$ is added by hand to make it relativistically correct.

Let the initial energy be E_0 and write

$$E = E_0 + \Delta E$$

The averages are

$$\langle E \rangle = E_0 + \langle \Delta E \rangle$$

Let x be the distance travelled. We have

$$\langle \Delta E \rangle \simeq \frac{dE}{dx} x \qquad \text{for small } \frac{dE}{dx}$$

$$\left\langle (\Delta E)^2 \right\rangle \simeq \frac{dE^2}{dx} x$$

so that

$$\langle E\rangle^2 = E_0{}^2 + 2E_0 \langle \Delta E\rangle + \langle \Delta E\rangle^2$$

$$\simeq E_0{}^2 + 2E_0 \frac{dE}{dx} x + \left(\frac{dE}{dx}\right)^2 x^2$$

$$\left\langle E^2\right\rangle = E_0{}^2 + 2E_0 \langle \Delta E\rangle + \left\langle (\Delta E)^2\right\rangle$$

$$\simeq E_0{}^2 + 2E_0 \frac{dE}{dx} x + \left(\frac{dE^2}{dx}\right) x$$

Hence, the mean deviation in energy is

$$\Omega^2 \equiv \left\langle E^2\right\rangle - \langle E\rangle^2 = x \frac{dE^2}{dx} \qquad (13.51)$$

For a Gaussian distribution, we have

$$P(E_0, E, x) \simeq \frac{1}{\sqrt{2\pi}\,\Omega} e^{-(E-\langle E \rangle)/2\Omega^2} \qquad (13.52)$$

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4. Density Effect

For $\gamma \gg 1$, observed $\frac{dE}{dx}$ gives $B_q(\epsilon) \sim \gamma$ instead of $B_q(\epsilon) \sim \gamma^2$. [see fig.13.5]
This is known as the density effect [Fermi (1940)].
It is caused by the dielectric polarization of the media.
(For close collisions, such effects are negligible)

- Continuum approximation:

The Fourier transforms are defined by

$$F(\boldsymbol{x}, t) = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{2\pi} F(\boldsymbol{k}, \omega)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t} \qquad (13.53)$$

$$F(\boldsymbol{k}, \omega) = \int d^3 x \int dt\, F(\boldsymbol{x}, t)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x} + i\omega t}$$

Note that

$$F_{\text{Jackson}}(\boldsymbol{k}, \omega) = \frac{1}{(2\pi)^2} F_{\text{our}}(\boldsymbol{k}, \omega)$$

The transformed wave equations are

$$\left[k^2 - \frac{\omega^2}{c^2} \epsilon(\omega) \right] \Phi(\boldsymbol{k}, \omega) = \frac{4\pi}{\epsilon(\omega)} \rho(\boldsymbol{k}, \omega)$$

$$\left[k^2 - \frac{\omega^2}{c^2} \epsilon(\omega) \right] \boldsymbol{A}(\boldsymbol{k}, \omega) = \frac{4\pi}{c} \boldsymbol{J}(\boldsymbol{k}, \omega) \qquad (13.54)$$

For the incident charge particle (v = const)

$$\rho(\boldsymbol{x}, t) = z e\, \delta(\boldsymbol{x} - \boldsymbol{v}\, t)$$

and

$$\boldsymbol{J}(\boldsymbol{x}, t) = \boldsymbol{v}\, \rho(\boldsymbol{x}, t) = z e\, \boldsymbol{v}\, \delta(\boldsymbol{x} - \boldsymbol{v}\, t) \tag{13.55}$$

Hence

$$\begin{aligned}
\rho(\boldsymbol{k}, \omega) &= z e \int d^3 x \int d t\ \delta(\boldsymbol{x} - \boldsymbol{v}\, t)\, e^{-i \boldsymbol{k}\cdot\boldsymbol{x} + i \omega t} \\
&= z e \int d t\ e^{-i(\boldsymbol{k}\cdot\boldsymbol{v} - \omega) t} \\
&= 2 \pi z e\ \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \\
\boldsymbol{J}(\boldsymbol{k}, \omega) &= z e\, \boldsymbol{v} \int d^3 x \int d t\ \delta(\boldsymbol{x} - \boldsymbol{v}\, t)\, e^{-i \boldsymbol{k}\cdot\boldsymbol{x} + i \omega t} \\
&= 2 \pi z e\, \boldsymbol{v}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \\
&= \boldsymbol{v}\, \rho\, (\boldsymbol{k}, \omega)
\end{aligned} \tag{13.56}$$

(13.54) becomes

$$\Phi(\boldsymbol{k}, \omega) = \frac{8 \pi^2 z e}{\epsilon(\omega)} \frac{\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})}{k^2 - \frac{\omega^2}{c^2} \epsilon(\omega)}$$

and

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{k}, \omega) &= \frac{8 \pi^2}{c} z e\, \boldsymbol{v} \frac{\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})}{k^2 - \frac{\omega^2}{c^2} \epsilon(\omega)} \\
&= \epsilon(\omega) \frac{\boldsymbol{v}}{c} \Phi(k, \omega)
\end{aligned} \tag{13.57}$$

Now

$$\boldsymbol{E}(\boldsymbol{x}, t) = -\nabla \Phi(\boldsymbol{x}, t) - \frac{\partial}{c\, \partial t} \boldsymbol{A}(\boldsymbol{x}, t)$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{A}(x, t)$$

implies

$$\boldsymbol{E}(\boldsymbol{k}, \omega) = -i\, \boldsymbol{k}\, \Phi(\boldsymbol{k}, \omega) + i \frac{\omega}{c} \boldsymbol{A}(\boldsymbol{k}, \omega)$$

$$\boldsymbol{B}(\boldsymbol{k}, \omega) = i\, \boldsymbol{k} \times \boldsymbol{A}(\boldsymbol{k}, \omega)$$

Hence

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{k}, \omega) &= i \left[-\boldsymbol{k} + \epsilon(\omega) \frac{\omega}{c} \frac{\boldsymbol{v}}{c}\right] \Phi(\boldsymbol{k}, \omega) \\
&= i \left[-\boldsymbol{k} + \epsilon(\omega) \frac{\omega}{c} \frac{\boldsymbol{v}}{c}\right] \frac{8 \pi^2 z e}{\epsilon(\omega)} \frac{\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})}{k^2 - \frac{\omega^2}{c^2} \epsilon(\omega)} \\
\boldsymbol{B}(\boldsymbol{k}, \omega) &= i\, \epsilon(\omega)\ \boldsymbol{k} \times \frac{\boldsymbol{v}}{c} \Phi(\boldsymbol{k}, \omega)
\end{aligned} \tag{13.58}$$

Using

$$\boldsymbol{E}(\boldsymbol{x}, \omega) = \int \frac{d^3 k}{(2 \pi)^3}\ \boldsymbol{E}(\boldsymbol{k}, \omega)\, e^{i \boldsymbol{k}\cdot\boldsymbol{x}}$$

the field at $\boldsymbol{x} = (0, b, 0)$ is

$$\begin{aligned}
\boldsymbol{E}(\omega) &\equiv \boldsymbol{E}\, (\boldsymbol{x}, \omega)\, |_{\boldsymbol{x}=(0,b,0)} \\
&= \int \frac{d^3 k}{(2 \pi)^3}\ \boldsymbol{E}(\boldsymbol{k}, \omega)\, e^{i k_2 b}
\end{aligned} \tag{13.59}$$

Hence

$$E_1(\omega) = i\,\frac{8\,\pi^2\,z\,e}{\epsilon(\omega)} \int \frac{d^3 k}{(2\,\pi)^3}\; e^{i\,b\,k_2} \left[\frac{\omega\,\epsilon(\omega)\,v}{c^2} - k_1\right] \frac{\delta(\,\omega\, - \,v\,k_1)}{k^2\, - \,\frac{\omega^2}{c^2}\,\epsilon(\omega)}$$

(13.60)

Using

$$\delta(\,\omega\, - \,v\,k_1) = \frac{1}{v}\,\delta\!\left(\frac{\omega}{v} - k_1\right)$$

we have

$$E_1(\omega) = i\,\frac{z\,e}{\pi\,\epsilon(\omega)} \int_{-\infty}^{\infty} d\,k_2 \int_{-\infty}^{\infty} d\,k_3\; e^{i\,b\,k_2}$$

$$\times\left[\frac{\omega\,\epsilon(\omega)\,v}{c^2} - \frac{\omega}{v}\right] \frac{1\,/\,v}{\left(\frac{\omega}{v}\right)^2 + k_2{}^2\; + \;k_3{}^2 - \;\frac{\omega^2}{c^2}\,\epsilon(\omega)}$$

$$= \; i\,\frac{z\,e\,\omega}{\pi\,v^2}\left[\,\beta^2 - \frac{1}{\epsilon(\omega)}\right] \int_{-\infty}^{\infty} d\,k_2 \int_{-\infty}^{\infty} d\,k_3\; e^{i\,b\,k_2}\; \frac{1}{\left(\frac{\omega}{v}\right)^2 + k_2{}^2\; + \;k_3{}^2 - \;\frac{\omega^2}{c^2}\,\epsilon(\omega)}$$

$$= \; -i\,\frac{z\,e\,\omega}{\pi\,v^2}\left[\frac{1}{\epsilon(\omega)} - \beta^2\right] \int_{-\infty}^{\infty} d\,k_2\; e^{i\,b\,k_2} \int_{-\infty}^{\infty} \frac{d\,k_3}{k_2{}^2\; + \;k_3{}^2\; + \;\lambda^2}$$

$$= \; -i\,\frac{z\,e}{\pi\,\omega\,\epsilon(\omega)}\,\lambda^2 \int_{-\infty}^{\infty} d\,k_2\; e^{i\,b\,k_2} \int_{-\infty}^{\infty} \frac{d\,k_3}{k_2{}^2\; + \;k_3{}^2\; + \;\lambda^2}$$

(13.60)

where

$$\lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\,\epsilon(\omega) = \frac{\omega^2}{v^2}\left[\,1 - \beta^2\,\epsilon(\omega)\,\right] \qquad (13.61)$$

$$= \frac{\omega^2}{v^2}\,\epsilon(\omega)\left[\,\frac{1}{\epsilon(\omega)} - \beta^2\,\right]$$

Using

$$\int \frac{d\,x}{x^2\; + \;a^2} = \frac{1}{a}\,\tan^{-1}\,\frac{x}{a}$$

we have

$$\int_{-\infty}^{\infty} \frac{d\,k_3}{k_2{}^2\; + \;k_3{}^2\; + \;\lambda^2} = \frac{1}{\sqrt{k_2{}^2\; + \;\lambda^2}} \left(\tan^{-1}\,\frac{k_3}{\sqrt{k_2{}^2\; + \;\lambda^2}}\right)_{-\infty}^{\infty}$$

$$= \; \frac{1}{\sqrt{k_2{}^2\; + \;\lambda^2}}\,\pi$$

Hence

$$E_1(\omega) = -i\,\frac{z\,e\,\omega}{v^2}\Big[\frac{1}{\epsilon(\omega)} - \beta^2\Big]\int_{-\infty}^{\infty} d\,k_2\;\frac{e^{i\,b\,k_2}}{\sqrt{\lambda^2 + k_2{}^2}} \quad (13.62)$$

$$= \;-2\,i\,\frac{z\,e\,\omega}{v^2}\Big[\frac{1}{\epsilon(\omega)} - \beta^2\Big] K_0(\lambda\,b)$$

$$= \;-2\,i\,\frac{z\,e}{\omega\,\epsilon(\omega)}\,\lambda^2\,K_0(\lambda\,b) \qquad (13.63)$$

where from

$$\int_{-\infty}^{\infty} d\,x\;\frac{e^{i\,\xi\,x}}{\left(x^2 + a^2\right)^{\nu+\frac{1}{2}}} = 2\,\frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{\xi}{2\,a}\right)^{\nu} K_\nu(a\,\xi)$$

we have

$$\int_{-\infty}^{\infty} d\,k_2\;\frac{e^{i\,b\,k_2}}{\sqrt{\lambda^2 + k_2{}^2}} = 2\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}\right)}\,K_0\,(\lambda\,b) = 2\,K_0(\lambda\,b)$$

Similarly

$$E_2(\omega) = 2\,z\,\frac{e}{v}\,\frac{\lambda}{\epsilon(\omega)}\,K_1(\lambda\,b)$$

$$B_3(\omega) = \epsilon(\omega)\,\beta\,E_2(\omega) \qquad (13.64)$$

For $\epsilon(\omega) \to 1$,

$$\lambda \to \frac{\omega}{v\,\gamma} = \frac{1}{b_{\max}}$$

(13.63) & (13.64) becomes (13.30) & (13.29), respectively.

The generalization of (13.23) is

$$\Delta\,E(b) = 2\,e\sum_j f_j\;\mathrm{Re}\int_0^{\infty}\frac{d\,\omega}{2\,\pi}\,i\,\omega\,\boldsymbol{x}_j(\omega)\cdot\boldsymbol{E}^*\,(\omega)$$

where $\boldsymbol{x}_j\,(\omega)$ is the amplitude of the jth type of electron in the atom.

The polarization $\boldsymbol{P}$ can be written as

$$\boldsymbol{P}(\omega) = -e\,N\sum_j f_j\;\boldsymbol{x}_j(\omega) = \frac{1}{4\,\pi}\,[\,\epsilon(\omega) - 1\,]\,\boldsymbol{E}(\omega)$$

so that

$$\sum_j f_j\;\boldsymbol{x}_j(\omega) = -\frac{1}{4\,\pi\,e\,N}\,[\,\epsilon(\omega) - 1\,]\;\boldsymbol{E}(\omega)$$

and

$$\Delta\,E(b) = -\frac{1}{2\,\pi\,N}\,\mathrm{Re}\int_0^{\infty}\frac{d\,\omega}{2\,\pi}\,i\,\omega\;[\,\epsilon(\omega) - 1\,]\,\boldsymbol{E}(\omega)\cdot\boldsymbol{E}^*(\omega)$$

$$= \;-\frac{1}{2\,\pi\,N}\,\mathrm{Re}\int_0^{\infty}\frac{d\,\omega}{2\,\pi}\,i\,\omega\;\epsilon(\omega)\,|\,\boldsymbol{E}(\omega)\,|^2 \qquad (13.65)$$

$$= \;\frac{1}{2\,\pi\,N}\,\mathrm{Re}\int_0^{\infty}\frac{d\,\omega}{2\,\pi}\,\omega\;\epsilon_2(\omega)\;|\,\boldsymbol{E}(\omega)\,|^2$$

where $\epsilon(\omega) = \epsilon_1(\omega) + i\,\epsilon_2(\omega)$.

Using (16.63) & (16.64), we have

$$| E_1(\omega) |^2 = \left(\frac{2\, z\, e}{\omega}\right)^2 \left| \frac{\lambda^2}{\epsilon} K_0(\lambda\, b) \right|^2$$

$$| E_2(\omega) |^2 = \left(\frac{2\, z\, e}{v}\right)^2 \left| \frac{\lambda}{\epsilon} K_1(\lambda\, b) \right|^2$$

$$| \boldsymbol{E}(\omega) |^2 = | E_1(\omega) |^2 + | E_2(\omega) |^2$$

$$= \left(\frac{2\, z\, e}{v}\right)^2 \left| \frac{\lambda}{\epsilon} \right|^2 \left[\left(\frac{v}{\omega}\right)^2 |\lambda|^2\ |K_0(\lambda\, b)|^2 + | K_1(\lambda\, b)|^2 \right]$$

Define

$$\left(\frac{d\, E}{d\, x}\right)_{b>a} = 2\, \pi\, N \int_a^\infty \Delta\, E(b)\, b\, d\, b \qquad (13.66)$$

we have

$$\left(\frac{d\, E}{d\, x}\right)_{b>a} = \int_a^\infty b\, d\, b\, \mathrm{Re} \int_0^\infty \frac{d\, \omega}{2\, \pi}\, i\, \omega\, \epsilon\ | \boldsymbol{E}(\omega) |^2$$

$$= \left(\frac{2\, z\, e}{v}\right)^2 \mathrm{Re} \int_0^\infty \frac{d\, \omega}{2\, \pi}\, i\, \omega\, \epsilon \left| \frac{\lambda}{\epsilon} \right|^2$$

$$\times \int_a^\infty b\, d\, b \left[\left(\frac{v}{\omega}\right)^2 |\lambda|^2\ | K_0(\lambda\, b)|^2 + | K_1(\lambda\, b)|^2 \right]$$

$$= \left(\frac{2\, z\, e}{v}\right)^2 \mathrm{Re} \int_0^\infty \frac{d\, \omega}{2\, \pi}\, i\, \omega\, \epsilon \left| \frac{\lambda}{\epsilon} \right|^2 \left[\left(\frac{v}{\omega}\right)^2 |\lambda|^2 I_0 + I_1 \right]$$

The integrals

$$I_0 = \int_a^\infty b\, d\, b\, | K_0(\lambda\, b)|^2$$

$$I_1 = \int_a^\infty b\, d\, b\, | K_1(\lambda\, b)|^2$$

are evaluated as follows.

Using

$$K_\nu(z)^* = K_\nu(z^*)$$

$$\int^z d\, t\ t\, \mathbb{C}_\mu(k\, t)\, \mathbb{D}_\mu(l\, t) = \frac{z}{k^2 - l^2} \left\{ k\, \mathbb{C}_{\mu+1}(k\, z)\, \mathbb{D}_\mu(l\, z) - l\, \mathbb{C}_\mu(k\, z)\, \mathbb{D}_{\mu+1}(l\, z) \right\}$$

we have

$$I_\mu = \int_a^\infty b\, d\, b \left| K_\mu(\lambda\, b) \right|^2$$

$$= \int_a^\infty b\, d\, b\, K_\mu(\lambda\, b)\, K_\mu(\lambda^*\, b)$$

$$= \frac{b}{\lambda^2 - \lambda^{*2}} \left\{ \lambda\, K_{\mu+1}(\lambda\, b)\, K_\mu(\lambda^*\, b) - \lambda^*\, K_\mu(\lambda\, b)\, K_{\mu+1}(\lambda^*\, b) \right\} \Big|_a^\infty$$

$$= -\frac{a}{\lambda^2 - \lambda^{*2}} \left\{ \lambda\, K_{\mu+1}(\lambda\, a)\, K_\mu(\lambda^*\, a) - \lambda^*\, K_\mu(\lambda\, a)\, K_{\mu+1}(\lambda^*\, a) \right\}$$

Writing

$$\lambda = \lambda_1 + i\, \lambda_2$$

we have

$$\lambda^2 - \lambda^{*2} = (\lambda + \lambda^*)\,(\lambda - \lambda^*) = 4\, i\, \lambda_1\, \lambda_2$$

$$I_\mu = -\frac{a}{4\, i\, \lambda_1\, \lambda_2}\, 2\, i\, \mathrm{Im}\left[\lambda\, K_{\mu+1}(\lambda\, a)\, K_\mu(\lambda^*\, a) \right]$$

$$= \frac{1}{2\, \lambda_1\, \lambda_2}\, \mathrm{Im}\left[\lambda^*\, a\, K_{\mu+1}(\lambda^*\, a)\, K_\mu(\lambda\, a) \right]$$

Now

$$\lambda^2 = \left(\frac{\omega}{v}\right)^2 \left[1 - \beta^2\, \epsilon \right]$$

$$= \left(\frac{\omega}{v}\right)^2 \left[1 - \beta^2\, \epsilon_1 - i\, \beta^2\, \epsilon_2 \right]$$

$$= \lambda_1{}^2 - \lambda_2{}^2 + 2\, i\, \lambda_1\, \lambda_2$$

$\longrightarrow$

$$\lambda_1{}^2 - \lambda_2{}^2 = \left(\frac{\omega}{v}\right)^2 \left[1 - \beta^2\, \epsilon_1 \right]$$

$$2\, \lambda_1\, \lambda_2 = -\left(\frac{\omega}{v}\right)^2 \beta^2\, \epsilon_2 = -\left(\frac{\omega}{c}\right)^2 \epsilon_2$$

$$|\, \lambda\, |^2 = \left(\frac{\omega}{v}\right)^2 \left| \, 1 - \beta^2\, \epsilon \right|$$

Thus

$$I_\mu = -\frac{c^2}{\omega^2\, \epsilon_2}\, \mathrm{Im}\left[\lambda^*\, a\, K_{\mu+1}(\lambda^*\, a)\, K_\mu(\lambda\, a) \right]$$

ie.

$$I_0 = -\frac{c^2}{\omega^2\, \epsilon_2}\, \mathrm{Im}[\, \lambda^*\, a\, K_1(\lambda^*\, a)\, K_0(\lambda\, a)\,]$$

$$\left(\frac{v}{\omega}\right)^2 |\, \lambda|^2\, I_0 = -\frac{c^2}{\omega^2\, \epsilon_2} \left| \, 1 - \epsilon\, \beta^2 \right|\, \mathrm{Im}[\, \lambda^*\, a\, K_1(\lambda^*\, a)\, K_0(\lambda\, a)\,]$$

$$I_1 = -\frac{c^2}{\omega^2\, \epsilon_2}\, \mathrm{Im}[\, \lambda^*\, a\, K_2(\lambda^*\, a)\, K_1(\lambda\, a)\,]$$

Using

$$K_2 = K_0 + \frac{2}{z} K_1$$

we have

$$\lambda^* a\, K_2(\lambda^* a)\, K_1(\lambda a) = [\, \lambda^* a\, K_0(\lambda^* a) + 2\, K_1(\lambda^* a)\,]\, K_1(\lambda a)$$
$$= \lambda^* a\, K_1(\lambda a)\, K_0(\lambda^* a) + 2\, |\, K_1\, (\lambda\, b)|^2$$

$$\mathrm{Im}[\, \lambda^* a\, K_2(\lambda^* a)\, K_1(\lambda a)\,] = \mathrm{Im}[\, \lambda^* a\, K_1(\lambda a)\, K_0(\lambda^* a)\,]$$
$$= -\mathrm{Im}[\, \lambda\, a\, K_1(\lambda^* a)\, K_0(\lambda a)\,]$$
$$I_1 = \frac{c^2}{\omega^2 \epsilon_2} \mathrm{Im}[\, \lambda\, a\, K_1(\lambda^* a)\, K_0(\lambda a)\,]$$

Hence

$$\left(\frac{v}{\omega}\right)^2 |\, \lambda|^2\, I_0 + I_1 = -\frac{c^2}{\omega^2 \epsilon_2} \mathrm{Im}\left\{ \left[\left(\frac{v}{\omega}\right)^2 |\, \lambda|^2\, \lambda^* - \lambda \right] a\, K_1(\lambda^* a)\, K_0(\lambda a) \right\}$$
$$= -\frac{c^2}{\omega^2 \epsilon_2} \mathrm{Im}\left\{ \left[\left(\frac{v}{\omega}\right)^2 \lambda^{*2} - 1 \right] a\, \lambda\, K_1(\lambda^* a)\, K_0(\lambda a) \right\}$$
$$= \frac{c^2}{\omega^2 \epsilon_2} \mathrm{Im}\left\{ \beta^2 \epsilon^* a\, \lambda\, K_1(\lambda^* a)\, K_0(\lambda a) \right\}$$

Using

$$|\, \lambda\, |^2\, \lambda = \lambda^2\, \lambda^* = \left(\frac{\omega}{v}\right)^2 \left[\, 1 - \beta^2\, \epsilon \right] \lambda^*$$

we have

$$\epsilon \left| \frac{\lambda}{\epsilon} \right|^2 \left[\left(\frac{v}{\omega}\right)^2 |\, \lambda|^2\, I_0 + I_1 \right]$$
$$= \frac{c^2}{\omega^2 \epsilon_2 \epsilon^*} \mathrm{Im}\left\{ |\, \lambda|^2\, \beta^2\, \epsilon^* a\, \lambda\, K_1(\lambda^* a)\, K_0(\lambda a) \right\}$$
$$= \frac{c^2}{\omega^2 \epsilon_2 \epsilon^*} \mathrm{Im}\left\{ \beta^2\, \epsilon^* \left(\frac{\omega}{v}\right)^2 \left[\, 1 - \beta^2\, \epsilon \right] a\, \lambda^* K_1(\lambda^* a)\, K_0(\lambda a) \right\}$$
$$= \frac{1}{\epsilon_2 \epsilon^*} |\, \epsilon\, |^2\, \mathrm{Im}\left\{ \left(\frac{1}{\epsilon} - \beta^2 \right) a\, \lambda^* K_1(\lambda^* a)\, K_0(\lambda a) \right\}$$
$$= \frac{\epsilon}{\epsilon_2} \mathrm{Im}\left\{ \left(\frac{1}{\epsilon} - \beta^2 \right) a\, \lambda^* K_1(\lambda^* a)\, K_0(\lambda a) \right\}$$
$$= \left(\frac{\epsilon_1}{\epsilon_2} + i \right) \mathrm{Im}\left\{ \left(\frac{1}{\epsilon} - \beta^2 \right) a\, \lambda^* K_1(\lambda^* a)\, K_0(\lambda a) \right\}$$

$$Re\int_0^\infty \frac{d\,\omega}{2\,\pi}\; i\,\omega\;\epsilon\left|\frac{\lambda}{\epsilon}\right|^2\left[\left(\frac{v}{\omega}\right)^2 |\,\lambda|^2\, I_0 + I_1\right]$$

$$= \;\; \mathrm{Re}\int_0^\infty \frac{d\,\omega}{2\,\pi}\; i\,\omega\left(\frac{\epsilon_1}{\epsilon_2}+i\right)\mathrm{Im}\left\{\left(\frac{1}{\epsilon}-\beta^2\right)a\,\lambda^*\,K_1(\lambda^*\,a)\,K_0(\lambda\,a)\right\}$$

$$= \; -\mathrm{Re}\int_0^\infty \frac{d\,\omega}{2\,\pi}\;\omega\,\mathrm{Im}\left\{\left(\frac{1}{\epsilon}-\beta^2\right)a\,\lambda^*\,K_1(\lambda^*\,a)\,K_0(\lambda\,a)\right\}$$

$$= \; -\mathrm{Im}\int_0^\infty \frac{d\,\omega}{2\,\pi}\;\omega\left(\frac{1}{\epsilon}-\beta^2\right)a\,\lambda^*\,K_1(\lambda^*\,a)\,K_0(\lambda\,a)$$

$$= \;\; \mathrm{Re}\int_0^\infty \frac{d\,\omega}{2\,\pi}\; i\,\omega\left(\frac{1}{\epsilon}-\beta^2\right)a\,\lambda^*\,K_1(\lambda^*\,a)\,K_0(\lambda\,a)$$

$$\left(\frac{d\,E}{d\,x}\right)_{b>a} = \left(\frac{2\,z\,e}{v}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\,\omega}{2\,\pi}\, i\,\omega\,\lambda^*\,a\,K_1(\lambda^*\,a)\,K_0(\lambda\,a)\left(\frac{1}{\epsilon}-\beta^2\right) \qquad (13.67)$$

This result can be obtained more elegantly by calculating the electromagnetic energy flow through a cylinder of radius a around the path of the incident particle. By conservation of energy this is the energy lost per unit time by the incident particle. Thus

$$\left(\frac{d\,E}{d\,x}\right)_{b>a} = \frac{1}{v}\,\frac{d\,E}{d\,t}$$

Now,

$$\frac{d\,E}{d\,t} = -\frac{c}{4\,\pi}\int_{\text{cylinder}} d\,\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{B})$$

Let the particle moves along the z axis, we have

$$d\,\boldsymbol{\sigma} = a\,d\,\phi\,d\,z\,\hat{\boldsymbol{\rho}}$$

Now, the only non-zero component of $\boldsymbol{B}$ is B_ρ (B_3 in previous notations when particle moves along x –axis). This means only E_z (E_1) contributes to the integral.

$$\frac{d\,E}{d\,t} = -\frac{c}{4\,\pi}\int_0^{2\pi} d\,\phi\int_{-\infty}^{\infty} d\,z\,a\,E_z\,B_\rho$$

$$= \;\; -\frac{c}{2}\,a\int_{-\infty}^{\infty} d\,z\;E_z\,B_\rho$$

$$= \;\; -\frac{c}{2}\,a\int_{-\infty}^{\infty} d\,x\;E_1\,B_3 \qquad \text{(particle moves along } x \text{ –axis)}$$

Hence

$$\left(\frac{d\,E}{d\,x}\right)_{b>a} = -\frac{c\,a}{2\,v}\int_{-\infty}^{\infty} d\,x\;E_1\,B_3$$

Using $d\,x = v\,d\,t$, we have

$$\left(\frac{d\,E}{d\,x}\right)_{b>a} = -\frac{c\,a}{2}\int_{-\infty}^{\infty} B_3(t)\,E_1(t)\,d\;t$$

Now

$$\int_{-\infty}^{\infty} d\,t\, A(t)\, B(t) = \int_{-\infty}^{\infty} d\,t \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi} \int_{-\infty}^{\infty} \frac{d\,\omega'}{2\,\pi}\, e^{-i\,(\omega+\omega')\,t}\, A(\omega)\, B(\omega\,')$$

$$= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi} \int_{-\infty}^{\infty} d\,\omega'\, \delta(\omega+\omega')\, A(\omega)\, B(\omega\,')$$

$$= \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, A(\omega)\, B(-\omega\,)$$

$$= \left[\int_{-\infty}^{0} + \int_{0}^{\infty}\right] \frac{d\,\omega}{2\,\pi}\, A(\omega)\, B(-\omega\,)$$

$$= \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, [A(\omega)\, B(-\omega\,) + A(-\omega)\, B(\omega\,)\,]$$

$$= \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, [A(\omega)\, B^*(\omega\,) + A^*(\omega)\, B(\omega\,)\,] \qquad [\; A(t),\, B(t) \text{ real }]$$

$$= \text{Re} \int_{0}^{\infty} \frac{d\,\omega}{\pi}\, A(\omega)\, B^*(\omega\,)$$

Thus

$$\left(\frac{d\,E}{d\,x}\right)_{b>a} = -c\,a\,\text{Re} \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, B_3^{*}(\omega)\, E_1(\omega) \qquad (13.68)$$

With fields (13.63) and (13.64) this gives the Fermi result (13.67).

For $\beta \ll 1$,

$$\lambda = \frac{\omega}{v} \sqrt{1-\beta^2\,\epsilon} \approx \frac{\omega}{v}$$

is real and independent of $\epsilon(\omega)$. Hence, $K_1(\lambda^*\,a)$ and $K_0(\lambda\,a)$ are also real. (13.67) becomes

$$\left(\frac{d\,E}{d\,x}\right)_{b>a} = \left(\frac{2\,z\,e}{v}\right)^2 \int_{0}^{\infty} \frac{d\,\omega}{2\,\pi}\, \frac{\omega^2\,a}{v}\, K_1\!\left(\frac{\omega\,a}{v}\right) K_0\!\left(\frac{\omega\,a}{v}\right) \text{Re}\left(\frac{i}{\epsilon}\right)$$

If we neglect the local field correction, ϵ can be written as

$$\epsilon(\omega) \simeq 1 + \frac{4\,\pi\,N\,e^2}{m} \sum_j \frac{f_j}{\omega_j^{\,2} - \omega^2 - i\,\omega\,\Gamma_j} \qquad (13.69)$$

$$= 1 + \frac{4\,\pi\,N\,e^2}{m} \sum_j \frac{f_j\left[\,\omega_j^{\,2} - \omega^2 + i\,\omega\,\Gamma_j\,\right]}{\left(\omega_j^{\,2} - \omega^2\right)^2 + \left(\omega\,\Gamma_j\right)^2}$$

Assuming that the second term is small,

$$\frac{1}{\epsilon(\omega)} \simeq 1 - \frac{4\,\pi\,N\,e^2}{m} \sum_j \frac{f_j\left[\,\omega_j^{\,2} - \omega^2 + i\,\omega\,\Gamma_j\,\right]}{\left(\omega_j^{\,2} - \omega^2\right)^2 + \left(\omega\,\Gamma_j\right)^2}$$

$$\text{Re}\left(\frac{i}{\epsilon}\right) \simeq \frac{4\,\pi\,N\,e^2}{m} \sum_j f_j \,\frac{\omega\,\Gamma_j}{\left(\omega_j^{\,2} - \omega^2\right)^2 + \left(\omega\,\Gamma_j\right)^2}$$

Now

$$\frac{\omega\,\Gamma_j}{\left(\omega_j^2-\omega^2\right)^2+\left(\omega\,\Gamma_j\right)^2}$$

$$=\frac{1}{2\,i}\left[\frac{1}{\omega_j^2-\omega^2-i\,\omega\,\Gamma_j}-\frac{1}{\omega_j^2-\omega^2+i\,\omega\,\Gamma_j}\right]$$

$$\underset{\Gamma_j\to 0}{=}\frac{1}{2\,i}\left\{\frac{1}{2\,\omega_j}\left(\frac{1}{\omega_j-\omega-i\,\frac{\Gamma_j}{2}}+\frac{1}{\omega_j+\omega+i\,\frac{\Gamma_j}{2}}\right)\right.$$

$$\left.-\frac{1}{2\,\omega_j}\left(\frac{1}{\omega_j-\omega+i\,\frac{\Gamma_j}{2}}+\frac{1}{\omega_j+\omega-i\,\frac{\Gamma_j}{2}}\right)\right\}$$

$$\underset{\Gamma_j\to 0}{=}\frac{1}{4\,i\,\omega_j}\left(\frac{1}{\omega_j-\omega-i\,\frac{\Gamma_j}{2}}-\frac{1}{\omega_j-\omega+i\,\frac{\Gamma_j}{2}}\right.$$

$$\left.+\frac{1}{\omega_j+\omega+i\,\frac{\Gamma_j}{2}}-\frac{1}{\omega_j+\omega-i\,\frac{\Gamma_j}{2}}\right)$$

$$=\frac{\pi}{2\,\omega_j}\left[\delta\left(\omega_j-\omega\right)-\delta\left(\omega_j+\omega\right)\right]$$

where we've used

$$\lim_{\delta\to 0}\frac{1}{x\mp i\,\delta}=P\,\frac{1}{x}\pm i\,\pi\,\delta(x)$$

Hence

$$\mathrm{Re}\left(\frac{i}{\epsilon}\right)\to\frac{4\,\pi\,N\,e^2}{m}\sum_j f_j\left[\delta\left(\omega-\omega_j\right)-\delta\left(\omega+\omega_j\right)\right]\frac{\pi}{2\,\omega_j}$$

so that

$$\left(\frac{d\,E}{d\,x}\right)_{b>a}=\left(\frac{z\,e}{v}\right)^2\frac{4\,\pi\,N\,e^2}{m}\sum_j f_j\,\frac{\omega_j\,a}{v}\,K_1\left(\frac{\omega_j\,a}{v}\right)K_0\left(\frac{\omega_j\,a}{v}\right)$$

which is (13.35) without the β^2 term.

Typically, we have

$\omega\sim 10^{15}$ Hz $a\sim 10^{-8}$ cm

$c\sim 3\times 10^{10}$ cm/s $\epsilon\sim 10$

so that

$$\frac{\omega\,a}{c}\sim 3\times 10^{15-8-11}\sim 3\times 10^{-4}$$

For $\beta\simeq 1$

$$\left|\lambda\,a\right|\simeq\frac{\omega\,a}{c}\left|\sqrt{1-\beta^2\,\epsilon}\right|\sim 3\times 10^{-4}\ll 1$$

For $\xi \ll 1$

$$K_0(\xi) \simeq \ln\left(\frac{1.123}{\xi}\right)$$

$$K_1(\xi) \simeq \frac{1}{\xi}$$

Hence

$$\lambda^* a K_1(\lambda^* a) K_0(\lambda a) \simeq \ln\left(\frac{1.123}{\lambda a}\right)$$

$$= \ln\left(\frac{1.123\, c}{\omega a}\right) - \ln\sqrt{1-\beta^2 \epsilon}$$

$$= \ln\left(\frac{1.123\, c}{\omega a}\right) - \frac{1}{2}\ln\left(1-\beta^2 \epsilon\right)$$

$$\left(\frac{dE}{dx}\right)_{b>a} \simeq \left(\frac{2ze}{v}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\omega}{2\pi}\, i\omega\left[\ln\left(\frac{1.123\, c}{\omega a}\right) - \frac{1}{2}\ln\left(1-\beta^2\epsilon\right)\right]\left(\frac{1}{\epsilon} - \beta^2\right)$$

$$\simeq \left(\frac{2ze}{v}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\omega}{2\pi}\, i\omega\left[\ln\left(\frac{1.123\, c}{\omega a}\right) - \frac{1}{2}\ln\left(1-\beta^2\epsilon\right)\right]\left(\frac{1}{\epsilon} - 1\right) \qquad (13.70)$$

If $\epsilon \simeq 1$,

$$\frac{1}{2}\ln\left(1-\beta^2\epsilon\right) \simeq \frac{1}{2}\ln\left(1-\beta^2\right) = -\ln\gamma$$

so that

$$\left(\frac{dE}{dx}\right)_{b>a} \simeq \left(\frac{2ze}{c}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\omega}{2\pi}\, i\omega\left[\ln\left(\frac{1.123\, c}{\omega a}\right) - \frac{1}{2}\ln(1-\epsilon)\right]\left(\frac{1}{\epsilon} - 1\right)$$

$$\simeq \left(\frac{2ze}{c}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\omega}{2\pi}\, i\omega\left[\ln\left(\frac{1.123\, c}{\omega a\sqrt{1-\epsilon}}\right)\right]\left(\frac{1}{\epsilon} - 1\right)$$

$$\simeq \left(\frac{2ze}{c}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\omega}{2\pi}\, i\omega\left[\ln\left(\frac{1.123\, c\gamma}{\omega a}\right)\right]\left(\frac{1}{\epsilon} - 1\right)$$

$$= \left(\frac{2ze}{c}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\omega}{2\pi}\, i\omega\left[\ln\left(\frac{1.123\, c\gamma}{\omega a}\right)\right]\frac{1}{\epsilon}$$

where

$$\epsilon(\omega) \simeq 1 + \frac{4\pi N e^2}{m}\sum_j \frac{f_j}{\omega_j{}^2 - \omega^2 - i\omega\Gamma_j}$$

$$= 1 + \frac{\omega_p{}^2}{Z}\sum_j \frac{f_j}{\omega_j{}^2 - \omega^2 - i\omega\Gamma_j}$$

$$\omega_p{}^2 = \frac{4\pi N Z e^2}{m}$$

The integral is evaluated choosing a contour consists of

real axis → quarter circle at infinity → imaginary axis

For the imaginary axis, set

$$\omega = i\, y$$

so that

$$i\,\omega\, d\,\omega = -i\, y\, d\, y$$

$$\epsilon(i\,y) = 1 + \omega_p{}^2 \sum_j \frac{f_j}{\omega_j{}^2 + y^2 + y\,\Gamma_j} \text{ is real}$$

$$\sqrt{1-\epsilon(i\,y)} = \sqrt{-\omega_p{}^2 \sum_j \frac{f_j}{\omega_j{}^2 + y^2 + y\,\Gamma_j}} \quad \text{is imaginary}$$

$$\ln\left[\omega\sqrt{1-\epsilon}\right] \text{ is real}$$

Hence, the integral is imaginary & doesn't contribution to $\left(\frac{d\,E}{d\,x}\right)_{b>a}$.

For the quarter circle

$$\omega = R\, e^{i\,\phi} \qquad\qquad R \to \infty$$

$$d\,\omega = i\, R\, e^{i\,\phi}\, d\,\phi = i\,\omega\, d\,\phi$$

$$\epsilon(\omega) \simeq 1 - \frac{\omega_p{}^2}{Z} \sum_j \frac{f_j}{\omega^2 - i\,\omega\,\Gamma_j}$$

$$\simeq 1 - \frac{\omega_p{}^2}{Z} \sum_j \frac{f_j}{\omega^2 - i\,\omega\,\Gamma} \qquad\qquad (\text{ if } \Gamma_j = \Gamma\ \forall\ j\,)$$

$$= \ 1 - \frac{\omega_p{}^2}{\omega\,(\,\omega - i\,\Gamma\,)}$$

$$\simeq 1 - \frac{\omega_p{}^2}{\omega^2}$$

$$\frac{1}{\epsilon} \simeq 1 + \frac{\omega_p{}^2}{\omega^2}$$

$$\frac{1}{\epsilon} - 1 \simeq \frac{\omega_p{}^2}{\omega^2}$$

$$\omega\sqrt{1-\epsilon} \simeq \omega_p$$

$$\int_{Q\,\text{circle}} \frac{d\,\omega}{2\,\pi}\, i\,\omega\left[\ln\!\left(\frac{1.123\,c}{\omega\,a\sqrt{1-\epsilon}}\right)\right]\left(\frac{1}{\epsilon} - 1\right)$$

$$= \ -\frac{1}{2\,\pi}\int_0^{\pi/2} d\,\phi\,\omega^2 \left[\ln\!\left(\frac{1.123\,c}{\omega_p\,a}\right)\right] \frac{\omega_p{}^2}{\omega^2}$$

$$= \ -\frac{1}{4}\left[\ln\!\left(\frac{1.123\,c}{\omega_p\,a}\right)\right]\omega_p{}^2$$

Since there is no poles enclosed by the contour, we have

$$\text{Re}\int_0^\infty \frac{d\,\omega}{2\,\pi}\, i\,\omega\left[\ln\!\left(\frac{1.123\,c}{\omega\,a\sqrt{1-\epsilon}}\right)\right]\left(\frac{1}{\epsilon} - 1\right) = \frac{1}{4}\,\omega_p{}^2 \ln\!\left(\frac{1.123\,c}{\omega_p\,a}\right)$$

and

$$\left(\frac{d\,E}{d\,x}\right)_{b>a} \simeq \left(\frac{z\,e}{c}\right)^2 \omega_p{}^2 \ln\!\left(\frac{1.123\,c}{\omega_p\,a}\right) \qquad\qquad (13.71)$$

The corresponding relativistic expression without the density effect is, from (13.36)

$$\left(\frac{dE}{dx}\right)_{b>a} = \left(\frac{z\,e\,\omega_p}{c}\right)^2 \left[\ln\left(\frac{1.123\,\gamma\,c}{a\,\langle\omega\rangle}\right) - \frac{1}{2}\right] \qquad (13.73)$$

The difference between (13.73) and (13.71) is

$$\lim_{\beta\to 1} \Delta\left(\frac{dE}{dx}\right) = \left(\frac{z\,e\,\omega_p}{c}\right)^2 \left[\ln\left(\frac{\gamma\,\omega_p}{\langle\omega\rangle}\right) - \frac{1}{2}\right] \qquad (13.74)$$

whose numerical values are tabulated.

For photographic emulsions, the relevant energy loss is given by (13.46) and (13.47) with $\epsilon \simeq 10$ keV. With the density correction applied, this becomes constant at high energies with the value,

$$\frac{dE(\epsilon)}{dx} \to \frac{1}{2}\left(\frac{z\,e\,\omega_p}{c}\right)^2 \ln\left(\frac{2\,m\,c^2\,\epsilon}{\hbar^2\,\omega_p{}^2}\right) \qquad (13.75)$$

5. Cherenkov Radiation

For $\xi \gg 1$

$$K_0(\xi) \approx K_1(\xi) \approx \sqrt{\frac{\pi}{2\,\xi}}\; e^{-\xi}$$

The fields

$$E_1(\omega) = -2\,i\,\frac{z\,e}{\omega\,\epsilon}\,\lambda^2\,K_0(\lambda\,b) \qquad (13.63)$$

$$E_2(\omega) = 2\,\frac{z\,e}{v}\,\frac{\lambda}{\epsilon}\,K_1(\lambda\,b) \qquad (13.64)$$

$$B_3(\omega) = \epsilon(\omega)\,\beta\,E_2(\omega)$$

$$\lambda^2 = \left(\frac{\omega}{v}\right)^2 \left(1 - \beta^2\,\epsilon\right) = \left(\frac{\omega}{c}\right)^2 \epsilon \left(\frac{1}{\beta^2\,\epsilon} - 1\right)$$

thus becomes, for $|\,\lambda\,a\,| \gg 1$,

$$E_1(\omega) \simeq -2\,i\,\frac{z\,e}{\omega\,\epsilon}\,\lambda^2 \sqrt{\frac{\pi}{2\,\lambda\,b}}\; e^{-\lambda\,b}$$

$$\simeq \sqrt{2\,\pi}\; i\,\frac{z\,e\,\omega}{c^2}\left(1 - \frac{1}{\beta^2\,\epsilon}\right)\frac{e^{-\lambda\,b}}{\sqrt{\lambda\,b}}$$

$$E_2(\omega) \simeq 2\,\frac{z\,e}{v}\,\frac{\lambda}{\epsilon}\sqrt{\frac{\pi}{2\,\lambda\,b}}\; e^{-\lambda\,b}$$

$$\simeq \sqrt{2\,\pi}\;\frac{z\,e}{v\,\epsilon}\sqrt{\frac{\lambda}{b}}\; e^{-\lambda\,b}$$

$$B_3(\omega) = \epsilon\,\beta\,E_2(\omega) \qquad (13.76)$$

$$\simeq \sqrt{2\,\pi}\;\frac{z\,e}{c}\sqrt{\frac{\lambda}{b}}\; e^{-\lambda\,b}$$

Hence

$$B_3^{*}(\omega)\, E_1(\omega) \simeq 2\,\pi\, i \left(\frac{z\,e}{c}\right)^2 \frac{\omega}{c}\left(1-\frac{1}{\beta^2\,\epsilon}\right)\sqrt{\frac{\lambda^*}{\lambda}}\ \frac{e^{-(\lambda+\lambda^*)\,b}}{b}$$

so that

$$\left(\frac{d\,E}{d\,x}\right)_{b>a} = -c\,a\,\mathrm{Re}\int_0^\infty \frac{d\,\omega}{2\,\pi}\, B_3^{*}(\omega)\, E_1(\omega) \qquad (13.68)$$

$$\simeq -c\ \mathrm{Re}\int_0^\infty d\,\omega\ i\left(\frac{z\,e}{c}\right)^2 \frac{\omega}{c}\left(1-\frac{1}{\beta^2\,\epsilon}\right)\sqrt{\frac{\lambda^*}{\lambda}}\ e^{-(\lambda+\lambda^*)\,a} \qquad (13.77)$$

From

$$\lambda^2 = \left(\frac{\omega}{c}\right)^2 \epsilon\left(\frac{1}{\beta^2\,\epsilon} - 1\right)$$

we see that for ϵ real (no absorption), λ is imaginary if

$$\beta^2\,\epsilon > 1$$

ie

$$v > \frac{c}{\sqrt{\epsilon}} \qquad (13.78)$$

Writing

$$\lambda = -i\ |\,\lambda\,|$$

we have

$$\frac{\lambda^*}{\lambda} = -1 \qquad\qquad \sqrt{\frac{\lambda^*}{\lambda}} = \mathrm{i}$$

$$\lambda\ + \lambda^* = 0$$

so that the portion of (13.77) which satisfies $\beta^2\,\epsilon > 1$ becomes

$$\left(\frac{d\,E}{d\,x}\right)_{\mathrm{CR}} \simeq \left(\frac{z\,e}{c}\right)^2 \int_{\epsilon(\omega)>(1/\beta^2)} d\,\omega\ \omega\left(1-\frac{1}{\beta^2\,\epsilon}\right) \qquad (13.79)$$

This is independent of a (ie., observable very far away from the incident particle) and explains the radiation observed first by Cherenkov.

Since the only non-zero components of the fields are E_1, E_2, and B_3, we have

$$\boldsymbol{E}\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ E_1 & E_2 & 0 \\ 0 & 0 & B_3 \end{vmatrix} = E_2\,B_3\,\boldsymbol{e}_1 - E_1\,B_3\,\boldsymbol{e}_2$$

where $\boldsymbol{e}_1$ is the direction of the particle motion.

Direction of the propagation of the radiation is given by the Poynting vector

$$\boldsymbol{S} = \frac{c}{4\,\pi}\,\boldsymbol{E}\times\boldsymbol{B}$$

and hence makes an angle

$$\theta_c = \tan^{-1}\left(\frac{-E_1\,B_3}{E_2\,B_3}\right) = -\tan^{-1}\frac{E_1}{E_2} \qquad (13.80)$$

with respect to $\boldsymbol{e}_1$, the direction of the particle motion. (see figs.13.7-8)

From the far fields (13.76) we find

$$\frac{E_1}{E_2} \simeq i\,\frac{v\,\omega\,\epsilon}{c^2}\left(1-\frac{1}{\beta^2\,\epsilon}\right)\frac{1}{\lambda}$$

$$= \quad \frac{v\,\omega\,\epsilon}{c^2}\left(1-\frac{1}{\beta^2\,\epsilon}\right)\frac{1}{|\ \lambda\ |}$$

Using

$$\lambda = -i\left(\frac{\omega}{c}\right)\sqrt{\epsilon - \frac{1}{\beta^2}}$$

we have

$$\frac{E_1}{E_2} \simeq \beta\,\sqrt{\epsilon - \frac{1}{\beta^2}} = \sqrt{\beta^2\,\epsilon - 1}$$

$$\cos\theta_c = \frac{1}{\sqrt{1+\tan^2\theta_c}} = \frac{1}{\beta\,\sqrt{\epsilon(\omega)}} \qquad (13.81)$$

Using

$$\boldsymbol{A}(\boldsymbol{k},\,\omega) = \frac{8\,\pi^2}{c}\,z\,e\,\boldsymbol{v}\,\frac{\delta(\omega-\boldsymbol{k}\,\cdot\,\boldsymbol{v})}{k^2\,-\,\frac{\omega^2}{c^2}\,\epsilon(\omega)} \qquad (13.57)$$

we have

$$\boldsymbol{A}(\boldsymbol{x},\,t) = \int\frac{d^3\,k}{(2\,\pi)^3}\int\frac{d\,\omega}{2\,\pi}\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}-i\,\omega\,t}\,\boldsymbol{A}(\boldsymbol{k},\,\omega)$$

$$= \int\frac{d^3\,k}{(2\,\pi)^3}\int\frac{d\,\omega}{2\,\pi}\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}-i\,\omega\,t}\,\frac{8\,\pi^2}{c}\,z\,e\,\boldsymbol{v}\,\frac{\delta(\omega-\boldsymbol{k}\,\cdot\,\boldsymbol{v})}{k^2\,-\,\frac{\omega^2}{c^2}\,\epsilon(\omega)}$$

$$= \frac{z\,e\,\boldsymbol{v}}{2\,\pi^2\,c}\int d^3\,k\;e^{i\,\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{v}\,t)}\,\frac{1}{k^2\,-\,\frac{(\boldsymbol{k}\cdot\boldsymbol{v})^2}{c^2}\,\epsilon}$$

where ϵ is evaluated at $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$.

Let

$$\boldsymbol{v} = v\,\hat{\boldsymbol{e}}_1$$

$$\boldsymbol{x} = x\,\hat{\boldsymbol{e}}_1 + \boldsymbol{\rho} \qquad \text{with} \qquad \hat{\boldsymbol{e}}_1\cdot\boldsymbol{\rho} = 0$$

$$\boldsymbol{k} = k_1\ \hat{\boldsymbol{e}}_1 + \boldsymbol{k}_\perp \qquad\qquad \hat{\boldsymbol{e}}_1\cdot\boldsymbol{k}_\perp = 0$$

we have

$$\boldsymbol{k}\,\cdot\,\boldsymbol{v}\ = k_1\,v$$

$$\boldsymbol{k}\cdot(\,\boldsymbol{x}\ -\boldsymbol{v}\,t\,) = k_1\,(\,x - v\,t\,) + \boldsymbol{k}_\perp\cdot\,\boldsymbol{\rho}$$

$$k^2 - \frac{(\boldsymbol{k}\cdot\boldsymbol{v})^2}{c^2}\epsilon = k_1{}^2 + k_\perp{}^2 - \left(\frac{k_1 v}{c}\right)^2 \epsilon$$
$$= k_1{}^2\left(1-\beta^2\epsilon\right) + k_\perp{}^2$$

so that

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{2ze}{(2\pi)^2}\boldsymbol{\beta}\int d^3k \frac{e^{ik_1(x-vt)}e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{\rho}}}{k_1{}^2\left(1-\beta\epsilon^2\right)+k_\perp{}^2}$$
$$= \frac{2ze}{(2\pi)^2}\boldsymbol{\beta} I$$

where

$$I = \int d^3k \frac{e^{ik_1(x-vt)}e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{\rho}}}{k_1{}^2\left(1-\beta\epsilon^2\right)+k_\perp{}^2}$$
$$= \int_{-\infty}^{\infty} dk_1 \int_0^{\infty} k_\perp dk_\perp \int_0^{2\pi} d\phi \frac{e^{ik_1(x-vt)}e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{\rho}}}{k_1{}^2\left(1-\beta\epsilon^2\right)+k_\perp{}^2}$$

For the k_1 integral, if we neglect the k_1 dependence of ϵ, poles are at

$$k_\pm = \pm\frac{k_\perp}{\sqrt{\beta\epsilon^2-1}}$$

In the Cherenkov regime,

$$\beta\epsilon^2 > 1$$

so that $k_\pm$ are real and on the k_1 axis. This means I is improper and can only be calculated by limiting processes.

We displace the integration path infinitesimally into the upper plane.
Hence, for $x - vt > 0$, the path must be closed in the upper plane.
For $x - vt < 0$, it must be closed in the lower plane.
Let

$$J = \int_{-\infty}^{\infty} dk_1 \frac{e^{ik_1(x-vt)}}{k_1{}^2\left(1-\beta\epsilon^2\right)+k_\perp{}^2}$$

we have

$$J = 0 \qquad \text{for} \qquad x - vt > 0$$

For $x - vt < 0$, the contour goes in the clockwise sense so that

$$J = -2\pi i \sum_{k_1=k_\pm} \frac{e^{ik_1(x-vt)}}{2k_1\left(1-\beta\epsilon^2\right)}$$
$$= \frac{\pi i}{\sqrt{\beta\epsilon^2-1}\,k_\perp}\left[e^{i\frac{k_\perp(x-vt)}{\sqrt{\beta\epsilon^2-1}}} - e^{-i\frac{k_\perp(x-vt)}{\sqrt{\beta\epsilon^2-1}}}\right]$$

and

$$I = \int_0^\infty k_\perp\, d\,k_\perp \int_0^{2\pi} d\,\phi\; e^{i\,\mathbf{k}_\perp\cdot\boldsymbol{\rho}}\, J$$

$$= \pi\, i \int_0^\infty k_\perp\, d\,k_\perp \int_0^{2\pi} d\,\phi\; \frac{e^{i\,\mathbf{k}_\perp\cdot\boldsymbol{\rho}}}{\sqrt{\beta\,\epsilon^2 - 1}\; k_\perp} \left[e^{i\,\frac{k_\perp (x - v\,t)}{\sqrt{\beta\,\epsilon^2 - 1}}} - e^{-i\,\frac{k_\perp (x - v\,t)}{\sqrt{\beta\,\epsilon^2 - 1}}} \right]$$

$$= \int_0^{2\pi} d\,\phi\; K$$

where

$$K = \pi\, i \int_0^\infty k_\perp\, d\,k_\perp\, \frac{e^{i\,\mathbf{k}_\perp\cdot\boldsymbol{\rho}}}{\sqrt{\beta\,\epsilon^2 - 1}\; k_\perp} \left[e^{i\,\frac{k_\perp (x - v\,t)}{\sqrt{\beta\,\epsilon^2 - 1}}} - e^{-i\,\frac{k_\perp (x - v\,t)}{\sqrt{\beta\,\epsilon^2 - 1}}} \right]$$

$$= \frac{\pi\, i}{\sqrt{\beta\,\epsilon^2 - 1}} \int_0^\infty d\,k_\perp\, e^{i\,k_\perp\,\rho\cos\phi} \left[e^{i\,\frac{k_\perp (x - v\,t)}{\sqrt{\beta\,\epsilon^2 - 1}}} - e^{-i\,\frac{k_\perp (x - v\,t)}{\sqrt{\beta\,\epsilon^2 - 1}}} \right]$$

$$= \frac{\pi\, i}{\sqrt{\beta\,\epsilon^2 - 1}} \left[-\frac{1}{i\left(\rho\cos\phi + \frac{x - v\,t}{\sqrt{\beta\,\epsilon^2 - 1}}\right)} + \frac{1}{i\left(\rho\cos\phi - \frac{x - v\,t}{\sqrt{\beta\,\epsilon^2 - 1}}\right)} \right]$$

$$= \pi \left[-\frac{1}{\sqrt{\beta\,\epsilon^2 - 1}\;\rho\cos\phi + x - v\,t} + \frac{1}{\sqrt{\beta\,\epsilon^2 - 1}\;\rho\cos\phi - (x - v\,t)} \right]$$

$$= \frac{2\,\pi\,(x - v\,t)}{\left(\beta\,\epsilon^2 - 1\right)\rho^2\cos^2\phi - (x - v\,t)^2}$$

Hence

$$I = \int_0^{2\pi} d\,\phi\, \frac{2\,\pi\,(x - v\,t)}{\left(\beta\,\epsilon^2 - 1\right)\rho^2\cos^2\phi - (x - v\,t)^2}$$

$$= 4 \int_0^{\pi/2} d\,\phi\, \frac{2\,\pi\,(x - v\,t)}{\left(\beta\,\epsilon^2 - 1\right)\rho^2\cos^2\phi - (x - v\,t)^2}$$

$$= 8\,\pi\,(v\,t - x) \int_0^{\pi/2} d\,\phi\, \frac{1}{(v\,t - x)^2 - \left(\beta\,\epsilon^2 - 1\right)\rho^2\cos^2\phi}$$

Using

$$\int \frac{d\,\phi}{a + b\cos^2\phi} = \frac{1}{\sqrt{a\,(a + b)}} \tan^{-1}\left[\frac{\sqrt{a\,(a + b)}}{a + b} \tan\phi \right]$$

where

$\sqrt{a(a+b)}$ is real

with

$$a = (v\,t - x)^2 \qquad b = -\left(\beta\,\epsilon^2 - 1\right)\rho^2$$
$$a + b = (x - v\,t)^2 - \left(\beta\,\epsilon^2 - 1\right)\rho^2$$
$$\sqrt{a(a+b)} = (v\,t - x)\sqrt{(v\,t - x)^2 - \left(\beta\,\epsilon^2 - 1\right)\rho^2}$$

we have

$$I = 8\pi \frac{1}{\sqrt{(v\,t - x)^2 - \left(\beta\,\epsilon^2 - 1\right)\rho^2}} \frac{\pi}{2}$$
$$= \frac{4\pi^2}{\sqrt{(v\,t - x)^2 - \left(\beta\,\epsilon^2 - 1\right)\rho^2}}$$

Hence, for $v\,t > x$,

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{2\,z\,e}{(2\pi)^2}\boldsymbol{\beta}\,I$$
$$= \boldsymbol{\beta}\frac{2\,z\,e}{\sqrt{(x - v\,t)^2 - \left(\beta^2\,\epsilon - 1\right)\rho^2}} \qquad (13.82)$$

for $v\,t < x$,

$$\boldsymbol{A}(\boldsymbol{x}, t) = 0$$

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6. Electronic Plasma

See sections 10.8-9 for basic properties of electronic plasma.

The debye wave number is defined as

$$k_D = \frac{\omega_p}{\sqrt{\langle u^2 \rangle}} = \frac{\omega_p}{v_{\text{th}}} \qquad (10.91)$$

where

$$v_{\text{th}} = \sqrt{\langle u^2 \rangle}$$

is the thermal rms velocity.

For distances $b < k_D^{-1}$, individual- particle behavior dominates (screened potential). We have (Prob 13.3)

$$\Delta E \simeq \frac{2\,z^2\,e^4}{m\,v^2} k_D^2\,K_1^2(k_D\,b)$$
$$\left(\frac{d\,E}{d\,x}\right)_{b\,k_D < 1} \simeq \left(\frac{z\,e}{v}\right)^2 \omega_p^2 \ln\left(\frac{1}{1.47\,k_D\,b_{\min}}\right)$$

For distances $b > k_D^{-1}$, the plasma acts as a continuous medium (plasma oscillations). (13.67) becomes

$$\left(\frac{dE}{dx}\right)_{k_D b>1} \simeq \left(\frac{2ze}{v}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\omega}{2\pi}\, i\omega\, \frac{\lambda^*}{k_D}\, K_1\!\left(\frac{\lambda^*}{k_D}\right) K_0\!\left(\frac{\lambda}{k_D}\right)\left(\frac{1}{\epsilon}-\beta^2\right)$$

For non-relativistic motion

$$\lambda = \frac{\omega}{v}\sqrt{1-\beta^2\epsilon} \simeq \frac{\omega}{v}$$

so that

$$\left(\frac{dE}{dx}\right)_{k_D b>1} \simeq \left(\frac{2ze}{v}\right)^2 \mathrm{Re}\int_0^\infty \frac{d\omega}{2\pi}\, i\omega\, \frac{\omega}{k_D v}\, K_1\!\left(\frac{\omega}{k_D v}\right) K_0\!\left(\frac{\omega}{k_D v}\right)\left(\frac{1}{\epsilon}\right) \qquad (13.83)$$

where the term β^2 is dropped.

For $\omega \sim \omega_p$, we have

$$\epsilon(\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2 + i\omega\Gamma} \qquad (7.59)$$

so that $\epsilon(\omega_p) \simeq 0$ as required for longitudinal plasma oscikkations.

Hence

$$\frac{1}{\epsilon} = \frac{\omega^2 + i\omega\Gamma}{\omega^2 - \omega_p^2 + i\omega\Gamma}$$

$$= \frac{(\omega^2 + i\omega\Gamma)(\omega^2 - \omega_p^2 - i\omega\Gamma)}{(\omega^2 - \omega_p^2)^2 + (\omega\Gamma)^2}$$

$$= \frac{\omega^2(\omega^2 - \omega_p^2 + \Gamma^2) - i\omega\Gamma\,\omega_p^2}{(\omega^2 - \omega_p^2)^2 + (\omega\Gamma)^2}$$

$$\mathrm{Re}\left(\frac{i}{\epsilon}\right) = \frac{\omega\Gamma\,\omega_p^2}{(\omega^2 - \omega_p^2)^2 + (\omega\Gamma)^2}$$

$$\underset{\Gamma\to 0}{=} \frac{\pi\,\omega_p}{2}\left[\delta(\omega_p - \omega) - \delta(\omega_p + \omega)\right]$$

Therefore,

$$\left(\frac{dE}{dx}\right)_{k_D b>1} \simeq \left(\frac{ze}{v}\right)^2 \frac{\omega_p^3}{k_D v}\, K_1\!\left(\frac{\omega_p}{k_D v}\right) K_0\!\left(\frac{\omega_p}{k_D v}\right)$$

$$= \left(\frac{ze}{v}\right)^2 \omega_p^2\, \frac{v_{\mathrm{th}}}{v}\, K_1\!\left(\frac{v_{\mathrm{th}}}{v}\right) K_0\!\left(\frac{v_{\mathrm{th}}}{v}\right)$$

Thus, for $v \ll v_{\mathrm{th}}$

$$\left(\frac{dE}{dx}\right)_{k_D b>1} \propto e^{-2 v_{\mathrm{th}}/v}$$

and is negligible.

For $v \gg v$

$$\left(\frac{dE}{dx}\right)_{k_D b>1} \simeq \left(\frac{ze}{v}\right)^2 \omega_p^2 \ln\!\left(\frac{1.123\,v}{v_{\mathrm{th}}}\right) \qquad (13.87)$$

$$= \left(\frac{ze}{v}\right)^2 \omega_p^2 \ln\!\left(\frac{1.123\,k_D v}{\omega_p}\right)$$

Hence

$$\left(\frac{dE}{dx}\right)_{\text{total}} = \left(\frac{dE}{dx}\right)_{k_D b>1} + \left(\frac{dE}{dx}\right)_{k_D b<1}$$

$$\simeq \left(\frac{ze}{v}\right)^2 \omega_p{}^2 \ln\left(\frac{1.123\, k_D\, v}{\omega_p} \frac{1}{1.47\, k_D\, b_{\min}}\right)$$

$$\simeq \left(\frac{ze}{v}\right)^2 \omega_p{}^2 \ln\left(\frac{\Lambda\, v}{\omega_p\, b_{\min}}\right) \qquad (13.88)$$

where $\Lambda = \frac{1.123}{1.47}$.

Quantum mechanically, the presence of ω_p in the argument of the logarithm may be interpreted as quantization of the plasma energy in units of $\hbar\,\omega_p$.

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7. Scattering by Atoms

■ General consideration

Consider the (non-relativistic) scattering problem where
mass M moves with $\boldsymbol{V}$ is incident on stationary mass m.
Afterwards,
M moves with $\boldsymbol{V}'$ and m with $\boldsymbol{v}'$.

Conservation of momentum:

$$M\,(\boldsymbol{V} - \boldsymbol{V}') = m\,\boldsymbol{v}' \equiv \Delta\,\boldsymbol{p} \equiv -M\,\Delta\,\boldsymbol{V}$$

Conservation of energy:

$$\frac{1}{2} M\left(V^2 - V'^2\right) = \frac{1}{2} m\, v'^2 \equiv \Delta E = \frac{1}{2m} (\Delta\,\boldsymbol{p})^2$$

$$= \frac{1}{2} \frac{M^2}{m} (\Delta\,\boldsymbol{V})^2$$

$$= \frac{M}{m} \frac{1}{2} M (\Delta\,\boldsymbol{V})^2$$

Thus, for a given $\Delta\,\boldsymbol{V}$, the energy transfer ΔE is larger if $m < M$.
That is, energy loss is great if target is lighter than incident particle.

Conversely, for given amount of ΔE, $\Delta\,\boldsymbol{V}$ is greatest if $m = M$.
That is, deflection of the path of incident particle is greatest if the masses of target & incident particle are the same.

■ Rutherford Scattering

According to (13.3), a particle $z\,e$ with $p = \gamma\, M\, v$, passing nucleus Ze at b, will suffer an angular deflection,

$$\theta \simeq \frac{2\, z\, Z\, e^2}{p\, v\, b} \qquad (13.89)$$

The differential scattering cross section $d\,\sigma / d\,\Omega$ is defined by,

$$n\, b\, d\, b\, d\,\phi = n \frac{d\,\sigma}{d\,\Omega} \sin\theta\, d\,\theta\, d\,\phi \qquad (13.90)$$

where n is the number of particles incident on the atom per unit area per unit time. Thus

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{13.91}$$

(13.89) gives

$$d\theta = -\frac{2 z Z e^2}{p v b^2} db$$

or

$$\frac{db}{d\theta} = -\frac{p v}{2 z Z e^2} b^2 = -\frac{p v}{2 z Z e^2}\left(\frac{2 z Z e^2}{p v \theta}\right)^2$$
$$= -\frac{2 z Z e^2}{p v}\frac{1}{\theta^2}$$

For small θ,

$$\frac{d\sigma}{d\Omega} \simeq \frac{b}{\theta}\left|\frac{db}{d\theta}\right|$$
$$\simeq \frac{2 z Z e^2}{p v \theta^2}\frac{2 z Z e^2}{p v}\frac{1}{\theta^2}$$
$$\simeq \left(\frac{2 z Z e^2}{p v}\right)^2 \frac{1}{\theta^4} \tag{13.92}$$

which is also true quantum mechanically.

Similarly, the Z electrons in the atom contributes

$$\left(\frac{d\sigma}{d\Omega}\right)_{Ze} \simeq Z\left(\frac{2 z e^2}{p v}\right)^2 \frac{1}{\theta^4} = \frac{1}{Z}\left(\frac{d\sigma}{d\Omega}\right)$$

which, for large Z, can be neglected.

For large θ,

$$\frac{d\sigma}{d\Omega} = \left(\frac{z Z e^2}{2 M v^2}\right)^2 \operatorname{cosec}^4\frac{\theta}{2} \tag{13.93}$$

which follows from (13.4), holds quantum mechanically as well.

■ Screening (Large *b*) Correction

Charge of the nucleus are screened by the atomic electrons. In the Fermi- Thomas model:

$$V(r) \simeq \frac{z Z e^2}{r} e^{-r/a} \tag{13.94}$$

where the atomic radius is

$$a \simeq 1.4\, a_0 Z^{-1/3} \tag{13.95}$$
$$a_0 = \frac{\hbar^2}{m e^2} = \text{hydrogenic Bohr radius}$$

Thus, for $b > a$, $\frac{d\sigma}{d\Omega} \simeq 0$ except for $\theta \simeq 0$.

This is equivalent to setting

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2 z Z e^2}{p v}\right)^2 \frac{1}{\left(\theta^2 + \theta_{min}{}^2\right)^2} \tag{13.96}$$

where θ_{min} is a cutoff angle.

Classically $\theta_{\min}$ can be estimated by putting $b = a$ in (13.89). This gives

$$\theta_{\min}{}^{(c)} \simeq \frac{2\, z\, Z\, e^2}{p\, v\, a} \qquad (13.97)$$

Quantum mechanically, the finite size of the scatterer implies that the approximately classical trajectory must be localized to within $\Delta x < a$; the incident particle must have a minimum uncertainty in transverse momentum $\Delta p \gtrsim \hbar / a$. Using $\Delta p \simeq p\, \theta$, we have

$$\theta_{\min}{}^{(q)} \simeq \frac{\hbar}{p\, a} \qquad (13.98)$$

Note that

$$\frac{\theta_{\min}{}^{(c)}}{\theta_{\min}{}^{(q)}} = \frac{2\, z\, Z\, e^2}{\hbar\, v} = \eta$$

For most fast particles, $\eta < 1$ so that $\theta_{\min}{}^{(q)}$ is used.

Using (13.95) for the screening radius a, (13.98) becomes

$$\theta_{\min}{}^{(q)} \simeq \frac{\hbar}{1.4\, a_0\, Z^{-1/3}\, p} \qquad a_0 = \frac{\hbar^2}{m\, e^2}$$

$$\simeq \frac{m\, e^2}{1.4\, \hbar\, Z^{-1/3}\, p}$$

$$\simeq \frac{Z^{1/3}\, m\, c}{1.4\;\; p} \frac{e^2}{\hbar\, c} \qquad \frac{e^2}{\hbar\, c} = \frac{1}{137}$$

$$\simeq \frac{Z^{1/3}}{192} \left(\frac{m\, c}{p} \right) \qquad (13.99)$$

where p is the incident momentum ($p = \gamma\, M\, v$), and m is the electronic mass.

■ Finite Size (Large θ) Correction

At comparatively large angles the cross section departs from (13.92) because of the finite size of the nucleus. We will consider only the electromagnetic aspect.

Assuming uniform charge distribution inside a sphere of radius R, we have

$$V(r) = \begin{cases} \frac{3}{2} \frac{z\, Z\, e^2}{R} \left(1 - \frac{r^2}{3\, R^2} \right) & \text{for } r < R \\ \frac{z\, Z\, e^2}{r} & r > R \end{cases} \qquad (13.100)$$

The de Broglie wavelength of the incident particle is $\lambda = \hbar / p$. For diffraction of waves by a spherical object (see Chapter 9), the scattering is all confined to angles less than $\sim (\lambda / R)$. Defining

$$\theta_{\max} \simeq \frac{\hbar}{p\, R} \qquad (13.101)$$

the nucleus size effect is therefore to restrict (13.92) to $\theta < \theta_{\max}$.

Now

$$R \simeq \frac{1}{2} \frac{e^2}{m c^2} A^{1/3} \simeq 1.4 \times 10^{-13} A^{1/3} \text{ cm}$$

so that

$$\theta_{\max} \simeq \frac{274}{A^{1/3}} \frac{m c}{p}$$

Since

$$A < 3 Z, \qquad Z < 110$$

$$\theta_{\min}^{q} = \frac{Z^{1/3}}{192} \frac{m c}{p}$$

we have

$$\frac{\theta_{\max}}{\theta_{\min}^{q}} \simeq \frac{244 \times 192}{(A Z)^{1/3}} > 10^{4-\frac{4}{3}} \gg 1$$

The general behavior of the cross section is shown in Fig. 13.9.

The total scattering cross section is

$$\begin{aligned} \sigma &= \int \frac{d\sigma}{d\Omega} \sin\theta \, d\theta \, d\phi \\ &\simeq 2\pi \left(\frac{2 z Z e^2}{p v} \right)^2 \int_0^\infty \frac{\theta \, d\theta}{\left(\theta^2 + \theta_{\min}{}^2\right)^2} \end{aligned} \qquad (13.103)$$

This yields

$$\sigma \simeq \pi \left(\frac{2 z Z e^2}{p v} \right)^2 \frac{1}{\theta_{\min}{}^2} = \pi a^2 \left(\frac{2 z Z e^2}{\hbar v} \right)^2 \qquad (13.104)$$

Thus, at high velocities the total cross section can be far smaller than the classical value of geometrical area πa^2.

8. Angular Distribution

Rutherford scattering favors small θ ($\frac{d\sigma}{d\Omega} \propto \frac{1}{\theta^4}$).

$$\longrightarrow \quad \begin{pmatrix} \text{large} \\ \text{intermediate} \\ \text{small} \end{pmatrix} \text{ final } \theta \text{ is due to } \begin{pmatrix} \text{single} \\ \text{plural} \\ \text{multiple} \end{pmatrix} \text{scattering}$$

The mean square angle for a single scattering is defined by

$$\langle\theta^2\rangle = \frac{\int \theta^2 \frac{d\sigma}{d\Omega}\, d\Omega}{\int \frac{d\sigma}{d\Omega}\, d\Omega} \qquad (13.105)$$

For

$$\frac{d\sigma}{d\Omega} = \frac{A}{\theta^4} \qquad \& \qquad \int d\Omega = 2\pi \int \sin\theta\, d\theta = \int_{\theta_{\min}}^{\theta_{\max}} \theta\, d\theta$$

we have

$$\int \frac{d\sigma}{d\Omega}\, d\Omega = 2\pi A \int_{\theta_{\min}}^{\theta_{\max}} d\theta \frac{1}{\theta^3}$$

$$= -\pi A \left(\frac{1}{\theta_{\max}{}^2} - \frac{1}{\theta_{\min}{}^2} \right)$$

$$\simeq \frac{\pi A}{\theta_{\min}{}^2} \qquad \text{for} \qquad \theta_{\max} \gg \theta_{\min}$$

$$\int \theta^2 \left(\frac{d\sigma}{d\Omega} \right) d\Omega = 2\pi A \int_{\theta_{\min}}^{\theta_{\max}} d\theta \frac{1}{\theta}$$

$$= 2\pi A \ln \frac{\theta_{\max}}{\theta_{\min}}$$

so that

$$\langle\theta^2\rangle = 2\,\theta_{\min}{}^2 \ln\left(\frac{\theta_{\max}}{\theta_{\min}} \right) \qquad (13.106)$$

Using $\theta_{\min}^q$ and $A \simeq 2Z$, we have

$$\langle\theta^2\rangle \simeq 4\,\theta_{\min}{}^2 \ln\left(204\, Z^{-1/3} \right) \qquad (13.107)$$

If nuclear size is unimportant (generally only of interest for electrons, and perhaps other particles at very low energies), $\theta_{\max}$ should be put equal to unity in (13.106). Then the argument of the logarithm in (13.107) becomes $\left(\frac{192}{Z^{1/3}} \frac{p}{m c} \right)^{1/2}$, instead of

$\left(204\, Z^{-1/3}\right)$.

It is often desirable to use the projected angle of scattering θ', the projection being made on some convenient plane such as the plane of a photographic emulsion or a bubble chamber, as shown in Fig. 13.10. For small angles it is easy to show that

$$\left\langle \theta'^2 \right\rangle = \frac{1}{2} \left\langle \theta^2 \right\rangle \qquad (13.108)$$

In each collision the angular deflections obey the Rutherford formula (13.92) suitably cut off at θ_{min} and θ_{max} with average value zero (when viewed relative to the forward direction, or as a projected angle) and mean square angle $\left\langle \theta^2 \right\rangle$ given by (13.106). Since the successive collisions are independent events, the central limit theorem of statistics can be used to show that for a large number n of such collisions the distribution in angle will be approximately Gaussian around the forward direction with a mean square angle $\left\langle \Theta^2 \right\rangle = n \left\langle \theta^2 \right\rangle$. The number of collisions occurring as the particle traverses a thickness t of material containing N atoms per unit volume is

$$n = N\, \sigma\, t \simeq \pi\, N \left(\frac{2\, z\, Z\, e^2}{p\, v} \right)^2 \frac{1}{\theta_{\text{min}}^{\ 2}} \qquad (13.109)$$

This means that the mean square angle of the Gausssian is

$$\left\langle \Theta^2 \right\rangle \simeq 2\, \pi\, N \left(\frac{2\, z\, Z\, e^2}{p\, v} \right)^2 \ln\left(\frac{\theta_{\text{max}}}{\theta_{\text{min}}} \right) t \qquad (13.110)$$

Or, using (13.107) for $\left\langle \theta^2 \right\rangle$,

$$\left\langle \Theta^2 \right\rangle \simeq 4\, \pi\, N \left(\frac{2\, z\, Z\, e^2}{p\, v} \right)^2 \ln\left(204\, Z^{-1/3} \right) t \qquad (13.111)$$

The mean square angle increases linearly with the thickness t. But for reasonable thicknesses such that the particle does not lose appreciable energy, the Gaussian will still be peaked at very small forward angles.

The multiple- scattering distribution for the projected angle of scattering is

$$P_M(\theta')\, d\theta' = \frac{1}{\sqrt{\pi \left\langle \Theta^2 \right\rangle}}\, e^{-\frac{\theta'^2}{\left\langle \Theta^2 \right\rangle}}\, d\theta' \qquad (13.112)$$

where both positive and negative values of θ' are considered. The small- angle Rutherford formula (13.92) can be expressed in terms of the projected angle as

$$\frac{d\sigma}{d\theta'} = \frac{\pi}{2} \left(\frac{2\, z\, Z\, e^2}{p\, v} \right)^2 \frac{1}{\theta'^3} \qquad (13.113)$$

This gives a single- scattering distribution for the projected angle:

$$P_S(\theta')\, d\theta' = N\, t\, \frac{d\sigma}{d\theta'}\, d\theta' = \frac{\pi}{2} N\, t \left(\frac{2\, z\, Z\, e^2}{p\, v} \right)^2 \frac{d\theta'}{\theta'^3} \qquad (13.114)$$

The single- scattering distribution is valid only for angles large compared to $\left\langle \Theta^2 \right\rangle^{1/2}$, and contributes a tail to the Gaussian distribution.

If we express angles in terms of the relative projected angle,

$$\alpha = \frac{\theta '}{\langle \Theta^2 \rangle^{1/2}} \tag{13.115}$$

the multiple- and single- scattering distribution can be written

$$P_M(\alpha)\, d\,\alpha = \frac{1}{\sqrt{\pi}}\, e^{-\alpha^2}\, d\,\alpha$$

$$P_S(\sigma)\, d\,\alpha = \frac{1}{8\ \ln\left(204\, Z^{-1/3}\right)}\, \frac{d\,\alpha}{\alpha^3} \tag{13.116}$$

Thickness effect:

for t large, the Gaussian P_m broadens & overshadows P_S.