15
Bremsstrahlung, Method of Virtual Quanta, Radiative Beta Processes

1. Collision Radiation

Collision ⟶ Acceleration ⟶ Radiation
$P \propto a^2$ ⟶ light particle emit more.

■ (a) Low Frequency Limit

From

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{z^2 e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt\, e^{i\omega\{t - \boldsymbol{n}\cdot\boldsymbol{r}(t)/c\}} \frac{\boldsymbol{n}\times\left[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^2} \right|^2 \tag{14.65}$$

$$\frac{\boldsymbol{n}\times\left[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^2} = \frac{d}{dt}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right] \tag{14.66}$$

we have

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{z^2 e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt\, e^{i\omega\{t - \boldsymbol{n}\cdot\boldsymbol{r}(t)/c\}} \frac{d}{dt}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right] \right|^2 \tag{15.1}$$

For $\omega \to 0$,

$$\lim_{\omega\to 0} \frac{d^2 I}{d\omega\, d\Omega} = \frac{z^2 e^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt\, \frac{d}{dt}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right] \right|^2$$

$$= \frac{z^2 e^2}{4\pi^2 c} \left| \left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right]_0^\tau \right|^2 \qquad \tau = \text{collision time}$$

$$= \frac{z^2 e^2}{4\pi^2 c} \left| \frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta}')}{1-\boldsymbol{\beta}'\cdot\boldsymbol{n}} - \frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}} \right|^2$$

where $c\boldsymbol{\beta}'$ is the final and $c\boldsymbol{\beta}$ the initial velocity of the particle.
Note that the details of collision is not relevant.

Let $\boldsymbol{\epsilon}$ be the polarization of the radiation

$$\boldsymbol{\epsilon}^*\cdot\boldsymbol{n} = 0$$

$$\lim_{\omega\to 0} \frac{d^2 I}{d\omega\, d\Omega} = \frac{z^2 e^2}{4\pi^2 c} \left| \left[\boldsymbol{\epsilon}^*\cdot\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right]_0^\tau \right|^2$$

Using

$$\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta}) = \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\beta}) - \boldsymbol{\beta}$$

we have

$$\boldsymbol{\epsilon}^*\cdot[\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})] = (\boldsymbol{\epsilon}^*\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{\beta}) - \boldsymbol{\epsilon}^*\cdot\boldsymbol{\beta} = -\boldsymbol{\epsilon}^*\cdot\boldsymbol{\beta}$$

Hence

$$\lim_{\omega\to 0} \frac{d^2 I}{d\omega\, d\Omega} = \frac{z^2 e^2}{4\pi^2 c} \left| \left[-\frac{\boldsymbol{\epsilon}^*\cdot\boldsymbol{\beta}}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right]_0^\tau \right|^2$$

$$= \frac{z^2 e^2}{4\pi^2 c} \left| \boldsymbol{\epsilon}^*\cdot\left(\frac{\boldsymbol{\beta}'}{1-\boldsymbol{\beta}'\cdot\boldsymbol{n}} - \frac{\boldsymbol{\beta}}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right) \right|^2 \tag{15.2}$$

For quantized radiation energy

$$d\,I = \hbar\,\omega\,d\,N$$

so that

$$\frac{d^2\,I}{d\,\omega\,d\,\Omega} = \hbar\,\omega\left(\frac{d^2\,N}{d\,\omega\,d\,\Omega_\gamma}\right) = \hbar^2\,\omega\left(\frac{d^2\,N}{d\,(\,\hbar\,\omega\,)\,d\,\Omega_\gamma}\right)$$

Here,

$d\,N$ = number of photons of frequency ω

$\dfrac{d^2\,N}{d\,(\,\hbar\,\omega\,)\,d\,\Omega_\gamma}$ = number spectrum per energy per solid angle of the photon γ.

Hence

$$\lim_{\hbar\,\omega\to 0}\frac{d^2\,N}{d\,(\hbar\,\omega)\,d\,\Omega_\gamma} = \frac{z^2\,\alpha}{4\,\pi^2\,\hbar\,\omega}\left|\boldsymbol{\epsilon}^*\cdot\left(\frac{\boldsymbol{\beta}\,'}{1-\boldsymbol{\beta}\,'\cdot\boldsymbol{n}} - \frac{\boldsymbol{\beta}}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right)\right|^2 \quad (15.3)$$

where

$$\alpha = \frac{e^2}{\hbar\,c} \simeq \frac{1}{137} = \text{ fine structure constant}$$

Let

$\dfrac{d\,\sigma}{d\,\Omega_p}$ = cross section for particle scattering ($c\,\beta \to c\,\beta\,'$)

then

$$\frac{d^3\,\sigma}{d\,\Omega_p\,d\,(\hbar\,\omega)\,d\,\Omega_\gamma} = \left[\lim_{\hbar\,\omega\to 0}\frac{d^2\,N}{d\,(\hbar\,\omega)\,d\,\Omega_\gamma}\right]\frac{d\,\sigma}{d\,\Omega_p} \quad (15.4)$$

Let 4-momentum of photon be

$$\hbar\,k^\mu = \hbar\left(\frac{\omega}{c},\,\boldsymbol{k}\right) = \frac{\hbar\,\omega}{c}\,(\,1,\,\boldsymbol{n}\,) \qquad \text{ie.} \qquad \boldsymbol{k} = k_0\,\boldsymbol{n} = \frac{\omega}{c}\,\boldsymbol{n}$$

and 4-momentum of particle be

$$p^\mu = M\,c\,\gamma\,(\,1,\,\boldsymbol{\beta}\,)$$

As shown at the end of this section, $\dfrac{d^3\,k}{k_0}$ is Lorentz invariant. Since N is a scalar, the number spectrum that is Lorentz invariant is $\dfrac{d^3\,N}{\left(d^3\,k/k_0\right)}$. Now

$$\frac{d^3\,k}{k_0} = \frac{k^2}{k_0}\,d\,k\,d\,\Omega_\gamma$$

$$= \frac{1}{c^2}\,\hbar\,\omega\;d\,(\hbar\,\omega)\,d\,\Omega_\gamma$$

where $k = |\,\boldsymbol{k}\,| = k_0 = \dfrac{\hbar\,\omega}{c}$.

Therefore

$$\frac{d^3\,N}{\left(d^3\,k/k_0\right)} = \frac{c^2}{\hbar\,\omega}\left(\frac{d^2\,N}{d\,(\hbar\,\omega)\,d\,\Omega_\gamma}\right) = \frac{c^2}{\hbar\,(\hbar\,\omega)^2}\left(\frac{d^2\,I}{d\,\omega\,d\,\Omega_\gamma}\right) \quad (15.5)$$

Since,

$$k^\mu = \frac{\hbar\,\omega}{c}\,(1, \boldsymbol{n})$$

$$p^\mu = \gamma\, M\, c\,(1, \boldsymbol{\beta})$$

$$\epsilon^\mu = (0, \boldsymbol{\epsilon}) \qquad [\text{ radiation gauge }]$$

we have

$$k\cdot p = \hbar\,\omega\,\gamma\, M\,(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})$$

$$\epsilon\cdot p = \gamma\, M\, c\, \boldsymbol{\epsilon}\cdot\boldsymbol{\beta}$$

$$\frac{\boldsymbol{\epsilon}^*\cdot\boldsymbol{\beta}}{1 - \boldsymbol{\beta}\cdot\boldsymbol{n}} = \frac{\hbar\,\omega}{c}\,\frac{\epsilon\cdot p}{k\cdot p}$$

Eq (15.3) thus becomes

$$\lim_{\hbar\,\omega\to 0} \frac{d^2 N}{d(\hbar\,\omega)\, d\,\Omega_\gamma} = \frac{z^2\,\alpha\,\hbar\,\omega}{4\,\pi^2\, c^2}\left|\frac{\epsilon^*\cdot p'}{k\cdot p'} - \frac{\epsilon^*\cdot p}{k\cdot p}\right|^2$$

Hence

$$\lim_{\hbar\,\omega\to 0} \frac{d^3 N}{\left(d^3 k / k_0\right)} = \frac{z^2\,\alpha}{4\,\pi^2}\left|\frac{\epsilon^*\cdot p'}{k\cdot p'} - \frac{\epsilon^*\cdot p}{k\cdot p}\right|^2 \qquad (15.6)$$

Consider the Feynman diagrams in the 2nd row of Fig 15.1.
All three of them involves products of the particle propagators

$$\frac{1}{p^2 - M^2\, c^2}\cdot\frac{1}{p'^2 - M^2\, c^2}$$

which is finite as $\omega \to 0$.

However, the 1st diagram involves the propagator

$$\frac{1}{(p - k)^2 - M^2\, c^2}$$

and the 2nd one

$$\frac{1}{(p + k)^2 - M^2\, c^2}$$

Now,

$$\begin{aligned}(p \pm k)^2 - M^2\, c^2 &= p^2 + k^2 \pm 2\, p\cdot k - M^2\, c^2 \\ &= \pm 2\, p\cdot k \\ &= \pm\, 2\,\hbar\,\omega\,\gamma\, M\,(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})\end{aligned}$$

where we've used

$$p^2 = M^2\, c^2 \qquad k^2 = 0$$

Hence, these propagtors diverge as $\omega \to 0$. The 3rd digram is therefore negligible in this limit.

In other words, soft photon emissions come from external lines with results same as the classical ones.

- $\frac{d^3 k'}{k_0'} = \frac{d^3 k}{k_0}$

Consider the Lorentz transform

$$k_0' = \gamma\left(k_0 - \beta k_{||}\right) \quad \text{or} \quad \frac{\omega'}{c} = \gamma \frac{\omega}{c}(1 - \boldsymbol{\beta}\cdot\boldsymbol{n})$$
$$= \gamma \frac{\omega}{c}(1 - \beta\cos\theta)$$
$$k_{||}' = \gamma\left(k_{||} - \beta k_0\right) \quad \text{or} \quad \frac{\omega'}{c}\hat{\boldsymbol{\beta}}\cdot\boldsymbol{n}' = \gamma\frac{\omega}{c}\left(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{n} - \beta\right)$$
$$\frac{\omega'}{c}\cos\theta' = \gamma\frac{\omega}{c}(\cos\theta - \beta)$$
$$\boldsymbol{k}_\perp' = \boldsymbol{k}_\perp \quad \text{or} \quad \frac{\omega'}{c}\sin\theta' = \frac{\omega}{c}\sin\theta$$

where θ is the angle between $\boldsymbol{\beta}$ & $\boldsymbol{k}$.

Hence

$$d^3 k' = d k_{||}' d^2 k_\perp' = d k_{||}' d^2 k_\perp = \frac{d k_{||}'}{d k_{||}} d k_{||} d^2 k_\perp = \frac{d k_{||}'}{d k_{||}} d^3 k$$

where

$$\frac{d k_{||}'}{d k_{||}} = \gamma\left(1 - \beta\frac{d k_0}{d k_{||}}\right)$$

is to be evaluated with $\boldsymbol{k}_\perp$ kept constant.

From

$$\boldsymbol{k}^2 = \boldsymbol{k}_{||}^2 + \boldsymbol{k}_\perp^2 = \left(\frac{\omega}{c}\right)^2 = k_0^2$$

we have

$$2 k_{||} d k_{||} = 2 k_0 d k_0 \qquad (\boldsymbol{k}_\perp \text{ held constant})$$
$$\frac{d k_0}{d k_{||}} = \frac{k_{||}}{k_0}$$

Using

$$\frac{k_0'}{k_0} = \gamma\left(1 - \beta\frac{k_{||}}{k_0}\right)$$

we have

$$\frac{d k_{||}'}{d k_{||}} = \gamma\left(1 - \beta\frac{k_{||}}{k_0}\right)$$
$$= \frac{k_0'}{k_0}$$

or

$$d^3 k' = \frac{k_0'}{k_0} d^3 k$$
$$\frac{d^3 k'}{k_0'} = \frac{d^3 k}{k_0}$$

A simpler derivation of this result is as follows.
Since k^μ is restricted by the light cone condition

$$k^2 = k_0{}^2 - \boldsymbol{k}^2 = 0$$

an integration over the 4-dimensional space k is restricted to the 3-dimensional light cone. In other words, the integration of a function f over the photon k space is

$$\int d^4 k \;\delta\left(k^2\right) f(k)$$

Now

$$\begin{aligned}
\delta\left(k^2\right) &= \delta\left(k_0{}^2 - \boldsymbol{k}^2\right) \\
&= \delta[\,(k_0 - |\boldsymbol{k}|)\,(k_0 + |\boldsymbol{k}|)\,] \\
&= \frac{1}{2\,|\,\boldsymbol{k}\,|}[\,\delta(k_0 - |\boldsymbol{k}|) + \delta(k_0 + |\boldsymbol{k}|)\,] \\
&= \frac{1}{2\,k_0}\,\delta(k_0 - |\boldsymbol{k}|) \qquad \text{if we restrict to } k_0 > 0
\end{aligned}$$

Hence

$$\int d^4 k \;\delta\left(k^2\right) f(k) = \int \frac{d^3 k}{2\,k_0}\; f(k) \quad \text{where} \quad k^\mu = \frac{\omega}{c}\,(1, \boldsymbol{n})$$

Since $d^4 k$ is a Lorentz invariant , we have

$$\frac{d^3 k'}{k_0'} = \frac{d^3 k}{k_0}$$

- (b) Polarization and Spectrum Integrated over Angles

For non-relativistic motion

$$\beta\,,\ \beta' \ll 1$$

$$1 - \boldsymbol{n}\cdot\boldsymbol{\beta} \simeq 1$$

so that (15.2) becomes

$$\lim_{\omega\to 0} \frac{d^2 I_{\text{NR}}}{d\,\omega\, d\,\Omega} = \frac{z^2\, e^2}{4\,\pi^2\, c} |\,\boldsymbol{\epsilon}^* \cdot \Delta\,\boldsymbol{\beta}\,|^2 \qquad (15.7)$$

where $\Delta\,\boldsymbol{\beta} = \boldsymbol{\beta}' - \boldsymbol{\beta}$.

Exercise.
Show that

$$\lim_{\omega\to 0} \frac{d\, I_{\text{NR}}}{d\,\omega} = \frac{2\, z^2\, e^2}{3\,\pi\, c} |\,\Delta\,\boldsymbol{\beta}\,|^2 \qquad (15.8)$$

Compare this with Larmor's formula. What is the $\frac{d\,I}{d\,\omega}$ corresponding to Larmor's formula?

For relativistic motion in which the change in velocity $\Delta\,\beta$ is small,

$$\beta,\ \beta' \simeq 1 \qquad\qquad \Delta\,\beta \ll 1$$

we have

$$\frac{\boldsymbol{\beta}'}{1-\boldsymbol{\beta}'\cdot\boldsymbol{n}} = \frac{\boldsymbol{\beta}+\Delta\boldsymbol{\beta}}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}-\Delta\boldsymbol{\beta}\cdot\boldsymbol{n}}$$

$$\simeq \frac{1}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}(\boldsymbol{\beta}+\Delta\boldsymbol{\beta})\left[1+\frac{\Delta\boldsymbol{\beta}\cdot\boldsymbol{n}}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right]$$

$$\simeq \frac{1}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\left[\boldsymbol{\beta}+\Delta\boldsymbol{\beta}+\frac{\Delta\boldsymbol{\beta}\cdot\boldsymbol{n}}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\boldsymbol{\beta}\right]$$

$$\frac{\boldsymbol{\beta}'}{1-\boldsymbol{\beta}'\cdot\boldsymbol{n}} - \frac{\boldsymbol{\beta}}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}} \simeq \frac{1}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\left[\Delta\boldsymbol{\beta}+\frac{\Delta\boldsymbol{\beta}\cdot\boldsymbol{n}}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\boldsymbol{\beta}\right]$$

$$\simeq \frac{1}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^2}[\Delta\boldsymbol{\beta}-(\boldsymbol{\beta}\cdot\boldsymbol{n})\Delta\boldsymbol{\beta}+(\Delta\boldsymbol{\beta}\cdot\boldsymbol{n})\boldsymbol{\beta}]$$

$$\simeq \frac{1}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^2}[\Delta\boldsymbol{\beta}+\boldsymbol{n}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})]$$

(15.2) thus becomes

$$\lim_{\omega\to 0}\frac{d^2 I}{d\omega\, d\Omega} \simeq \frac{z^2 e^2}{4\pi^2 c}\left|\boldsymbol{\epsilon}^*\cdot\left(\frac{\Delta\boldsymbol{\beta}+\boldsymbol{n}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^2}\right)\right|^2 \qquad (15.9)$$

$$\boldsymbol{\beta} = (0, 0, \beta)$$
$$\boldsymbol{n} = (\sin\theta, 0, \cos\theta)$$
$$\Delta\boldsymbol{\beta} = \Delta\beta(\cos\phi, \sin\phi, 0)$$
$$\boldsymbol{\epsilon}_{||} = (-\cos\theta, 0, \sin\theta)$$
$$\boldsymbol{\epsilon}_{\perp} = (0, 1, 0)$$

$$\boldsymbol{n}\cdot\boldsymbol{\beta} = \beta\cos\theta$$
$$\boldsymbol{\epsilon}_{||}\cdot\Delta\boldsymbol{\beta} = -\Delta\beta\cos\theta\cos\phi$$
$$\boldsymbol{\epsilon}_{\perp}\cdot\Delta\boldsymbol{\beta} = \Delta\beta\sin\phi$$
$$\boldsymbol{\beta}\times\Delta\boldsymbol{\beta} = \beta\Delta\beta(-\sin\phi, \cos\phi, 0)$$
$$\boldsymbol{n}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta}) = \beta\Delta\beta(-\cos\theta\cos\phi, -\cos\theta\sin\phi, \sin\theta\cos\phi)$$
$$\boldsymbol{\epsilon}_{||}\cdot[\boldsymbol{n}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})] = \beta\Delta\beta\left[\cos^2\theta\cos\phi+\sin^2\theta\cos\phi\right]$$
$$= \beta\Delta\beta\cos\phi$$
$$\boldsymbol{\epsilon}_{\perp}\cdot[\boldsymbol{n}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})] = -\beta\Delta\beta\cos\theta\sin\phi$$

$$\lim_{\omega\to 0}\frac{d^2 I_{||}}{d\omega\, d\Omega} = \frac{z^2 e^2}{4\pi^2 c}\int_0^{2\pi}\frac{d\phi}{2\pi}\left|\boldsymbol{\epsilon}_{||}\cdot\left(\frac{\Delta\boldsymbol{\beta}+\boldsymbol{n}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^2}\right)\right|^2$$

$$= \frac{z^2 e^2}{4\pi^2 c}\int_0^{2\pi}\frac{d\phi}{2\pi}\left(\frac{-\Delta\beta\cos\theta\cos\phi+\beta\Delta\beta\cos\phi}{(1-\beta\cos\theta)^2}\right)^2$$

$$= \frac{z^2 e^2}{4\pi^2 c}(\Delta\beta)^2\frac{(-\cos\theta+\beta)^2}{(1-\beta\cos\theta)^4}\int_0^{2\pi}\frac{d\phi}{2\pi}\cos^2\phi$$

$$= \frac{z^2 e^2}{8\pi^2 c}(\Delta\beta)^2\frac{(-\cos\theta+\beta)^2}{(1-\beta\cos\theta)^4}$$

where

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\cos^2\phi = \frac{1}{2}$$

$$\lim_{\omega\to 0}\frac{d^2 I_\perp}{d\omega\, d\Omega} = \frac{z^2 e^2}{4\pi^2 c}\int_0^{2\pi}\frac{d\phi}{2\pi}\left|\epsilon_\perp\cdot\left(\frac{\Delta\boldsymbol{\beta} + \boldsymbol{n}\times(\boldsymbol{\beta}\times\Delta\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n})^2}\right)\right|^2$$

$$= \frac{z^2 e^2}{4\pi^2 c}\int_0^{2\pi}\frac{d\phi}{2\pi}\left(\frac{\Delta\beta\sin\phi - \beta\Delta\beta\cos\theta\sin\phi}{(1-\beta\cos\theta)^2}\right)^2$$

$$= \frac{z^2 e^2}{4\pi^2 c}(\Delta\beta)^2\frac{1}{(1-\beta\cos\theta)^2}\int_0^{2\pi}\frac{d\phi}{2\pi}\sin^2\phi$$

$$= \frac{z^2 e^2}{8\pi^2 c}(\Delta\beta)^2\frac{1}{(1-\beta\cos\theta)^2}$$

where

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\sin^2\phi = \frac{1}{2}$$

The polarization factor is

$$P(\theta) = \frac{d^2 I_\perp - d^2 I_{||}}{d^2 I_\perp + d^2 I_{||}}$$

$$= \frac{\frac{1}{(1-\beta\cos\theta)^2} - \frac{(-\cos\theta+\beta)^2}{(1-\beta\cos\theta)^4}}{\frac{1}{(1-\beta\cos\theta)^2} + \frac{(-\cos\theta+\beta)^2}{(1-\beta\cos\theta)^4}}$$

$$= \frac{(1-\beta\cos\theta)^2 - (-\cos\theta+\beta)^2}{(1-\beta\cos\theta)^2 + (-\cos\theta+\beta)^2}$$

$$= \frac{1+\beta^2\cos^2\theta - \cos^2\theta - \beta^2}{1+\beta^2\cos^2\theta + \cos^2\theta + \beta^2 - 4\beta\cos\theta}$$

$$= \frac{\frac{1}{\gamma^2}\sin^2\theta}{(1+\beta^2)(1+\cos^2\theta) - 4\beta\cos\theta}$$

Thus

$$P(0) = 0$$

ie., radiation is unpolarized in the forward direction.

The extrema of $P(\theta)$ are at

$$\frac{dP}{d\theta} = 0$$

ie

$$\left[(1+\beta^2)(1+\cos^2\theta) - 4\beta\cos\theta\right]2\sin\theta\cos\theta - \sin^2\theta\left[-2\cos\theta\sin\theta(1+\beta^2) + 4\beta\sin\theta\right] = 0$$

The roots are

$$\sin\theta = 0 \qquad (\text{minimum})$$

or

$$\left[(1+\beta^2)(1+\cos^2\theta) - 4\beta\cos\theta\right]2\cos\theta - \sin^2\theta\left[-2\cos\theta(1+\beta^2) + 4\beta\right] = 0$$

Setting $t = \cos\theta$, we have

$$\left[\left(1 + \beta^2\right)\left(1 + t^2\right) - 4\beta\, t\right] t - \left(1 - t^2\right)\left[-t\left(1 + \beta^2\right) + 2\beta\right] = 0$$

$$\left(1 + \beta^2\right) 2t - 2\beta\left(1 + t^2\right) = 0$$

$$\beta t^2 - \left(1 + \beta^2\right) t + \beta = 0$$

$$(\beta t - 1)(t - \beta) = 0$$

Since $\beta \le 1$, $\cos\theta = \dfrac{1}{\beta}$ cannot be satisfied for real θ. We therefore are left with

$$\cos\theta_m = \beta$$

Now

$$P\left(\cos^{-1}\beta\right) = \frac{\frac{1}{\gamma^2}\left(1 - \beta^2\right)}{\left(1 + \beta^2\right)\left(1 + \beta^2\right) - 4\beta^2}$$

$$= \frac{\left(1 - \beta^2\right)^2}{\left(1 - \beta^2\right)^2}$$

$$= 1$$

which is obviously the maximum of P.

For $\gamma \gg 1$,

$$\beta \simeq 1 - \frac{1}{2\gamma^2}$$

so that

$$\theta_m \simeq \cos^{-1}\left(1 - \frac{1}{2\gamma^2}\right) \simeq \frac{1}{\gamma} \ll 1$$

Thus, θ is small for ultra- relativistic motion. An approximation of

$$P(\theta) = \frac{\frac{1}{\gamma^2}\sin^2\theta}{\left(1 + \beta^2\right)\left(1 + \cos^2\theta\right) - 4\beta\cos\theta}$$

can be obtained as follows.

Up to the 4th order,

$$\beta \simeq 1 - \frac{1}{2\gamma^2} - \frac{1}{8\gamma^4} - \ldots$$

$$\beta^2 \simeq 1 - \frac{1}{\gamma^2} - \ldots \qquad \left(\frac{1}{4} - \frac{2}{8} = 0\right)$$

$$\sin\theta \simeq \theta - \frac{\theta^3}{3!} + \ldots$$

$$\sin^2\theta \simeq \theta^2 - \frac{\theta^4}{3} + \ldots$$

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \ldots$$

$$\cos^2\theta = 1 - \theta^2 + \frac{\theta^4}{3} - \ldots \qquad \left(\frac{1}{4} + \frac{2}{4!} = \frac{1}{3}\right)$$

Thus, up to order γ^{-4}, [$\theta \sim \gamma^{-1}$],

$$\frac{1}{\gamma^2}\sin^2\theta \simeq \frac{1}{\gamma^2}\theta^2 + \ldots$$

$$\left(1+\beta^2\right)\left(1+\cos^2\theta\right)-4\beta\cos\theta$$
$$\simeq\left(2-\frac{1}{\gamma^2}\right)\left(2-\theta^2+\frac{\theta^4}{3}\right)-4\left(1-\frac{1}{2\gamma^2}-\frac{1}{8\gamma^4}\right)\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4!}\right)$$
$$\simeq 4-2\theta^2+\frac{2}{3}\theta^4-\frac{2}{\gamma^2}+\frac{1}{\gamma^2}\theta^2+\ldots$$
$$-4\left(1-\frac{1}{2\gamma^2}-\frac{1}{8\gamma^4}-\frac{\theta^2}{2}+\frac{1}{4\gamma^2}\theta^2+\frac{\theta^4}{4!}+\ldots\right)$$
$$\simeq\left(\frac{2}{3}-\frac{1}{6}\right)\theta^4+\frac{1}{2\gamma^4}+\ldots$$
$$\simeq\frac{1}{2\gamma^4}\left(\gamma^4\theta^4+1\right)+\ldots$$

Hence

$$P(\theta)\simeq\frac{\frac{1}{\gamma^2}\theta^2}{\frac{1}{2\gamma^4}\left(\gamma^4\theta^4+1\right)}=\frac{2\gamma^2\theta^2}{1+\gamma^4\theta^4}$$

$$\lim_{\omega\to 0}\frac{d^2 I}{d\omega\, d\Omega}=\lim_{\omega\to 0}\frac{d^2 I_{||}}{d\omega\, d\Omega}+\lim_{\omega\to 0}\frac{d^2 I_{\perp}}{d\omega\, d\Omega}$$
$$=\frac{z^2 e^2}{8\pi^2 c}(\Delta\beta)^2\left[\frac{(-\cos\theta+\beta)^2}{(1-\beta\cos\theta)^4}+\frac{1}{(1-\beta\cos\theta)^2}\right]$$
$$=\frac{z^2 e^2}{8\pi^2 c}(\Delta\beta)^2\frac{(-\cos\theta+\beta)^2+(1-\beta\cos\theta)^2}{(1-\beta\cos\theta)^4}$$

For $\gamma \gg 1$, we've already shown that

$$(-\cos\theta+\beta)^2+(1-\beta\cos\theta)^2$$
$$=\left(1+\beta^2\right)\left(1+\cos^2\theta\right)-4\beta\cos\theta$$
$$\simeq\frac{1}{2\gamma^4}\left(\gamma^4\theta^4+1\right)$$

together with

$$1-\beta\cos\theta\simeq 1-\left(1-\frac{1}{2\gamma^2}-\frac{1}{8\gamma^4}\right)\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4!}\right)$$
$$\simeq\frac{1}{2\gamma^2}+\frac{\theta^2}{2}+\ldots$$

we have

$$\lim_{\omega\to 0}\frac{d^2 I}{d\omega\, d\Omega}\simeq\frac{z^2 e^2}{8\pi^2 c}(\Delta\beta)^2\frac{\frac{1}{2\gamma^4}\left(\gamma^4\theta^4+1\right)}{\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^4}$$
$$=\frac{z^2 e^2}{\pi^2 c}(\Delta\beta)^2\gamma^4\frac{1+\gamma^4\theta^4}{1+\gamma^2\theta^2}$$

Next, we calculate

$$\lim_{\omega\to 0}\frac{dI}{d\omega}=\lim_{\omega\to 0}\int d\Omega\;\frac{d^2 I}{d\omega\, d\Omega}$$

$$=\quad \frac{z^2 e^2}{8\pi^2 c}(\Delta\beta)^2\, 2\pi\int_{-1}^{1} d\cos\theta\left[\frac{(-\cos\theta\ +\ \beta)^2}{(1\ -\ \beta\cos\theta)^4}+\frac{1}{(1\ -\ \beta\cos\theta)^2}\right]$$

$$J=\int_{-1}^{1} d\cos\theta\;\frac{(-\cos\theta\ +\ \beta)^2}{(1\ -\ \beta\cos\theta)^4}$$

Let

$$x=1-\beta\cos\theta \qquad\qquad x\in\ [\,1+\beta,\ 1-\beta\,]$$

so that

$$\cos\theta=\ \frac{1}{\beta}(1-x)$$

$$dx=-\beta\, d\cos\theta \qquad\qquad d\cos\theta=\ -\frac{dx}{\beta}$$

$$\beta-\cos\theta=\frac{1}{\beta}\left(\beta^2-1+x\right)=\frac{1}{\beta}\left(-\frac{1}{\gamma^2}+x\right)$$

$$(\beta-\cos\theta)^2=\ \frac{1}{\beta^2}\left(\frac{1}{\gamma^4}-\frac{2x}{\gamma^2}+x^2\right)$$

and

$$J=-\int_{1+\beta}^{1-\beta}\frac{dx}{\beta}\;\frac{1}{\beta^2 x^4}\left(\frac{1}{\gamma^4}-\frac{2x}{\gamma^2}+x^2\right)$$

$$=\ -\frac{1}{\beta^3}\int_{1+\beta}^{1-\beta} dx\left(\frac{1}{\gamma^4 x^4}-\frac{2}{\gamma^2 x^3}+\frac{1}{x^2}\right)$$

$$=\ \frac{1}{\beta^3}\left[\frac{1}{3\gamma^4 x^3}-\frac{1}{\gamma^2 x^2}+\frac{1}{x}\right]_{1+\beta}^{1-\beta}$$

Using

$$\frac{1}{(1\ -\ \beta)^3}-\frac{1}{(1\ +\ \beta)^3}=\frac{1}{\left(1\ -\ \beta^2\right)^3}\left[(1\ +\ \beta)^3-(1\ -\ \beta)^3\right]$$

$$=\ \frac{2\left(3\beta\ +\ \beta^3\right)}{\left(1\ -\ \beta^2\right)^3}$$

$$\frac{1}{(1-\beta)^2} - \frac{1}{(1+\beta)^2} = \frac{1}{\left(1-\beta^2\right)^2}\left[(1+\beta)^2 - (1-\beta)^2\right]$$

$$= \frac{4\beta}{\left(1-\beta^2\right)^2}$$

$$\frac{1}{1-\beta} - \frac{1}{1+\beta} = \frac{2\beta}{1-\beta^2}$$

we have

$$J = \frac{1}{\beta^3}\left[\frac{2\left(3\beta+\beta^3\right)}{3\gamma^4\left(1-\beta^2\right)^3} - \frac{4\beta}{\gamma^2\left(1-\beta^2\right)^2} + \frac{2\beta}{1-\beta^2}\right]$$

$$= \frac{1}{\beta^3}\left[\frac{2\left(3\beta+\beta^3\right)}{3\left(1-\beta^2\right)} - \frac{4\beta}{1-\beta^2} + \frac{2\beta}{1-\beta^2}\right]$$

$$= \frac{2}{3\left(1-\beta^2\right)}$$

$$= \frac{2}{3}\gamma^2$$

Similarly

$$K = \int_{-1}^{1} d\cos\theta \, \frac{1}{(1-\beta\cos\theta)^2}$$

$$= -\int_{1+\beta}^{1-\beta} \frac{dx}{\beta} \frac{1}{x^2}$$

$$= \frac{1}{\beta}\left[\frac{1}{x}\right]_{1+\beta}^{1-\beta}$$

$$= \frac{2}{1-\beta^2}$$

$$= 2\gamma^2$$

Hence

$$J + K = \frac{8}{3}\gamma^2$$

$$\lim_{\omega\to 0} \frac{dI}{d\omega} = \frac{z^2 e^2}{8\pi^2 c} (\Delta\beta)^2 \, 2\pi \, (J+K)$$

$$= \frac{2 z^2 e^2}{3\pi c} (\Delta\beta)^2 \gamma^2$$

In the non-relativistic limit, $\gamma \simeq 1$,

$$\lim_{\omega\to 0} \frac{dI}{d\omega} \simeq \frac{2 z^2 e^2}{3\pi c} (\Delta\beta)^2$$

Now

$$\boldsymbol{\beta} = \frac{1}{\gamma M c} \boldsymbol{p}$$

$$\Delta \boldsymbol{\beta} = \frac{1}{\gamma M c} (\boldsymbol{p}' - \boldsymbol{p})$$

$$(\Delta \boldsymbol{\beta})^2 = \frac{1}{(\gamma M c)^2} (\boldsymbol{p}' - \boldsymbol{p})^2 = \frac{Q^2}{(\gamma M c)^2}$$

so that

$$\lim_{\omega\to 0} \frac{d I}{d \omega} \simeq \frac{2 z^2 e^2}{3 \pi c^3 M^2} Q^2$$

Finite Frequencies:

$$\boldsymbol{r}(t) = \boldsymbol{r}(0) + \int_0^t d t' c \boldsymbol{\beta}(t')$$

The phase change is

$$\Phi(t) = \omega \left[t - \frac{1}{c} \boldsymbol{n}\cdot\boldsymbol{r}(t) \right]$$

$$= \omega \left[t - \frac{1}{c} \boldsymbol{n}\cdot\boldsymbol{r}(0) - \frac{1}{c} \boldsymbol{n}\cdot \int_0^t d t' c \boldsymbol{\beta}(t')\right]$$

$$= \omega \left[t - \boldsymbol{n}\cdot \int_0^t d t' \boldsymbol{\beta}(t')\right] - \frac{\omega}{c} \boldsymbol{n}\cdot\boldsymbol{r}(0)$$

The last term is a constant & can be dropped.

Let the collision time be τ,

$$\Phi(\tau) = \omega \left[t - \boldsymbol{n}\cdot \int_0^\tau d t' \boldsymbol{\beta}(t')\right]$$

$$= \tau \omega [1 - \boldsymbol{n}\cdot\langle \boldsymbol{\beta} \rangle]$$

where

$$\langle \boldsymbol{\beta} \rangle = \frac{1}{\tau} \int_0^\tau d t' \boldsymbol{\beta}(t')$$

Criterion for appreciable radiation is

$$\Phi(\tau) < 1$$

ie

$$\tau \omega [1 - \boldsymbol{n}\cdot\langle \boldsymbol{\beta} \rangle] < 1$$

For non-relativistic motion,

$$\beta \ll 1$$

so that the criterion becomes

$$\tau \omega < 1$$

ie, radiation restricted to $\omega < \frac{1}{\tau}$.

For relativistic motion

$$\gamma \gg 1$$

so that the criterion becomes

$$\Phi(\tau) = \frac{\omega\,\tau}{2\,\gamma^2}\left(1 + \theta^2\,\gamma^2\right) < 1 \qquad (14.77)$$

Neglecting the factor 2, we have

$$\theta^2 < -\frac{1}{\gamma^2} + \frac{1}{\omega\,\tau} = \theta_m{}^2$$

For $\omega\,\tau < 1$,

$$\theta_m{}^2 > 1 - \frac{1}{\gamma^2} = \beta^2$$

Since most radiation is confined to

$$\theta \lesssim \frac{1}{\gamma} \ll \beta$$

the coherence criterion is satisfied.

For $\omega\,\tau > \gamma^2$,

$$\theta_m{}^2 = \frac{-\omega\,\tau + \gamma^2}{\gamma^2\,\omega\,\tau} < 0$$

which cannot be satisfied for any θ.

Let $\theta_{\max}{}^2 = \dfrac{1}{\omega\,\tau}$

we have

$$\theta^2 < \theta_{\max}{}^2 - \frac{1}{\gamma^2} < \theta_{\max}{}^2$$

For

$$1 < \omega\,\tau < \gamma^2$$

we have

$$1 > \theta_{\max}{}^2 > \frac{1}{\gamma^2}$$

Therefore, θ ranges from 1 to $\dfrac{1}{\gamma}$.

---

2. Bremsstrahlung

Bremsstrahlung = Braking Radiation

For an elastic Coulomb collision with

incident particle of charge $z\,e$, mass M,

initial velocity $\boldsymbol{v} = \boldsymbol{\beta}\,c$, scattering angle $\theta\,'$.

target of charge $Z\,e$, mass $\to \infty$.

the Rutherford formula gives

$$\frac{d\,\sigma_s}{d\,\Omega'} = \left(\frac{2\,z\,Z\,e^2}{p\,v}\right)^2 \frac{1}{(2\sin\theta\,'/2)^4} \qquad (15.17)$$

The validity of (15.17) is:

non-relativistic case: same as QM result.

relativistic case: good for small θ'.

Now, for elastic scattering,

$$\begin{aligned} Q^2 &= |\boldsymbol{p}' - \boldsymbol{p}|^2 \\ &= \left(2\, p \sin\frac{\theta'}{2}\right)^2 \\ &= 4\, p^2 \sin^2\frac{\theta'}{2} \\ &= 2\, p^2 (1 - \cos\theta') \end{aligned} \qquad (15.18)$$

so that

$$2\, Q\, d\, Q = -2\, p^2\, d \cos\theta'$$

and

$$\begin{aligned} d\,\Omega' &= d\,\phi'\, d\cos\theta' \\ &= -\frac{Q}{p^2}\, d\,Q\, d\,\phi' \end{aligned}$$

Hence

$$\begin{aligned} \frac{d\,\sigma_s}{d\,\Omega'} &= \left(\frac{2\, z\, Z\, e^2}{p\, v}\right)^2 \frac{p^4}{Q^4} \\ &= \frac{p^2}{Q}\left(\frac{d^2\,\sigma_s}{d\,\phi'\, d\,Q}\right) \end{aligned}$$

$$\begin{aligned} \frac{d\,\sigma_s}{d\,Q} &= \int_0^{2\pi} d\,\phi'\, \frac{d^2\,\sigma_s}{d\,\phi'\, d\,Q} \\ &= 2\pi\, \frac{d^2\,\sigma_s}{d\,\phi'\, d\,Q} \\ &= 2\pi \left(\frac{2\, z\, Z\, e^2}{p\, v}\right)^2 \frac{p^2}{Q^3} \\ &= 8\pi \left(\frac{z\, Z\, e^2}{v}\right)^2 \frac{1}{Q^3} \end{aligned} \qquad (15.19)$$

The radiation differential cross section is

$$\frac{d^2\,\chi}{d\,\omega\, d\,Q} = \frac{d\, I(\omega,\ Q)}{d\,\omega}\left(\frac{d\,\sigma_s}{d\,Q}\right) \qquad (15.20)$$

$$= \left[\frac{\text{energy}}{\text{freqency}} \cdot \frac{\text{area}}{\text{momentum}}\right]$$

Using

$$\frac{dI}{d\omega} \xrightarrow[\omega\to 0\ ,\ Q<2Mc]{} \frac{2}{3\pi}\frac{z^2 e^2}{M^2 c^3} Q^2$$

$$\frac{d\sigma_s}{dQ} = 8\pi \left(\frac{zZe^2}{\beta c}\right)^2 \frac{1}{Q^3}$$

we have

$$\frac{d^2\chi}{d\omega\, dQ} \xrightarrow[\omega\to 0\ ,\ Q<2Mc]{} \frac{16}{3}\left(\frac{z^2 e^2}{c}\right)\left(\frac{zZe^2}{Mc^2}\right)^2 \frac{1}{\beta^2}\frac{1}{Q} \qquad (15.21)$$

$$\frac{d\chi}{d\omega} = \int_{Q_{min}}^{Q_{max}} dQ\, \frac{d^2\chi}{d\omega\, dQ}$$

$$\xrightarrow[\omega\to 0\ ,\ Q<2Mc]{} \frac{16}{3}\left(\frac{z^2 e^2}{c}\right)\left(\frac{zZe^2}{Mc^2}\right)^2 \frac{1}{\beta^2} \ln\frac{Q_{max}}{Q_{min}} \qquad (15.22)$$

■ Supplement

	Q_{max}	Q_{min}
Elastic Kinematics	$2\gamma M v$ (back scattering)	0
Uncertainty Principle	$2\eta\gamma M v$	–
$\frac{dI}{d\omega} \propto Q^2$	$2Mc$	$\frac{2\eta\hbar\omega}{\gamma^3 v}$
Inelastic Kinematics	$2\gamma M v$	$\frac{\hbar\omega}{v}$ for $\beta\to 0$ $\frac{\hbar\omega}{2c\gamma\gamma'}$ for $\beta,\ \beta'\to 1$
Choice	$2Mv$	$\frac{2\eta\hbar\omega}{v}\frac{1}{\lambda}$ for $\eta>1,\ \beta\to 0$ $\frac{\hbar\omega}{v}\frac{1}{\lambda'}$ for $\eta<1,\ \beta\to 0$ $\frac{\hbar\omega}{2c\gamma\gamma'}\frac{1}{\lambda''}$ for $\eta<1,\ \beta\to 1$

where

$$\lambda = 2, \quad \lambda' = 1.$$

$$\eta = \frac{zZe^2}{v\hbar} = zZ\frac{\alpha}{\beta} \quad \text{with minimum at } \beta = 1 \text{ whereupon } \eta = zZ\alpha.$$

For $\beta = zZ\alpha = \frac{zZ}{137}$, we have $\eta = 1$.

Note that $\beta = \frac{1}{2}$ gives $\gamma \simeq 1.15$.

Some results from chap 13.

$$Q = \frac{2\, z\, Z\, e^2}{v\, b} = \frac{2\, \eta\, \hbar}{b}$$

The uncertainty principle gives

$$b_{\min} = \frac{\hbar}{p} = \frac{\hbar}{\gamma\, m\, v}$$

$$Q_{\max} = \frac{2\, \eta\, \hbar}{b_{\min}} = 2\, \eta\, \gamma\, m\, v$$

Validity of $\frac{d\, I}{d\, \omega} \propto Q^2$:

1. $Q < 2\, M\, c$

   [else $\boldsymbol{\beta}$ ' & $\boldsymbol{\beta}$ terms in eq(15.2) become incoherent]

2. $\omega\, \tau < \gamma^2$ eq(15.15,16)

   Using $\tau = \frac{b}{\gamma\, v} = \frac{2\, \eta\, \hbar}{\gamma\, v\, Q}$

   we have $Q > \frac{2\, \eta\, \hbar\, \omega}{\gamma^3\, v}$

For inelastic scattering, $Q_{\min} \neq 0$.

Proof:

$$E = E\,' + \hbar\, \omega \quad \longrightarrow \quad \gamma = \gamma\,' + \frac{\hbar\, \omega}{M\, c^2} \qquad (1)$$

From

$$Q^2 = (\, \mathbf{p} - \mathbf{p}' - \mathbf{k}\,)^2$$

we have

$$Q_{\min}{}^2 = \left(\gamma\, v - \gamma\,' v' - \frac{\hbar\, \omega}{M\, c} \right)^2 M^2$$

If we set

$$Q_{\min}{}^2 = 0$$

we have

$$\gamma\, v - \gamma\,' v' - \frac{\hbar\, \omega}{M\, c} = 0 \qquad (2)$$

(1) → (2) gives

$$\gamma\, v - \gamma\,' v' - c\, (\, \gamma - \gamma\,'\,) = 0$$

or

$$\gamma\, (\, c - v\,) = \gamma\,' (\, c - v\,'\,)$$

$$\sqrt{\frac{c - v}{c + v}} = \sqrt{\frac{c - v\,'}{c + v\,'}}$$

$$(\, c - v\,)\, (\, c + v'\,) = (\, c - v'\,)\, (\, c + v\,)$$

$$-v + v' = -v' + v$$

$$v\,' = v$$

ie., the only case for $Q_{\min} = 0$ is when the scattering is elastic with

$\gamma = \gamma\,'$ $E = E\,'$ & $\omega = 0$.

Classical Bremsstrahlung

From § 13.7 (p.645)

$$Q_{\min}^{(c)} \simeq \frac{2\, z\, Z\, e^2}{p\, v\, a}$$

$$Q_{\min}^{(q)} \simeq \frac{\hbar}{p\, a}$$

$$\eta = \frac{Q_{\min}^{(c)}}{Q_{\min}^{(q)}} = \frac{z\, Z\, e^2}{\hbar\, v}$$

Classical region means $\eta > 1$.

For z, Z not too large, this implies $\beta \ll 1$, ie, non-relativistic motion.

Since

$$Q = 2\, p \sin \frac{\theta'}{2}$$

we have

$$Q \leq 2\, p = 2\, M\, v \qquad (\text{ for } \gamma \simeq 1)$$

ie

$$Q_{\max} = 2\, M\, v$$

The coherence criterion $Q < 2\, M\, c$ is therefore always satisfied.

For significant radiation, the other coherence criterion $\omega\, \tau < 1$ need also be satisfied. [τ is the collision time]

For Coulomb collision, (see chap 13)

$$\tau \simeq \frac{b}{\gamma\, v} \xrightarrow[\gamma \simeq 1]{} \frac{b}{v}$$

and

$$Q = \Delta\, p = \frac{2\, z\, Z\, e^2}{b\, v}$$

$$b = \frac{2\, z\, Z\, e^2}{v\, Q}$$

$$\tau = \frac{2\, z\, Z\, e^2}{v^2\, Q}$$

so that

$$\omega\, \tau < 1 \qquad \longrightarrow \qquad Q > \frac{2\, z\, Z\, e^2}{v^2} \equiv Q_{\min} = 2\, \eta\, \frac{\hbar\, \omega}{v}$$

The classical radiation cross section is

$$\frac{d\, \chi_c}{d\, \omega} \simeq \frac{16}{3} \left(\frac{z^2\, e^2}{c} \right) \left(\frac{z\, Z\, e^2}{M\, c^2} \right)^2 \frac{1}{\beta^2} \ln \left(\frac{M\, v^3\, \lambda}{z\, Z\, e^2\, \omega} \right)$$

where $\lambda \simeq 1$ is a numerical correction factor.

On physical ground, we must have

$$\frac{d\chi_c}{d\omega} \geq 0$$

so that

$$\frac{M v^3 \lambda}{z Z e^2 \omega} \geq 1$$

ie

$$\omega \leq \frac{M v^3 \lambda}{z Z e^2} \equiv \omega_{\max}^{(c)}$$

or

$$\hbar\,\omega_{\max}^{(c)} = 2\frac{\lambda}{\eta}\left(\frac{1}{2} M v^2\right)$$

Since $\eta \gg 1$, this is always much smaller than the kinetic energy, $\frac{1}{2} M v^2$, of the incident particle. The radiation is soft.

▪ **Non-Relativistic** Bremsstrahlung

Non- relativistic conservation of energy & momentum gives

$$\frac{\boldsymbol{p}^2}{2M} = \frac{\boldsymbol{p}'^2}{2M} + \hbar\omega \qquad (E = E' + \hbar\omega)$$

$$\boldsymbol{p} = \boldsymbol{p}' + \boldsymbol{k} + \boldsymbol{Q} \qquad \boldsymbol{k} = \frac{\hbar\omega}{c}\boldsymbol{n}$$

$$Q^2 = (\boldsymbol{p} - \boldsymbol{p}' - \boldsymbol{k})^2$$

Now

$$k = \frac{\hbar\omega}{c} = \frac{1}{2Mc}\left(\boldsymbol{p}^2 - \boldsymbol{p}'^2\right)$$

$$= \frac{1}{2Mc}(\boldsymbol{p} - \boldsymbol{p}')\cdot(\boldsymbol{p} + \boldsymbol{p}')$$

$$\lesssim |\boldsymbol{p} - \boldsymbol{p}'|\,\frac{\max(p,\; p')}{2Mc}$$

$$\ll |\boldsymbol{p} - \boldsymbol{p}'|$$

Hence

$$Q^2 \simeq (\boldsymbol{p} - \boldsymbol{p}')^2$$

$$= p^2 + p'^2 - 2\,p\,p'\cos\theta'$$

and

$$Q_{\max} = p + p' \qquad (\theta' = \pi)$$

$$Q_{\min} = p - p' \qquad (\theta' = 0)$$

$$\frac{Q_{\max}}{Q_{\min}} = \frac{p + p'}{p - p'}$$

Using

$$p = \sqrt{2\,M\,E}$$

$$p' = \sqrt{p^2 - 2\,M\,\hbar\,\omega}$$

$$= \sqrt{2\,M}\;\sqrt{E - \hbar\,\omega}$$

we have

$$\frac{Q_{\text{max}}}{Q_{\text{min}}} = \frac{\sqrt{E} + \sqrt{E - \hbar\,\omega}}{\sqrt{E} - \sqrt{E - \hbar\,\omega}}$$

$$= \frac{\left(\sqrt{E} + \sqrt{E - \hbar\,\omega}\right)^2}{\hbar\,\omega}$$

Now

$$p - p' = \sqrt{2\,M\,E}\left(1 - \sqrt{1 - \frac{\hbar\,\omega}{E}}\right)$$

$$\simeq \sqrt{\frac{2\,M}{E}}\;\frac{\hbar\,\omega}{2} \qquad \text{for soft photons}$$

$$= \frac{\hbar\,\omega}{v} \qquad \text{for } \gamma \simeq 1$$

whereas

$$Q^{(c)}_{\text{min}} = 2\,\eta\,\frac{\hbar\,\omega}{v} \qquad \text{valid for} \qquad \eta > 1$$

Thus

$$Q = p - p' < Q^{(c)}_{\text{min}}$$

so that Q_{min} should be determined by $Q^{(q)}_{\text{min}}$.

$$Q^{(q)}_{\text{min}} \simeq p\,\theta^{(q)}_{\text{min}} \simeq \frac{\hbar}{b}$$

With the collision given by

$$\tau \simeq \frac{b}{v} = \frac{\hbar}{v\,Q^{(q)}_{\text{min}}}$$

the coherence condition $\omega\,\tau < 1$ becomes simply

$$\frac{\hbar\,\omega}{v} = Q < Q^{(q)}_{\text{min}}$$

Thus, with $\eta < 1$,

$$\frac{d\,\chi_{\text{NR}}}{d\,\omega} \simeq \frac{16}{3}\left(\frac{z^2\,e^2}{c}\right)\left(\frac{z\,Z\,e^2}{M\,c^2}\right)^2 \frac{1}{\beta^2}\ln\left(\frac{\lambda'\left(\sqrt{E} + \sqrt{E - \hbar\,\omega}\right)^2}{\hbar\,\omega}\right)$$

where λ' is a numerical correction factor of order 1.

Note: $\lambda' = 1$ gives the quantum (Born approximation) result.

For $\eta = \frac{z Z e^2}{\hbar v} = 1$,

$$\frac{d\chi_c}{d\omega} \simeq \frac{16}{3}\left(\frac{z^2 e^2}{c}\right)\left(\frac{z Z e^2}{M c^2}\right)^2 \frac{1}{\beta^2} \ln\left(\frac{2\lambda E}{\hbar\omega}\right) \qquad E = \frac{1}{2} M v^2$$

$$\frac{d\chi_{\mathrm{NR}}}{d\omega} \underset{\omega\to 0}{\simeq} \frac{16}{3}\left(\frac{z^2 e^2}{c}\right)\left(\frac{z Z e^2}{M c^2}\right)^2 \frac{1}{\beta^2} \ln\left(\frac{4\lambda' E}{\hbar\omega}\right)$$

Hence, these results match if $\lambda = 2\lambda'$.

Since

$$\frac{d\chi}{d\omega} \propto \left(\frac{z^2 Z}{M}\right)^2$$

the radiation is strongest for electron (smallest M) stopped by high Z material.

Let N be the number density of fixed charges $Z e$,

$$\frac{dE_{\mathrm{rad}}}{dx} = N \int_0^{\omega_{\max}} d\omega \, \frac{d\chi}{d\omega}$$

$$\simeq \frac{16}{3}\left(\frac{z^2 e^2}{c}\right)\left(\frac{z Z e^2}{M c^2}\right)^2 \frac{1}{\beta^2} N \int_0^{\omega_{\max}} d\omega \ln\left(\frac{\lambda'\left(\sqrt{E} + \sqrt{E - \hbar\omega}\right)^2}{\hbar\omega}\right)$$

From

$$E = E' + \hbar\omega$$

we have

$$\omega_{\max} = \frac{E}{\hbar} \qquad (\text{ at } E' = 0)$$

Let

$$x = \frac{\hbar\omega}{E} \qquad (x \in [0, 1])$$

we have

$$d\omega = \frac{E}{\hbar} dx$$

$$\sqrt{E} + \sqrt{E - \hbar\omega} = \sqrt{E}\left(1 + \sqrt{1 - x}\right)$$

$$\hbar\omega = E x$$

Hence, with $\lambda' = 1$,

$$\frac{dE_{\mathrm{rad}}}{dx} = A \int_0^1 dx \ln\left[\frac{\left(1 + \sqrt{1 - x}\right)^2}{x}\right]$$

where

$$A = \frac{16}{3} N \left(\frac{z^2 e^2}{c}\right)\left(\frac{z Z e^2}{M c^2}\right)^2 \frac{1}{\beta^2} \frac{E}{\hbar}$$

$$= \frac{8}{3} N \left(\frac{z^2 e^2}{\hbar M c}\right)\left(\frac{z Z e^2}{c}\right)^2 \qquad (E = \frac{1}{2} M v^2)$$

$$= \frac{8}{3} N Z \left(\frac{Z e^2}{\hbar c}\right)\left(\frac{z^4 e^4}{M c^2}\right)$$

The integral is evaluated as follows.

$$J = \int_0^1 d\,x \ln\left[\frac{\left(1+\sqrt{1-x}\,\right)^2}{x}\right]$$

$$= \; 2\int_0^1 d\,x\,\ln\left(1+\sqrt{1-x}\,\right) - \int_0^1 d\,x \ln x$$

For $\quad K = \int_0^1 d\,x \ln\left(1+\sqrt{1-x}\,\right)$

let

$$y = 1+\sqrt{1-x}$$

so that

$$d\,y = -\frac{1}{2\sqrt{1-x}}\,d\,x = -\frac{1}{2(y-1)}\,d\,x$$

$$d\,x = 2(1-y)\,d\,y$$

and

$$K = 2\int_2^1 d\,y\,(1-y)\ln y$$

$$= \; 2\left[y\ln y - y - y^2\left(\frac{1}{2}\ln y - \frac{1}{4}\right)\right]_2^1$$

$$= \; 2\left\{-1+\frac{1}{4}-\left[2\ln 2 - 2 - 4\left(\frac{1}{2}\ln 2 - \frac{1}{4}\right)\right]\right\}$$

$$= \; 2\left[-\frac{3}{4}+1\right]$$

$$= \; \frac{1}{2}$$

Next

$$L = \int_0^1 d\,x \ln x$$

$$= \; [\,x\ln x - x\,]_0^1$$

$$= \; -1$$

Hence

$$J = 2K - L = 2$$

and

$$\frac{d\,E_{\text{rad}}}{d\,x} = A\,J = \frac{16}{3}\,N\,Z\left(\frac{Z\,e^2}{\hbar\,c}\right)\left(\frac{z^4\,e^4}{M\,c^2}\right)$$

Now, the energy loss to the atomic electrons is

$$\frac{d\,E_{\text{col}}}{d\,x} = 4\,\pi\,N\,Z\left(\frac{z^2\,e^4}{m\,v^2}\right)\ln B_q \qquad (13.44)$$

Therefore

$$\frac{d E_{\text{rad}}}{d E_{\text{col}}} = \frac{4}{3\pi} z^2 Z \left(\frac{e^2}{\hbar c}\right) \frac{m}{M} \beta^2 \frac{1}{\ln B_q}$$

$$\ll 1 \qquad \text{for} \qquad \beta \to 0$$

ie, radiation loss is negligible for non- relativistic particles.

- **Relativistic** Bremsstrahlung

$$\frac{d I}{d \omega} \propto Q^2 \qquad \text{for} \qquad Q < 2 M c$$

$$\frac{d I}{d \omega} \text{ independ of } Q \qquad \text{for} \qquad Q > 2 M c$$

$$\frac{d \sigma_s}{d Q} \propto \frac{1}{Q^3}$$

Let

$$Q_{\text{max}} = 2 M c$$

we have

$$\frac{d \chi}{d \omega} \propto \int d Q \, \frac{d I}{d \omega} \frac{d \sigma_s}{d Q}$$

$$= A \int_{Q_{\text{min}}}^{Q_{\text{max}}} d Q \frac{1}{Q} + B \int_{Q_{\text{max}}}^{\infty} d Q \frac{1}{Q^3}$$

$$= A \ln\left(\frac{Q_{\text{max}}}{Q_{\text{min}}}\right) + \frac{B}{Q_{\text{max}}{}^2}$$

$$= A \ln\left(\frac{Q_{\text{max}}}{Q_{\text{min}}}\right)$$

where the term proportional to

$$\frac{1}{Q_{\text{max}}{}^2} = \left(\frac{1}{2 M c}\right)^2$$

is dropped.

Now

$$Q^2 = (\boldsymbol{p} - \boldsymbol{p}' - \boldsymbol{k})^2$$

implies

$$Q_{\text{min}} = p - p' - k \qquad (\boldsymbol{p} // \boldsymbol{p}' // \boldsymbol{k})$$

From

$$E^2 = p^2 c^2 + M^2 c^4$$

we have

$$\begin{aligned} p\,c &= \sqrt{E^2 - M^2 c^4} \\ &\simeq E\left(1 - \frac{M^2 c^4}{2 E^2}\right) \\ &= E - \frac{M^2 c^4}{2 E} \end{aligned}$$

Similarly,

$$p' c \simeq E' - \frac{M^2 c^4}{2 E'}$$

so that

$$\begin{aligned} Q_{\min} &\simeq \frac{1}{c}\left[E - \frac{M^2 c^4}{2E} - E' + \frac{M^2 c^4}{2E'} - \hbar\omega\right] \\ &= \frac{1}{c}\left[-\frac{M^2 c^4}{2E} + \frac{M^2 c^4}{2E'}\right] \qquad (E = E' + \hbar\omega) \\ &= \frac{M^2 c^3}{2 E E'}(E - E') \\ &= \frac{M^2 c^3}{2 E E'}\hbar\omega \end{aligned}$$

Hence

$$\frac{Q_{\max}}{Q_{\min}} \simeq \frac{4 E E'}{M c^2 \hbar\omega}$$

and

$$\frac{d\chi_R}{d\omega} \simeq \frac{16}{3}\left(\frac{z^2 e^2}{c}\right)\left(\frac{z Z e^2}{M c^2}\right)^2 \ln\left(\frac{\lambda'' E E'}{M c^2 \hbar\omega}\right)$$

where λ'' is a numerical correction factor of order 1.

$$\boldsymbol{p} = \gamma M c\,\boldsymbol{\beta} = \gamma M c\,\beta\,\hat{\boldsymbol{\beta}}$$

$\longrightarrow$

$$\Delta\boldsymbol{p} = M c\left[\boldsymbol{\beta}\,\Delta\gamma + \hat{\boldsymbol{\beta}}\,\gamma\,\Delta\beta + \gamma\,\beta\left(\Delta\hat{\boldsymbol{\beta}}\right)\right]$$

Using

$$\Delta\gamma = \beta\gamma^3\Delta\beta$$

we have

$$\begin{aligned} \Delta\boldsymbol{p} &= M c\left[\boldsymbol{\beta}\,\beta\gamma^3\Delta\beta + \hat{\boldsymbol{\beta}}\,\gamma\,\Delta\beta + \gamma\,\beta\left(\Delta\hat{\boldsymbol{\beta}}\right)\right] \\ &= M c\left[\hat{\boldsymbol{\beta}}\left(\beta^2\gamma^2 + 1\right)\gamma\,\Delta\beta + \gamma\,\beta\left(\Delta\hat{\boldsymbol{\beta}}\right)\right] \\ &= M c\left[\hat{\boldsymbol{\beta}}\,\gamma^3\Delta\beta + \gamma\,\beta\left(\Delta\hat{\boldsymbol{\beta}}\right)\right] \end{aligned}$$

Since

$$|\Delta\boldsymbol{p}| < Q_{\max} = 2 M c \;\&\; \qquad \gamma \gg 1$$

both $\Delta\beta$ and $\Delta\hat{\boldsymbol{\beta}}$ are small.

$$\max\left|\Delta \boldsymbol{\beta}_{||}\right| = \frac{2}{\gamma^3 \Delta \beta}$$

$$\max\left|\Delta \boldsymbol{\beta}_{\perp}\right| = \frac{2}{\gamma \beta}$$

so that

$$\max\left|\Delta \boldsymbol{\beta}_{\perp}\right| = \frac{\gamma^2}{\beta} (\Delta \beta) \max\left|\Delta \boldsymbol{\beta}_{||}\right| \gg \max\left|\Delta \boldsymbol{\beta}_{||}\right|$$

Hence, most $\Delta \boldsymbol{\beta}$ occurs in the $\perp$ direction.

Using the angular distribution for $\Delta \boldsymbol{\beta} \perp \boldsymbol{\beta}$, we have

$$\frac{d^2 \chi_R}{d\omega\, d\Omega_\gamma} \simeq \frac{3}{2\pi} \gamma^2 \frac{\left(1 + \gamma^4 \theta^4\right)}{\left(1 + \gamma^2 \theta^2\right)^4} \left(\frac{d\chi_R}{d\omega}\right)$$

Derivation by Lorentz Transform

Frame K : Laboratory frame, target $Z e$ initially at rest.
Relativistic motion for both incident & scattered particle.
Angle of deflection small.

Frame K' : Rest frame of incident particle $z e$.
Scattered particle motion non-relativistic.

Therefore

$$\frac{d^2 I'}{d\omega'\, d\Omega'} = \frac{z^2 e^2}{8\pi^2 c} (\Delta \boldsymbol{\beta}')^2 \left(1 + \cos^2\theta'\right)$$

where

$$\Delta \boldsymbol{\beta}' \simeq \frac{\Delta \boldsymbol{p}'}{M c}$$

is the velocity change of the incident particle as seen in frame K'.

Since

$$\Delta \boldsymbol{p}' \perp \beta$$

we have

$$\Delta \boldsymbol{p}' = \Delta \boldsymbol{p} \qquad \text{and} \qquad Q' = Q$$

Thus

$$\frac{d^2 I'}{d\omega'\, d\Omega'} = \frac{z^2 e^2}{8\pi^2 c} \left(\frac{Q}{M c}\right)^2 \left(1 + \cos^2\theta'\right)$$

and

$$\frac{d^3 \chi'}{d\omega'\, d\Omega'\, dQ'} = \frac{d^2 I'}{d\omega'\, d\Omega'} \frac{d\sigma_s}{dQ'}$$

$$= \frac{z^2 e^2}{8\pi^2 c} \left(\frac{Q}{M c}\right)^2 \frac{d\sigma_s}{dQ} \left(1 + \cos^2\theta'\right)$$

where

$$\frac{d\sigma_s}{dQ'} = \frac{d\sigma_s}{dQ}$$

Since

$$\frac{1}{\omega^2}\left(\frac{d^2 I}{d\,\omega\, d\,\Omega}\right) = \frac{1}{\omega'^2}\left(\frac{d^2 I'}{d\,\omega'\, d\,\Omega'}\right)$$

is Lorentz invariant, we have

$$\frac{d^3\chi}{d\,\omega\, d\,\Omega\, d\,Q} = \frac{\omega^2}{\omega'^2}\, d^3\, \frac{\chi'}{d\,\omega'\, d\,\Omega'\, d\,Q'}$$

$$= \frac{2\,z^2 e^2}{3\pi c}\left(\frac{Q}{M\,c}\right)^2 \frac{d\,\sigma_s}{d\,Q}\left[\frac{3}{16\pi}\left(\frac{\omega^2}{\omega'^2}\right)\left(1+\cos^2\theta'\right)\right]$$

Since

$$\omega = \gamma\omega'\,(1+\beta\cos\theta')$$

$$\omega' = \gamma\omega\,(1-\beta\cos\theta)$$

we have, from the 2nd eq,

$$\frac{\omega}{\omega'} = \frac{1}{\gamma\,(1-\beta\cos\theta)}$$

while, from the 1st,

$$\cos\theta' = \frac{1}{\beta}\left[\frac{\omega}{\gamma\,\omega'} - 1\right]$$

$$= \frac{1}{\beta}\left[\frac{\omega}{\gamma^2\,(1-\beta\cos\theta)} - 1\right]$$

$$= \frac{1-\gamma^2+\gamma^2\beta\cos\theta}{\beta\gamma^2\,(1-\beta\cos\theta)}$$

$$= \frac{\frac{1}{\beta}\left(\frac{1}{\gamma^2}-1\right)+\cos\theta}{1-\beta\cos\theta}$$

$$= \frac{-\beta+\cos\theta}{1-\beta\cos\theta}$$

where we've used

$$1-\frac{1}{\gamma^2} = \beta^2$$

For $\gamma \gg 1$ and $\theta \ll 1$,

$$\frac{\omega}{\omega'} \simeq \frac{1}{\gamma\left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\theta^2}{2}\right)\right]}$$

$$\simeq \frac{1}{\gamma\left[\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right]}$$

$$\simeq \frac{2\gamma}{1+\gamma^2\theta^2}$$

so that

$$\cos\theta' \simeq \left(1-\frac{1}{\gamma^2}\right)\left[\frac{2}{1+\gamma^2\theta^2}-1\right]$$

$$\simeq \frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2}$$

$$\left(\frac{\omega^2}{\omega'^2}\right)\left(1+\cos^2\theta'\right) \simeq \left(\frac{2\gamma}{1+\gamma^2\theta^2}\right)^2\left[1+\left(\frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2}\right)^2\right]$$

$$= \frac{4\gamma^2}{\left(1+\gamma^2\theta^2\right)^4}\left[\left(1+\gamma^2\theta^2\right)^2+\left(1-\gamma^2\theta^2\right)^2\right]$$

$$= \frac{8\gamma^2\left(1+\gamma^4\theta^4\right)}{\left(1+\gamma^2\theta^2\right)^4}$$

$$\frac{3}{16\pi}\left(\frac{\omega^2}{\omega'^2}\right)\left(1+\cos^2\theta'\right) \simeq \frac{3\gamma^2\left(1+\gamma^4\theta^4\right)}{2\pi\left(1+\gamma^2\theta^2\right)^4}$$

and

$$\frac{d^3\chi}{d\omega\, d\Omega\, dQ} \simeq \frac{2z^2e^2}{3\pi c}\left(\frac{Q}{Mc}\right)^2\frac{d\sigma_s}{dQ}\,3\gamma^2\,\frac{\left(1+\gamma^4\theta^4\right)}{2\pi\left(1+\gamma^2\theta^2\right)^4}$$

$$= \frac{z^2e^2}{\pi^2 c}\left(\frac{Q}{Mc}\right)^2\frac{d\sigma_s}{dQ}\,\gamma^2\,\frac{\left(1+\gamma^4\theta^4\right)}{\left(1+\gamma^2\theta^2\right)^4}$$

which is the relativistic Rutherford formula.

■ 3. Screening

Effects of atomic electrons in Brmsstrahlung are:

1. Coulomb collision: negligible since it's proportional to Z while that of nuclei, Z^2.
2. Screening of nuclear charge: noticeable.

Using the Thomas- Fermi model, the screening is described by

$$\frac{1}{r} \longrightarrow \frac{1}{r}e^{-r/a}$$

where the atomic radius a is given roughly by

$$a \simeq 1.4\, a_0\, Z^{-1/3} \qquad\qquad a_0 = \frac{\hbar^2}{m e^2} = \text{Bohr radius}$$

Accordingly, the (small angle) Rutherford scattering cross section [see Chap 13]

$$\frac{d\sigma_s}{d\Omega} = \left(\frac{2zZe^2}{pv}\right)^2\frac{1}{\theta^4}$$

is replaced by

$$\frac{d\sigma_s}{d\Omega} = \left(\frac{2zZe^2}{pv}\right)^2\frac{1}{\left(\theta^2+\theta_{\min}{}^2\right)^2}$$

where

$$\theta_{\min}^{(q)} = \frac{\hbar}{p\,a} \simeq \frac{Z^{1/3}}{192}\,\frac{m\,c}{p} \qquad (13.98)$$

Similarly

$$\frac{d\,\sigma_s}{d\,Q} = 8\,\pi \left(\frac{z\,Z\,e^2}{\beta\,c}\right)^2 \frac{1}{Q^3}$$

is replaced by

$$\frac{d\,\sigma_s}{d\,Q} = 8\,\pi \left(\frac{z\,Z\,e^2}{\beta\,c}\right)^2 \frac{Q}{\left(Q^2 + Q_s{}^2\right)^2}$$

where

$$Q_s = p\,\theta_{\min}^{(q)} = \frac{\hbar}{a} \simeq \frac{Z^{1/3}}{192}\,m\,c$$

Proof of this is left as an exercise.

Thus (15.21) is modified as

$$\frac{d^2\,\chi}{d\,\omega\,d\,Q} = \frac{16}{3}\left(\frac{z^2\,e^2}{c}\right)\left(\frac{z\,Z\,e^2}{M\,c^2}\right)^2 \frac{1}{\beta^2}\,\frac{Q^3}{\left(Q^2 + Q_s{}^2\right)^2}$$

and

$$\frac{d\,\chi}{d\,\omega} = \int_{Q_{\min}}^{Q_{\max}} d\,Q\,\frac{d^2\,\chi}{d\,\omega\,d\,Q}$$

$$= \frac{16}{3}\left(\frac{z^2\,e^2}{c}\right)\left(\frac{z\,Z\,e^2}{M\,c^2}\right)^2 \frac{1}{\beta^2}\int_{Q_{\min}}^{Q_{\max}} d\,Q\,\frac{Q^3}{\left(Q^2 + Q_s{}^2\right)^2}$$

The integral is evaluated as follows

$$J = \int_{Q_{\min}}^{Q_{\max}} d\,Q\,\frac{Q^3}{\left(Q^2 + Q_s{}^2\right)^2}$$

$$= \frac{1}{2}\int_{Q_{\min}{}^2}^{Q_{\max}{}^2} d\,q\,\frac{q}{\left(q + Q_s{}^2\right)^2} \qquad q = Q^2$$

$$= \frac{1}{2}\int_{Q_{\min}{}^2 + Q_s{}^2}^{Q_{\max}{}^2 + Q_s{}^2} d\,x\,\frac{x - Q_s{}^2}{x^2} \qquad x = q + Q_s{}^2$$

$$= \frac{1}{2}\left[\ln\left(\frac{Q_{\max}{}^2 + Q_s{}^2}{Q_{\min}{}^2 + Q_s{}^2}\right) + Q_s{}^2\left(\frac{1}{Q_{\max}{}^2 + Q_s{}^2} - \frac{1}{Q_{\min}{}^2 + Q_s{}^2}\right)\right]$$

For $Q_{\max} \gg Q_{\min}\,,\ Q_s$

$$J \simeq \ln\left(\frac{Q_{\max}}{\sqrt{Q_{\min}{}^2 + Q_s{}^2}}\right) - \frac{Q_s{}^2}{2\left(Q_{\min}{}^2 + Q_s{}^2\right)}$$

If we also have

$$Q_{\min} \gg Q_s$$

then

$$J \simeq \ln\left(\frac{Q_{\max}}{Q_{\min}}\right)$$

We recover previous Coulomb result & the screening effect is negligible.

On the other hand, if

$$Q_{\min} \ll Q_s$$

then

$$J \simeq \ln\left(\frac{Q_{\max}}{Q_s}\right) - \frac{1}{2}$$

and

$$\frac{d\chi}{d\omega} = \frac{16}{3}\left(\frac{z^2 e^2}{c}\right)\left(\frac{z Z e^2}{M c^2}\right)^2 \frac{1}{\beta^2}\left[\ln\left(\frac{Q_{\max}}{Q_s}\right) - \frac{1}{2}\right]$$

$$= \frac{16}{3}\left(\frac{Z^2 e^2}{c}\right)\left(\frac{z^2 e^2}{M c^2}\right)^2 \frac{1}{\beta^2}\left[\ln\left(\frac{Q_{\max}}{Q_s}\right) - \frac{1}{2}\right]$$

$$\simeq \frac{16}{3}\left(\frac{Z^2 e^2}{c}\right)\left(\frac{z^2 e^2}{M c^2}\right)^2 \left[\ln\left(\frac{Q_{\max}}{Q_s}\right) - \frac{1}{2}\right] \qquad \text{for} \qquad \beta \simeq 1.$$

To summarize,

$$Q_{\max} \simeq 2 M v \qquad (15.23)$$

For $\omega \to 0$,

$$Q_{\min}^{NR} \simeq p - p' \simeq \frac{2\hbar\omega}{v} \qquad (15.28)$$

$$Q_{\min}^{R} \simeq \frac{\hbar\omega M^2 c^3}{2 E E'} = \frac{\hbar\omega}{2 c \gamma \gamma'} \simeq \frac{\hbar\omega}{2 c \gamma^2} \qquad (15.33)$$

$$Q_s \simeq \frac{Z^{1/3}}{192} m c$$

Since $Q_{\min} \propto \omega$, there is always an ω_s below which $Q_{\min} < Q_s$.
In other words, screening is always important for low enough frequencies.

For the non- relativistic case, let

$$Q_{\min}^{NR} = Q_s$$

at ω_s, we have

$$\frac{2\hbar\omega_s}{v} = \frac{Z^{1/3}}{192} m c$$

or

$$\hbar\omega_s = \frac{Z^{1/3}}{192\beta} \frac{m}{M}\left(\frac{1}{2} M v^2\right) = \frac{Z^{1/3}}{192\beta} \frac{m}{M} (\hbar\omega_{\max})$$

where

$$\hbar\omega_{\max} = \frac{1}{2} M v^2$$

Since $\frac{m}{M} \simeq 10^{-3}$, screening effect is unimportant for most frequencies.

For the relativistic case, let

$$Q^R_{\min} = Q_s$$

at ω_s, we have

$$\frac{\hbar\,\omega_s}{2\,c\,\gamma\,\gamma\,'} = \frac{Z^{1/3}}{192}\,m\,c$$

or

$$\hbar\,\omega_s = \frac{Z^{1/3}}{96}\,\frac{m}{M}\,\gamma\,'\left(\gamma\,M\,c^2\right) = \frac{Z^{1/3}}{96}\,\frac{m}{M}\,\gamma\,'\,(\hbar\,\omega_{\max})$$

where

$$\hbar\,\omega_{\max} = \gamma\,M\,c^2$$

Since

$$\hbar\,\omega = E - E\,' = (\gamma - \gamma\,')\,M\,c^2$$

we have

$$x = \frac{\omega}{\omega_{\max}} = \frac{\hbar\,\omega}{\gamma\,M\,c^2} = 1 - \frac{\gamma\,'}{\gamma}$$

or

$$\gamma\,' = \gamma\,(1 - x)$$

$$\frac{Q^R_{\min}}{Q_s} \simeq \frac{96\,M}{m\,Z^{1/3}}\,\frac{x}{\gamma\,(1\,-\,x)} \xrightarrow[x\to 0]{} 0$$

Let

$$\frac{Q^R_{\min}}{Q_s} = \frac{1}{2} \qquad \text{at } x = \frac{1}{2}$$

and call the corresponding γ, γ_s, we have

$$\frac{96\,M}{m\,Z^{1/3}}\,\frac{\frac{1}{2}}{\gamma_s\left(1\,-\,\frac{1}{2}\right)} = \frac{1}{2}$$

we have

$$\gamma_s = \frac{192\,M}{m\,Z^{1/3}}$$

$$E_s = \gamma_s\,M\,c^2 = \frac{192\,M}{m\,Z^{1/3}}\,M\,c^2$$

For $\gamma > \gamma_s$ or $E > E_s$, we have $Q^R_{\min} < Q_s$ for most ω. (complete screening)

Now

$$\frac{Q^R_{\min}}{Q_s} = \frac{1}{2}\,\frac{\gamma_s}{\gamma}\,\frac{x}{1\,-\,x}$$

The condition for

$$\frac{Q_{\min}^R}{Q_s} = 1$$

is

$$\frac{1}{2}\,\frac{\gamma_s}{\gamma}\,\frac{x}{1-x} = 1$$

ie

$$x = \frac{2\gamma}{\gamma_s}(1-x) = \frac{2\gamma}{\gamma_s}\,\frac{1}{1+\frac{2\gamma}{\gamma_s}} = \frac{2\gamma}{\gamma_s + 2\gamma}$$

For $\gamma = \gamma_s$, we have $x = \frac{2}{3}$.

Thus, for complete screening, the screening formula applies strictly only up to $\frac{2}{3}$ of the spectrum.

$$\frac{d\chi}{d\omega} \simeq \frac{16}{3}\left(\frac{Z^2 e^2}{c}\right)\left(\frac{z^2 e^2}{M c^2}\right)^2 \left[\ln\left(\frac{Q_{\max}}{Q_s}\right) - \frac{1}{2}\right]$$

$$= \frac{16}{3}\left(\frac{Z^2 e^2}{c}\right)\left(\frac{z^2 e^2}{M c^2}\right)^2 \ln\left(\frac{2 M c \times 192}{Z^{1/3} m c}\, e^{-1/2}\right)$$

$$\simeq \frac{16}{3}\left(\frac{Z^2 e^2}{c}\right)\left(\frac{z^2 e^2}{M c^2}\right)^2 \ln\left(\frac{2 \times 33 M}{Z^{1/3} m}\right)$$

Note that $\frac{d\chi}{d\omega}$ is independent of ω for complete screening $(E > E_s)$.

Since

$$\frac{dN}{d\hbar\omega} = \frac{1}{\hbar^2 \omega}\,\frac{d\chi}{d\omega}$$

the photon spectrum is proportional to $\frac{1}{\hbar\omega}$ for $E > E_s$.

For $\gamma \gg 1$,

$$\frac{dE_{\text{rad}}}{dx} = N \int_0^{\omega_{\max}} d\omega\, \frac{d\chi}{d\omega}$$

$$\simeq \frac{16}{3} N \left(\frac{Z^2 e^2}{c}\right)\left(\frac{z^2 e^2}{M c^2}\right)^2 \int_0^{\gamma M c^2/\hbar} d\omega \ln\left(\frac{Q_{\max}}{\sqrt{Q_{\min}{}^2 + Q_s{}^2}}\right)$$

For negligible screening,

$$Q_{\min} \gg Q_s$$

and

$$K = \int_0^{\gamma M c^2/\hbar} d\omega \ln\left(\frac{Q_{\max}}{\sqrt{Q_{\min}{}^2 + Q_s{}^2}}\right)$$

$$\simeq \int_0^{\gamma M c^2/\hbar} d\omega \ln\left(\frac{Q_{\max}}{Q_{\min}}\right)$$

$$\simeq \int_0^{\gamma M c^2/\hbar} d\omega \ln\left(\frac{1}{A\omega}\right)$$

where

$$\frac{Q_{\max}}{Q_{\min}} = \frac{2 M c}{\hbar\omega / 2\gamma^2 c} = \frac{1}{A\omega}$$

$$A = \frac{\hbar}{4\gamma^2 M c^2}$$

Hence

$$K = -\int_0^{\gamma M c^2/\hbar} d\omega \; [\ln A + \ln\omega]$$

$$= -[\omega \ln A + \omega \ln\omega - \omega]_0^{\gamma M c^2/\hbar}$$

$$= -\left[\omega \ln\left(\frac{A\omega}{e}\right)\right]_0^{\gamma M c^2/\hbar}$$

Note: e is the exponential $\ln e = 1$

$$= -\frac{\gamma M c^2}{\hbar} \ln\left(\frac{A}{e}\gamma M \frac{c^2}{\hbar}\right)$$

$$= -\frac{\gamma M c^2}{\hbar} \ln\left(\frac{1}{4\gamma e}\right)$$

$$= \frac{\gamma M c^2}{\hbar} \ln(\lambda\gamma)$$

where λ is a numerical factor of order 1.

Hence

$$\frac{d E_{\text{rad}}}{d x} \simeq \frac{16}{3} N \left(\frac{Z^2 e^2}{c}\right)\left(\frac{z^2 e^2}{M c^2}\right)^2 \frac{\gamma M c^2}{\hbar} \ln(\lambda\gamma)$$

For complete screening (high energy)

$$\frac{d\chi}{d\omega} \simeq \frac{16}{3}\left(\frac{Z^2 e^2}{c}\right)\left(\frac{z^2 e^2}{M c^2}\right)^2 \ln\left(\frac{2 \times 33 M}{Z^{1/3} m}\right)$$

is independent of ω.

Hence

$$\frac{d E_{\text{rad}}}{d x} = N \int_0^{\omega_{\max}} d\omega \; \frac{d\chi}{d\omega}$$

$$= \quad N\, \omega_{\max} \frac{d\chi}{d\omega} \qquad\qquad \omega_{\max} = \gamma M c^2 / \hbar$$

$$\simeq \frac{16}{3} N \left(\frac{Z^2 e^2}{c} \right) \left(\frac{z^2 e^2}{M c^2} \right)^2 \; \frac{\gamma M c^2}{\hbar} \ln\left(\frac{2 \times 33 M}{Z^{1/3} m} \right)$$

Recalling the energy loss for ultra- relativistic particles

$$\frac{d E_{\text{coll}}}{d x} \simeq 4 \pi \text{N} Z \frac{z^2 e^4}{m v^2} \ln B_q$$

we have

$$\frac{d E_{\text{rad}}}{d E_{\text{coll}}} \simeq \frac{4}{3\pi} \left(\frac{Z z^2 e^2}{\hbar c} \right) \frac{m}{M} \frac{\ln\left(\frac{2 \times 33 M}{Z^{1/3} m} \right)}{\ln B_q} \gamma$$

For electron screening,

$$\frac{d E_{\text{rad}}}{d E_{\text{coll}}} = \quad 1$$

for

$\gamma \simeq 200$ in air with $\langle Z \rangle \simeq 7$

$\gamma \simeq 20$ in lead with $Z = 96$

Define the radiation length X_0 by

$$E_{\text{rad}}(x) = E_0 \, e^{-x/X_0}$$

or

$$\frac{d E_{\text{rad}}(x)}{d x} = \quad -\frac{1}{X_0} E_{\text{rad}}(x)$$

Using

$E_{\text{rad}} = \gamma M c^2$ (γ a function of x through β)

we have

$$X_0 = \frac{1}{\frac{16}{3} N \left(\frac{Z^2 e^2}{\hbar c} \right) \left(\frac{z^2 e^2}{M c^2} \right)^2 \ln\left(\frac{2 \times 33 M}{Z^{1/3} m} \right)}$$

■ 4. Virtual Quanta

As mentioned in § 11.12, the fields of a ultra- relativistic particle look like a radiation pulse to a stationary observer.

The method of virtual quanta (Weizsacker- Williams) is to replace the incident particle in a collision problem with an equivalent radiation pulse. The particle process is thereby replaced by a radiative one. (see table below)

Particle Process	Incident Particle	Struck System	Radiative process	b_{min}
Bremsstrahlung in electron (light particle) – nucleus collision	Nucleus	Electron (light particle)	Scattering of virtual photons of nuclear Coulomb field by the electron (light particle)	$\hbar/2\,M\,v$
Collisional ioniza – tion of atoms (in distant collisions)	Incident Particle	Atom	Photoejection of atomic electrons by virtual quanta	a
Electron disintegra – tion of nuclei	Electron	Nucleus	Photodisintegration of nuclei by virtual quanta	Larger of $\hbar/\gamma\,m\,v$ and R
Production of pions in electron – nuclear collisions	Electron	Nucleus	Photoproduction of pions by virtual quanta interactions with nucleus	

Uncertainty Principle $\longrightarrow\ b \simeq \dfrac{\hbar}{Q}$

$$b_{\min} \simeq \frac{\hbar}{Q_{\max}} \xrightarrow[\text{Bremsstrhlung}]{} \frac{\hbar}{2\,M\,v} \qquad (15.50)$$

$$b_{\max} \simeq \frac{\hbar}{Q_{\min}} \quad \text{incorporated into virtual quanta automatically}$$

Equivalent radiation fields are (see fig 15.6)

$$E_2(t) = q\,\frac{\gamma\,b}{\left(b^2 + \gamma^2\,v^2\,t^2\right)^{3/2}}$$

$$B_3(t) = \beta\,E_2(t)$$

$$E_1(t) = -q\,\frac{\gamma\,v\,t}{\left(b^2 + \gamma^2\,v^2\,t^2\right)^{3/2}}$$

For pulse P_1:

$$\frac{d\,I_1}{d\,\omega} = \frac{c}{2\,\pi}\left|\,E_2(\omega)\,\right|^2 \qquad (15.51a)$$

For pulse P_2:

$$\frac{d\,I_2}{d\,\omega} = \frac{c}{2\,\pi}\left|\,E_1(\omega)\,\right|^2 \qquad (15.51b)$$

Using (13.29) & (13.30), we have

$$\left.\begin{array}{l}\frac{d I_1(\omega,b)}{d\omega}\\ \frac{d I_2(\omega,b)}{d\omega}\end{array}\right\} = \frac{1}{\pi^2}\frac{q^2}{c}\left(\frac{c}{v}\right)^2\frac{1}{b^2}\left\{\begin{array}{l}\xi^2 K_1{}^2(\xi)\\ \frac{\xi^2}{\gamma^2} K_0{}^2(\xi)\end{array}\right. \tag{15.52}$$

$$= A\left\{\begin{array}{l}\xi^2 K_1{}^2(\xi)\\ \frac{\xi^2}{\gamma^2} K_0{}^2(\xi)\end{array}\right.$$

where

$$\xi = \frac{\omega b}{\gamma v}$$

$$A = \frac{1}{\pi^2}\frac{q^2}{c}\left(\frac{c}{v}\right)^2\frac{1}{b^2}$$

For $\xi \gg 1$,

$$K_i(\xi) \to \sqrt{\frac{\pi}{2\xi}}\ e^{-\xi}$$

For $\xi \ll 1$,

$$K_0(\xi) \to \ln\left(\frac{1.123}{\xi}\right)$$

$$K_1(\xi) \to \frac{1}{\xi}$$

Hence

$$\frac{d I_1}{d\omega} \longrightarrow \left\{\begin{array}{c}A\\ A\,\frac{\pi}{2}\,\xi\, e^{-2\xi}\end{array}\right. \text{for} \quad \begin{array}{l}\xi \ll 1\\ \xi \gg 1\end{array}$$

$$\frac{d I_2}{d\omega} \longrightarrow \left\{\begin{array}{c}A\left(\frac{\xi}{\gamma}\ln\xi\right)^2\\ \frac{1}{\gamma^2}\,A\,\frac{\pi}{2}\,\xi\, e^{-2\xi}\end{array}\right. \quad \text{for} \quad \begin{array}{l}\xi \ll 1\\ \xi \gg 1\end{array}$$

In particular, for $\xi \gg 1$,

$$\frac{d I_2}{d\omega} \to \frac{1}{\gamma^2}\left(\frac{d I_1}{d\omega}\right)$$

$$\ll \frac{d I_1}{d\omega} \qquad \text{if} \qquad \gamma \gg 1$$

Assuming

1. Incoherent superposition

(valid for weak fields, ie., motion in K ' non-relativistic)

2. Existence of $b_{\min}$ to take care of the region $b < b_{\min}$.

we have

$$\frac{dI}{d\omega} = 2\pi \int_{b_{\min}}^{\infty} db\, b \left(\frac{dI_1}{d\omega} + \frac{dI_2}{d\omega} \right)$$

$$= \; 2\pi \tilde{A} \int_{b_{\min}}^{\infty} db\, b\, \xi^2 \left(K_1{}^2 + \frac{1}{\gamma^2} K_0{}^2 \right)$$

where

$$\tilde{A} = A\, b^2 = \frac{1}{\pi^2} \frac{q^2}{c} \left(\frac{c}{v} \right)^2$$

$$\xi = \frac{\omega b}{\gamma v}$$

Hence

$$\frac{dI}{d\omega} = 2\pi \tilde{A} \left[x K_0 K_1 - \frac{v^2}{2c^2} \left(K_1{}^2 - K_0{}^2 \right) \right]_{x = \frac{\omega b_{\min}}{\gamma v}}$$

so that

$$\frac{dI}{d\omega} \underset{x \ll 1}{\longrightarrow} 2\pi \tilde{A} \left[\ln\left(\frac{1.123\, \gamma v}{\omega b_{\min}} \right) - \frac{v^2}{2c^2} \right]$$

$$\frac{dI}{d\omega} \underset{x \gg 1}{\longrightarrow} \pi^2 \tilde{A} \left[1 - \frac{v^2}{2c^2} \right] e^{-2\omega b_{\min}/\gamma v}$$

$$\frac{dN}{d(\hbar\omega)} = \frac{1}{\hbar^2 \omega} \frac{dI}{d\omega}$$

$$\underset{\omega \to 0}{\longrightarrow} \frac{2\pi \tilde{A}}{\hbar \omega} \left[\ln\left(\frac{1.123\, \gamma v}{\omega b_{\min}} \right) - \frac{v^2}{2c^2} \right]$$

Choice of $b_{\min}$:

1. Bremsstrahlung:

$$\frac{\hbar}{2Mv}.$$

2. Collisional ionization of atom:

[Incident: particle. Struck: atom.]

a (atomic radius)

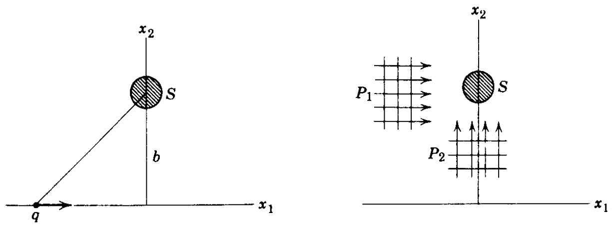
$b < b_{\min}$ treated as collision between particle & free e.

3. Electron disintegration of nuclei:

[Incident: e. Struck: nuclei.]

$$b_{\min} = \max \begin{pmatrix} \hbar/\gamma M V \\ R \end{pmatrix} = \begin{cases} b_{\min}^{(q)} \\ \text{nucleus radius} \end{cases}$$

- **Fig 15.6**



5. Bremsstrahlung & Virtual Quanta

Actual Process:

Incident particle Coulomb scattered by e^- in target.

Virtual Quanta Process:

VQ of e^- Thomson scattered by incident particle (struck system).

In K' (rest frame of incident particle):

$$\frac{d^2\chi'}{d\omega' d\Omega'} = \left(\frac{d\sigma'}{d\Omega'}\right)_{\text{TS}} \frac{dI'}{d\omega'}$$

with

$$\left(\frac{d\sigma'}{d\Omega'}\right)_{\text{TS}} = \left(\frac{z^2 e^2}{M c^2}\right)^2 \frac{1}{2}\left(1+\cos^2\theta'\right) \qquad (14.103)$$

$$\frac{dI'}{d\omega'} \underset{\omega'\to 0,\,\beta\to 1}{\simeq} \frac{2}{\pi c}\left(\frac{Zec}{v}\right)^2 \ln\left(\frac{\lambda M\gamma c^2}{\hbar\omega'}\right)$$

so that

$$\frac{d^2\chi'}{d\omega' d\Omega'} \underset{\omega'\to 0,\,\beta\to 1}{\simeq} \frac{1}{\pi c}\left(\frac{z^2 e^2}{M c^2}\right)^2 (Ze)^2 \left(1+\cos^2\theta'\right) \ln\left(\frac{\lambda M\gamma c^2}{\hbar\omega'}\right)$$

In K (laboratory frame):

$$\frac{d^2\chi}{d\omega\, d\Omega} = \frac{\omega^2}{\omega'^2}\frac{d^2\chi'}{d\omega' d\Omega'}$$

Now:

$$\frac{\omega}{\omega'} \simeq \frac{2\gamma}{1+\gamma^2\theta^2}$$

$$\cos\theta' \simeq \frac{1-\gamma^2\theta^2}{1+\gamma^2\theta^2}$$

$$\longrightarrow \quad \left(\frac{\omega}{\omega'}\right)^2 \left(1+\cos^2\theta'\right) \simeq \frac{4\gamma^2}{\left(1+\gamma^2\theta^2\right)^4}\left[\left(1+\gamma^2\theta^2\right)^2 + \left(1-\gamma^2\theta^2\right)^2\right]$$

$$= \frac{8\gamma^2\left(1+\gamma^4\theta^4\right)}{\left(1+\gamma^2\theta^2\right)^4}$$

Therefore

$$\frac{d^2\chi}{d\omega\, d\Omega} \underset{\omega'\to 0,\,\beta\to 1}{\simeq} \frac{8}{\pi c}\left(\frac{z^2 e^2}{M c^2}\right)^2 (Ze)^2 \frac{\gamma^2\left(1+\gamma^4\theta^4\right)}{\left(1+\gamma^2\theta^2\right)^4} \ln\left(\frac{2\lambda M\gamma^2 c^2}{\hbar\omega\left(1+\gamma^2\theta^2\right)}\right) \qquad (15.61)$$

To compare with eq(15.34) of which the argument of ln is

$$\frac{\lambda'' E E'}{M c^2 \hbar \omega}$$

we see that

$$E E' = \gamma M c^2 \gamma' M c^2 = \gamma \gamma' M^2 c^4$$

---

6. Beta Decay

(See Landau & Lifshitz, Mechanics)

$$Z \longrightarrow (Z \pm 1) + e^{\mp} + \nu$$

Let e be created at $t = 0$ at $\boldsymbol{x} = 0$ with $\boldsymbol{v} = c\,\boldsymbol{\beta} = \text{const.}$

Eq(15.2) becomes

$$\lim_{\omega \to 0} \frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c} \left| \boldsymbol{\epsilon}^* \cdot \frac{\boldsymbol{\beta}}{1 - \boldsymbol{\beta} \cdot \boldsymbol{n}} \right|^2$$

Assuming $\boldsymbol{v}$ goes from 0 to $c\,\boldsymbol{\beta}$ smoothly with time τ, the criterion for appreciable radiation is, from eq(15.15),

$$\omega \tau (1 - \boldsymbol{n} \cdot \langle \boldsymbol{\beta} \rangle) < 1$$

$$\langle \boldsymbol{\beta} \rangle = \frac{1}{\tau} \int_0^\tau d t\, \boldsymbol{\beta}(t)$$

For constant acceleration,

$$\beta(t) = a t$$

$$\beta(\tau) = a \tau$$

$$\langle \beta \rangle = \frac{1}{\tau} a \frac{\tau^2}{2} = \frac{1}{2} a \tau = \frac{1}{2} \beta(\tau) \le \frac{1}{2}$$

Since $\dot{\beta} = 0$ for $t > \tau$, this should serve as a good estimate of $\langle \beta \rangle$ forthe general case.

Thus

$$| \boldsymbol{n} \cdot \langle \boldsymbol{\beta} \rangle | \le \frac{1}{2} \beta \le \frac{1}{2}$$

$$\frac{1}{2} \omega \tau < \omega \tau (1 - \boldsymbol{n} \cdot \langle \boldsymbol{\beta} \rangle) < \frac{3}{2} \omega \tau$$

Therefore,

$$\omega \tau (1 - \boldsymbol{n} \cdot \langle \boldsymbol{\beta} \rangle) < 1$$

can be satisfied by

$$\omega \tau < \frac{2}{3}$$

Uncertainty principle gives

$$\tau \simeq \frac{\hbar}{E}$$

where E is the total energy of the electron (energy of ν neglected).

Under Fourier analysis, this means there is appreciable radiation only up to

$$\omega \simeq \frac{1}{\tau} \simeq \frac{E}{\hbar}$$

$\boldsymbol{E}$ field of radiation is in direction

$$\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta})$$

& is therefore in $(\boldsymbol{n}, \boldsymbol{\beta})$ plane.

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c}\left|\frac{-\beta\sin\theta}{1-\beta\cos\theta}\right|^2$$

$$= \frac{e^2}{4\pi^2 c}\beta^2\frac{\sin^2\theta}{(1-\beta\cos\theta)^2}$$

$$\frac{dI}{d\omega} = 2\pi\int_{-1}^{1} d\cos\theta\,\frac{d^2 I}{d\omega\, d\Omega}$$

$$= \frac{e^2}{2\pi c}\beta^2\int_{-1}^{1} dx\,\frac{1-x^2}{(1-\beta x)^2}$$

To evaluate

$$J = \int_{-1}^{1} dx\,\frac{1-x^2}{(1-\beta x)^2}$$

let

$$y = 1-\beta x$$

so that

$$dx = -\frac{1}{\beta}dy \qquad x = \frac{1}{\beta}(1-y)$$

and

$$J = -\frac{1}{\beta}\int_{1+\beta}^{1-\beta} dy\,\frac{1}{y^2}\left[1-\frac{1}{\beta^2}(1-y)^2\right]$$

$$= -\frac{1}{\beta^3}\int_{1+\beta}^{1-\beta} dy\,\frac{1}{y^2}\left[\beta^2-1+2y-y^2\right]$$

$$= -\frac{1}{\beta^3}\left[-\left(\beta^2-1\right)\frac{1}{y}+2\ln y-y\right]_{1+\beta}^{1-\beta}$$

With

$$\frac{1}{1-\beta}-\frac{1}{1+\beta} = \frac{2\beta}{1-\beta^2}$$

we have

$$J = -\frac{1}{\beta^3}\left[2\beta+2\ln\left(\frac{1-\beta}{1+\beta}\right)+2\beta\right]$$

$$= \frac{2}{\beta^2}\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2\right]$$

Finally

$$\frac{dI}{d\omega} = \frac{e^2}{\pi c}\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right)-2\right]$$

For $\beta \ll 1$, (low energy)

$$\ln\left(\frac{1+\beta}{1-\beta}\right) \longrightarrow 2\beta + \frac{2}{3}\beta^3$$

$$\frac{dI}{d\omega} \longrightarrow \frac{e^2}{\pi c} \cdot \frac{2}{3}\beta^2 \qquad \text{(negligible radiation)}$$

For the photon spectrum,

$$\frac{dN}{d(\hbar\omega)} = \frac{1}{\hbar^2\omega}\frac{dI}{d\omega}$$

$$= \frac{e^2}{\pi\hbar c}\frac{1}{\hbar\omega}\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\right]$$

The $\frac{1}{\hbar\omega}$ dependency leads to the name "inner Bremsstrahlung".

$$E_{\text{rad}} = \int_0^{\omega_{\max}} d\omega \frac{dI}{d\omega}$$

$$= \frac{e^2}{\pi\hbar c}\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\right] E \qquad E = \hbar\omega_{\max}$$

For $\beta \to 1$,

$$\beta \simeq 1 - \frac{1}{2\gamma^2}$$

$$\frac{1+\beta}{1-\beta} \simeq \left(2 - \frac{1}{2\gamma^2}\right)(2\gamma^2) \simeq 4\gamma^2$$

Since

$$E = \gamma m c^2$$

we have

$$\frac{1+\beta}{1-\beta} \simeq \left(\frac{2E}{mc^2}\right)^2$$

and

$$\frac{E_{\text{rad}}}{E} \simeq 2\frac{e^2}{\pi\hbar c}\left[\ln\left(\frac{2E}{mc^2}\right) - 1\right]$$

Since $\frac{e^2}{\hbar c} \simeq \frac{1}{137}$, while $\gamma \lesssim 30$ for most beta decay,

$$\frac{E_{\text{rad}}}{E} \ll 1$$

Effects of ν :

1. e energy E is less than the total energy E_{tot} and can have a distribution.
2. $\frac{dI}{d\omega}$ must be averaged over this distribution.

Note that there's no radiation from ν since it's neutral.

7. Electron Capture

Orbital electron capture:

$$Z + e^- \longrightarrow (Z-1) + \nu \qquad (15.70)$$

Let e^- orbit be in $x-y$ plane with radius a, frequency ω_0, while $\boldsymbol{n}$ is in $x-z$ plane. (see Fig.15.10)

Observed radiation is specified by

$$\boldsymbol{n} = (\sin\theta,\ 0,\ \cos\theta)$$
$$\boldsymbol{\epsilon}_{||} = (-\cos\theta,\ 0,\ \sin\theta)$$
$$\boldsymbol{\epsilon}_{\perp} = (0,\ 1,\ 0)$$

Position of e^-:

$$\boldsymbol{x}(t) = a\,(\boldsymbol{\epsilon}_1 \cos\phi + \boldsymbol{\epsilon}_2 \sin\phi)$$

where

$$\boldsymbol{\epsilon}_1 = \hat{\boldsymbol{x}} \qquad \boldsymbol{\epsilon}_2 = \hat{\boldsymbol{y}}$$
$$\phi = \omega_0\, t + \alpha$$

with α a phase factor.

Velocity of e^-:

$$\boldsymbol{v}(t) = a\,\omega_0\,(-\boldsymbol{\epsilon}_1 \sin\phi + \boldsymbol{\epsilon}_2 \cos\phi)$$

■ Disappearance of charge

Let e^- be captured at $t = 0$. Eq(14.67) becomes, in the dipole approximation,

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c^3}\left|\int_{-\infty}^{0} \boldsymbol{n}\times[\boldsymbol{n}\times\boldsymbol{v}(t)]\, e^{i\omega t}\, d t\right|^2 \qquad (15.72)$$

Now

$$\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta}) = a\,\frac{\omega_0}{c}\,\boldsymbol{n}\times[\,\boldsymbol{n}\times(-\boldsymbol{\epsilon}_1 \sin\phi + \boldsymbol{\epsilon}_2\cos\phi)\,]$$

Using

$$\begin{aligned}
\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\epsilon}_1) &= \boldsymbol{n}\,(\boldsymbol{n}\cdot\boldsymbol{\epsilon}_1) - \boldsymbol{\epsilon}_1\\
&= \boldsymbol{n}\sin\theta - \boldsymbol{\epsilon}_1\\
&= \left(\sin^2\theta - 1,\ 0,\ \sin\theta\cos\theta\right)\\
&= \cos\theta\,(-\cos\theta,\ 0,\ \sin\theta)\\
&= \boldsymbol{\epsilon}_{||}\cos\theta
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\epsilon}_2) &= \boldsymbol{n}\,(\boldsymbol{n}\cdot\boldsymbol{\epsilon}_2) - \boldsymbol{\epsilon}_2\\
&= -\boldsymbol{\epsilon}_2\\
&= -\boldsymbol{\epsilon}_{\perp}
\end{aligned}$$

we have

$$\begin{aligned}
\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\beta}) &= -a\,\frac{\omega_0}{c}\left(\boldsymbol{\epsilon}_{||}\cos\theta\ \sin\phi + \boldsymbol{\epsilon}_{\perp}\cos\phi\right)\\
&= -a\,\frac{\omega_0}{c}\left[\boldsymbol{\epsilon}_{||}\cos\theta\ \sin(\omega_0 t + \alpha) + \boldsymbol{\epsilon}_{\perp}\cos(\omega_0 t + \alpha)\right]
\end{aligned}$$

Eq(15.72) thus becomes

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{1}{c}\left(\frac{e\,\omega\,\omega_0\, a}{2\pi c}\right)^2 |\boldsymbol{K}|^2$$

where

$$\begin{aligned}\boldsymbol{K} &= \int_{-\infty}^{0} dt\, e^{i\omega t}\left[\boldsymbol{\epsilon}_{||}\cos\theta\ \sin(\omega_0 t+\alpha) + \boldsymbol{\epsilon}_{\perp}\cos(\omega_0 t+\alpha)\right] \\ &= \boldsymbol{\epsilon}_{||}\cos\theta\ I_2 + \boldsymbol{\epsilon}_{\perp} I_1\end{aligned}$$

with

$$\begin{aligned}I_1 &= \int_{-\infty}^{0} dt\, e^{i\omega t}\ \cos(\omega_0 t+\alpha) \\ I_2 &= \int_{-\infty}^{0} dt\, e^{i\omega t}\ \sin(\omega_0 t+\alpha)\end{aligned} \qquad (15.74)$$

Consider

$$\begin{aligned}I_{\pm} &= \int_{-\infty}^{0} dt\, e^{i\omega t \pm i(\omega_0 t+\alpha)} \\ &= e^{\pm i\alpha}\left[\frac{e^{i(\omega\pm\omega_0)t}}{i(\omega\pm\omega_0)}\right]_{-\infty}^{0} \\ &= e^{\pm i\alpha}\frac{1}{i(\omega\pm\omega_0)}\end{aligned}$$

where the value at $t = -\infty$ is made to vanish by adding a positive infinitesimal imaginary part to ω.

Hence

$$\begin{aligned}I_1 &= \frac{1}{2}(I_+ + I_-) \\ &= \frac{1}{2i}\left[\frac{e^{i\alpha}}{\omega+\omega_0} + \frac{e^{-i\alpha}}{\omega-\omega_0}\right] \\ &= \frac{1}{2i\left(\omega^2-\omega_0^2\right)}\left[\omega\left(e^{i\alpha}+e^{-i\alpha}\right) - \omega_0\left(e^{i\alpha}-e^{-i\alpha}\right)\right] \\ &= \frac{1}{\omega^2-\omega_0^2}\left[-i\,\omega\cos\alpha - \omega_0\sin\alpha\right]\end{aligned}$$

$$\begin{aligned}I_2 &= \frac{1}{2i}(I_+ - I_-) \\ &= -\frac{1}{2}\left[\frac{e^{i\alpha}}{\omega+\omega_0} - \frac{e^{-i\alpha}}{\omega-\omega_0}\right] \\ &= -\frac{1}{2\left(\omega^2-\omega_0^2\right)}\left[\omega\left(e^{i\alpha}-e^{-i\alpha}\right) - \omega_0\left(e^{i\alpha}+e^{-i\alpha}\right)\right] \\ &= \frac{1}{\omega^2-\omega_0^2}\left[i\,\omega\sin\alpha + \omega_0\cos\alpha\right]\end{aligned}$$

And

$$\boldsymbol{K} = \frac{1}{\omega^2 - \omega_0^2}\left\{ \boldsymbol{\epsilon}_{||} \cos\theta \; [i\,\omega \sin\alpha + \omega_0 \cos\alpha] - \boldsymbol{\epsilon}_{\perp} [i\,\omega \cos\alpha + \omega_0 \sin\alpha] \right\}$$

$$= \frac{1}{\omega^2 - \omega_0^2}\left\{ i\,\omega \left[\epsilon_{||} \cos\theta \sin\alpha - \boldsymbol{\epsilon}_{\perp} \cos\alpha \right] + \omega_0 \left[\boldsymbol{\epsilon}_{||} \cos\theta \cos\alpha - \boldsymbol{\epsilon}_{\perp} \sin\alpha \right]\right\}$$

$$\boldsymbol{K}^{*} = \frac{1}{\omega^2 - \omega_0^2}\left\{ -i\,\omega \left[\epsilon_{||} \cos\theta \sin\alpha - \boldsymbol{\epsilon}_{\perp} \cos\alpha \right] + \omega_0 \left[\boldsymbol{\epsilon}_{||} \cos\theta \cos\alpha - \boldsymbol{\epsilon}_{\perp} \sin\alpha \right]\right\}$$

$$|\,\boldsymbol{K}|^2 = \frac{1}{\left(\omega^2 - \omega_0^2\right)^2}\left\{ \omega^2 \left[\epsilon_{||} \cos\theta \sin\alpha - \boldsymbol{\epsilon}_{\perp} \cos\alpha \right]^2 + \omega_0^2 \left[\boldsymbol{\epsilon}_{||} \cos\theta \cos\alpha - \boldsymbol{\epsilon}_{\perp} \sin\alpha \right]^2\right\}$$

$$= \frac{1}{\left(\omega^2 - \omega_0^2\right)^2}\left\{ \omega^2 \left[\cos^2\theta \sin^2\alpha + \cos^2\alpha \right] + \omega_0^2 \left[\cos^2\theta \cos^2\alpha + \sin^2\alpha \right]\right\}$$

$$= \frac{1}{\left(\omega^2 - \omega_0^2\right)^2}\left\{ \omega^2 \cos^2\alpha + \omega_0^2 \sin^2\alpha + \cos^2\theta \left[\omega^2 \sin^2\alpha + \omega_0^2 \cos^2\alpha \right]\right\}$$

Hence

$$\frac{d^2 I}{d\,\omega\, d\,\Omega} = \frac{1}{c}\left(\frac{e\,\omega\,\omega_0\, a}{2\,\pi\, c\left(\omega^2 - \omega_0^2\right)}\right)^2 \left\{ \omega^2 \cos^2\alpha + \omega_0^2 \sin^2\alpha + \cos^2\theta \left[\omega^2 \sin^2\alpha + \omega_0^2 \cos^2\alpha \right]\right\} \tag{15.75}$$

Averaging over α, we have

$$\langle\, f\,\rangle = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} d\,\alpha\, f(\alpha)$$

$$\left\langle \cos^2\alpha \right\rangle = \frac{1}{\pi}\int_{0}^{\pi} d\,\alpha \cos^2\alpha = \frac{1}{2} = \left\langle \sin^2\alpha \right\rangle$$

so that

$$\frac{d^2 I}{d\,\omega\, d\,\Omega} = \frac{1}{c}\left(\frac{e\,\omega\,\omega_0\, a}{2\,\pi\, c\left(\omega^2 - \omega_0^2\right)}\right)^2 \frac{1}{2}\left\{ \omega^2 + \omega_0^2 + \cos^2\theta \left[\omega^2 + \omega_0^2 \right]\right\}$$

$$= \frac{1}{c}\left(\frac{e\,\omega\,\omega_0\, a}{2\,\pi\, c\left(\omega^2 - \omega_0^2\right)}\right)^2 \left(\omega^2 + \omega_0^2\right) \frac{1}{2}\left(1 + \cos^2\theta\right)\Big\} \tag{15.76}$$

Using

$$\int d\,\Omega\, \frac{1}{2}\left(1 + \cos^2\theta\right) = \pi \int_{-1}^{1} d\cos\theta \left(1 + \cos^2\theta\right)$$

$$= \pi\left(2 + \frac{2}{3}\right)$$

$$= \frac{8\,\pi}{3}$$

we have

$$\frac{d\,I}{d\,\omega} = \frac{2}{3\,\pi\, c}\left(\frac{e\,\omega\,\omega_0\, a}{c\left(\omega^2 - \omega_0^2\right)}\right)^2 \left(\omega^2 + \omega_0^2\right)$$

$$= \frac{2\, e^2}{3\,\pi\, c}\left(\frac{\omega_0\, a}{c}\right)^2 \frac{\omega^2\left(\omega^2 + \omega_0^2\right)}{\left(\omega^2 - \omega_0^2\right)^2} \tag{15.77}$$

With

$$d\,I = \hbar\,\omega\,d\,N$$

we have

$$\frac{d\,N}{d\,(\hbar\,\omega)} = \frac{2\,e^2}{3\,\pi\,\hbar\,c}\left(\frac{\omega_0\,a}{c}\right)^2 \frac{\omega^2\left(\omega^2 + {\omega_0}^2\right)}{\left(\omega^2 - {\omega_0}^2\right)^2}\,\frac{1}{\hbar\,\omega} \qquad (15.78)$$

Note: Jackson used $N(\hbar\omega)$ to denote $\frac{d\,N}{d\,(\hbar\,\omega)}$.

For $\omega \gg \omega_0$,

$$\frac{\omega^2\left(\omega^2 + {\omega_0}^2\right)}{\left(\omega^2 - {\omega_0}^2\right)^2} \simeq \frac{\omega^4}{\omega^4} = 1$$

so that

$$\frac{d\,N}{d\,(\hbar\,\omega)} = \frac{2\,e^2}{3\,\pi\,\hbar\,c}\left(\frac{\omega_0\,a}{c}\right)^2 \frac{1}{\hbar\,\omega}$$

which is typical of Bresstrahlung in the complete screening limit (see p.717).

For $\omega \simeq \omega_0$,

$$\frac{d\,N}{d\,(\hbar\,\omega)} \propto \frac{1}{\left(\omega^2 - {\omega_0}^2\right)^2}$$

is singular at $\omega = \omega_0$.

Note:
Classically, e^- in circular orbit emits Lamor radiation with frequency ω_0.
This is suppressed in quantum mechanic.
From the view point of electromagnetics, electron capture is equivalent to stopping the motion of e^- (or applying the orbital acceleration in the opposite direction). Again, Lamor radiation with frequency ω_0 is expected.
This is not suppressed in quantum mechanic.

Experimentally, when e^- in $2\,p$ orbit is captured, radiation strongly peaked at the $2\,p \to 1\,s$ frequency is observed. The reason for this is that the e^- wavefunction is non-zero at the nucleus only if it has $l = 0$ ($s-$ waves). The capture of a $2\,p$ electron is therefore more probable if it made a (virtual) transition to the $1\,s$ state first. Hence, $\hbar\,\omega_0 = E_{2\,p} - E_{1\,s}$

$$\simeq \frac{Z^2\,e^2}{2\,a_0}\left(\frac{1}{1^2} - \frac{1}{2^2}\right) \qquad (\,H\text{ -like atom}\,)$$

$$= \frac{3\,Z^2\,e^2}{8\,a_0}$$

where

$$a_n = \frac{n^2}{Z}\,a_0 \qquad a_0 = \frac{\hbar^2}{m\,e^2}$$

$$E_n = -\frac{Z\,e^2}{2\,a_n} = -\frac{Z\,e^2}{2\,n^2\,a_0}$$

Hence

$$\omega_0 \, a_1 = \frac{3 \, Z \, e^2}{8 \, \hbar}$$

and (15.78) becomes

$$\frac{d N}{d (\hbar \omega)} = \frac{3 \, e^2}{32 \, \pi \, \hbar \, c} \left(\frac{Z \, e^2}{\hbar \, c} \right)^2 \frac{\omega^2 \left(\omega^2 + \omega_0{}^2 \right)}{\left(\omega^2 - \omega_0{}^2 \right)^2} \frac{1}{\hbar \, \omega} \qquad (15.79)$$

$$\propto Z^2$$

■ Disappearance of Magnetic Moment

Classically,

$$\boldsymbol{\mu} = \frac{e}{m \, c} \boldsymbol{L}$$

The electron spin can be treated semi- classically as an angular momentum with magnitude

$$L = \sqrt{\frac{1}{2} \left(\frac{1}{2} + 1 \right)} \, \hbar = \frac{\sqrt{3}}{2} \hbar$$

and component

$$L_z = \pm \frac{\hbar}{2}$$

Hence

$$\mu \simeq \frac{\sqrt{3}}{2} \, e \, \frac{\hbar}{m \, c}$$

$$\mu_z = \pm \frac{e \, \hbar}{2 \, m \, c}$$

The angle α between $\boldsymbol{\mu}$ and the z −axis is

$$\cos \alpha = \pm \frac{\mu_z}{\mu} = \pm \frac{1}{\sqrt{3}}$$

or

$$\tan \alpha = \pm \sqrt{\frac{1}{\cos^2 \alpha} - 1} = \pm \sqrt{2}$$

Consider (14.74),

$$\frac{d^2 I}{d \, \omega \, d \, \Omega} = \frac{\omega^4}{2 \, \pi \, c^3} \left| \int_{-\infty}^{\infty} d \, t \, e^{i \, \omega \left\{ t + \frac{1}{c} (\boldsymbol{x} - \boldsymbol{n} \cdot \boldsymbol{r}) \right\}} \boldsymbol{n} \times [\boldsymbol{\mu} + \boldsymbol{n} \times (\boldsymbol{\beta} \times \boldsymbol{\mu})] \right|^2$$

In the dipole approximation, ie,

$$\frac{\omega}{c} \boldsymbol{n} \cdot \boldsymbol{r} \ll 1$$

$$\boldsymbol{\mu} \text{ time independent}$$

with non-relativistic motion

$$\beta \ll 1$$

we have

$$\frac{d^2 I}{d \, \omega \, d \, \Omega} \simeq \frac{\omega^4}{2 \, \pi \, c^3} \left| \int_{-\infty}^{0} d \, t \, e^{i \, \omega \, t} \, \boldsymbol{n} \times \boldsymbol{\mu} \right|^2$$

Now

$$\int_{-\infty}^{0} d\,t\; e^{i\,\omega\,t} = \frac{1}{i\,\omega}$$

Since (see below)

$$\boldsymbol{n}\times\boldsymbol{\mu} = \boldsymbol{\epsilon}\,\mu\sin\Theta$$

we have

$$\frac{d^2 I}{d\,\omega\,d\,\Omega} \simeq \frac{\omega^2}{2\,\pi\,c^3}\,\mu^2 \left\langle \sin^2\Theta \right\rangle_{\text{time}}$$

Semiclassical treatment of precession

$$\boldsymbol{n} = (\sin\theta,\ 0,\ \cos\theta)$$
$$\boldsymbol{\mu} = \mu\,(\sin\alpha\cos\phi,\ \sin\alpha\sin\phi,\ \cos\alpha)$$

$$\cos\alpha = \frac{1}{\sqrt{3}} \qquad \sin\alpha = \sqrt{\frac{2}{3}}$$

$$\boldsymbol{n}\times\boldsymbol{\mu} = \mu\,(-\cos\theta\sin\alpha\sin\phi,\ -\sin\theta\cos\alpha + \cos\theta\sin\alpha\cos\phi,\ \sin\theta\sin\alpha\sin\phi)$$

$$|\,\boldsymbol{n}\times\boldsymbol{\mu}\,|^2 = \mu^2\left[\cos^2\theta\sin^2\alpha\sin^2\phi + (-\sin\theta\cos\alpha + \cos\theta\sin\alpha\cos\phi)^2 + \sin^2\theta\sin^2\alpha\sin^2\phi\right]$$

$$= \mu^2\left[\sin^2\alpha\sin^2\phi + \sin^2\theta\cos^2\alpha - 2\sin\theta\cos\theta\cos\alpha\sin\alpha\cos\phi + \cos^2\theta\sin^2\alpha\cos^2\phi\right]$$

$$\equiv \mu^2\sin^2\Theta$$

Now:

$$\langle\,\boldsymbol{\mu}\,\rangle_{\text{time}} = \langle\,\boldsymbol{\mu}\,\rangle_{\phi}$$

so that

$$\left\langle \sin^2\Theta \right\rangle_{\text{time}} = \frac{1}{2}\sin^2\alpha + \sin^2\theta\cos^2\alpha + \frac{1}{2}\cos^2\theta\sin^2\alpha$$

where we've used

$$\left\langle \sin^2\phi \right\rangle_{\phi} = \frac{1}{2} = \left\langle \cos^2\phi \right\rangle_{\phi}$$
$$\langle\,\cos\phi\,\rangle_{\phi} = 0$$

Hence

$$\left\langle \sin^2\Theta \right\rangle_{\text{time}} = \frac{1}{2}\cdot\frac{2}{3} + \sin^2\theta\left(\frac{1}{3}\right) + \frac{1}{2}\cos^2\theta\left(\frac{2}{3}\right)$$

$$= \frac{2}{3}$$

and

$$\frac{d^2 I}{d\,\omega\, d\,\Omega} \simeq \frac{\omega^2}{3\,\pi\, c^3}\,\mu^2$$

$$= \frac{\omega^2}{3\,\pi\, c^3}\,\frac{3}{4}\left(\frac{e\,\hbar}{m\,c}\right)^2$$

$$= \frac{e^2}{4\,\pi\, c}\left(\frac{\hbar\,\omega}{m\,c^2}\right)^2$$

is independent of Ω.

Hence

$$\frac{d\,I}{d\,\omega} = \frac{e^2}{c}\left(\frac{\hbar\,\omega}{m\,c^2}\right)^2$$

$$\frac{d\,N}{d\,(\hbar\,\omega)} = \frac{1}{\hbar\,\omega}\left(\frac{d\,I}{d\,\omega}\right) = \frac{e^2}{\hbar\,c}\;\frac{\hbar\,\omega}{\left(m\,c^2\right)^2}$$

which is proportional to $\hbar\,\omega$, in contrast to the $\frac{1}{\hbar\,\omega}$ behavior of Bremsstrahlung.