16
Multipole Fields

1. Scalar waves

■ Wave Equation

The source- free scalar wave equation is,

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \psi = 0 \qquad (16.1)$$

Using the Fourier transform

$$\psi(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \psi(\boldsymbol{x}, \omega) \qquad (16.2)$$

$$\psi(\boldsymbol{x}, \omega) = \int_{-\infty}^{\infty} d t \, e^{i\omega t} \psi(\boldsymbol{x}, t)$$

we have,

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[-\frac{\omega^2}{c^2} - \nabla^2 \right] \psi(x, \omega) = 0$$

ie

$$\left(\nabla^2 + k^2 \right) \psi(\boldsymbol{x}, \omega) = 0 \qquad \text{(Helmholtz equation)} \qquad (16.3)$$

where $k^2 = \omega^2 / c^2$.

▪ ∇^2

In spherical coordinates,

$$\nabla^2 = \nabla_r{}^2 - \frac{\boldsymbol{L}^2}{r^2}$$

$$\nabla_r{}^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

$$= \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)^2$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2} r$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

$$-\boldsymbol{L}^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\boldsymbol{L} = \frac{1}{i}\boldsymbol{r}\times\nabla$$

$$\boldsymbol{r}\cdot\boldsymbol{L} = \frac{1}{i}\boldsymbol{r}\cdot(\boldsymbol{r}\times\nabla) = 0$$

The spherical harmonic functions $Y_{l m}(\theta, \phi)$ are defined by

$$\boldsymbol{L}^2 Y_{l m} = l(l+1) Y_{l m}$$

▪ Helmholtz Eq

Let

$$\psi(\boldsymbol{x}, \omega) = \sum_{l m} f_{k l}(r) Y_{l m}(\theta, \phi) \qquad (16.4)$$

(16.3) becomes

$$\sum_{l m} \left(\nabla^2 + k^2\right) f_{k l} Y_{l m} = 0$$

$$= \sum_{l m} \left(\nabla_r{}^2 - \frac{\boldsymbol{L}^2}{r^2} + k^2\right) f_{k l} Y_{l m}$$

$$= \sum_{l m} \left(\nabla_r{}^2 - \frac{l(l+1)}{r^2} + k^2\right) f_{k l} Y_{l m}$$

$$= \sum_{l m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} + k^2\right) f_{k l} Y_{l m}$$

which implies

$$\left[\frac{d^2}{d r^2} + \frac{2}{r}\frac{d}{d r} - \frac{l(l+1)}{r^2} + k^2\right] f_{k l}(r) = 0 \qquad (16.5)$$

Let

$$f_{kl}(r) = \frac{1}{\sqrt{r}} u_{kl}(r) \qquad (16.6)$$

we have

$$\frac{d f_{kl}}{dr} = -\frac{1}{2 r^{3/2}} u_{kl} + \frac{1}{\sqrt{r}} \frac{d u_{kl}}{dr}$$

$$\frac{d^2 f_{kl}}{dr^2} = \frac{3}{4 r^{5/2}} u_{kl} - \frac{1}{r^{3/2}} \frac{d u_{kl}}{dr} + \frac{1}{\sqrt{r}} \frac{d^2 u_{kl}}{dr^2}$$

Hence, (16.5) becomes

$$\frac{3}{4 r^{5/2}} u_{kl} - \frac{1}{r^{3/2}} \frac{d u_{kl}}{dr} + \frac{1}{\sqrt{r}} \frac{d^2 u_{kl}}{dr^2} - \frac{1}{r^{5/2}} u_{kl} + \frac{2}{r^{3/2}} \frac{d u_{kl}}{dr} - \frac{l(l+1)}{r^{5/2}} u_{kl} + \frac{k^2}{\sqrt{r}} u_{kl} = 0$$

$$= \frac{1}{\sqrt{r}} \frac{d^2 u_{kl}}{dr^2} + \frac{1}{r^{3/2}} \frac{d u_{kl}}{dr} - \frac{l(l+1) + \frac{1}{4}}{r^{5/2}} u_{kl} + \frac{k^2}{\sqrt{r}} u_{kl}$$

or

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\left(l + \frac{1}{2}\right)^2}{r^2} + k^2 \right] u_{kl} = 0 \qquad (16.7)$$

This equation is just the Bessel equation (3.75) with $\nu = l + \frac{1}{2}$. Thus the solutions for $f_{kl}(r)$ are

$$f_{kl}(r) = \frac{A}{\sqrt{r}} J_{l+1/2}(k r) + \frac{B}{\sqrt{r}} N_{l+1/2}(k r) \qquad (16.8)$$

$$= A' j_l(k r) + B' n_l(k r)$$

$$= A'' h_l^{(1)}(k r) + B'' h_l^{(2)}(k r)$$

(16.4) thus becomes

$$\psi(x) = \sum_{l m} \left[A_{l m}{}^{(1)} h_l{}^{(1)}(k r) + A_{l m}{}^{(2)} h_l{}^{(2)}(k r) \right] Y_{l m}(\theta, \phi) \qquad (16.16)$$

where the coefficients $A_{l m}{}^{(1)}$ and $A_{l m}{}^{(2)}$ are determined by the boundary conditions.

■ Spherical Bessel Functions

■ Definitions

The spherical Bessel and Hankel functions are defined by

$$j_l(x) = \sqrt{\frac{\pi}{2 x}} J_{l+1/2}(x)$$

$$n_l(x) = \sqrt{\frac{\pi}{2 x}} N_{l+1/2}(x)$$

$$h_l{}^{(1,2)}(x) = \sqrt{\frac{\pi}{2 x}} \left[J_{l+1/2}(x) \pm i N_{l+1/2}(x) \right] \qquad (16.9)$$

For real x, $h_l^{(2)}(x)$ is the complex conjugate of $h_l^{(1)}(x)$. From the series expansions (3.82) and (3.83) one can show that

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right)$$

$$n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\cos x}{x}\right) \qquad (16.10)$$

- Explicit Forms

For the first few values of l the explicit forms are:

$$j_0(x) = \frac{\sin x}{x} \qquad n_0(x) = -\frac{\cos x}{x} h_0^{(1)}(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$h_1^{(1)}(x) = -\frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right)$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3\cos x}{x^2} \qquad n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3\sin x}{x^2}$$

$$h_2^{(1)}(x) = \frac{e^{ix}}{x}\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right)$$

$$j_3(x) = \left(\frac{15}{x^4} - \frac{6}{x^2}\right)\sin x - \left(\frac{15}{x^3} - \frac{1}{x}\right)\cos x$$

$$n_3(x) = -\left(\frac{15}{x^4} - \frac{6}{x^2}\right)\cos x - \left(\frac{15}{x^3} - \frac{1}{x}\right)\sin x$$

$$h_3^{(1)}(x) = \frac{e^{ix}}{x}\left(1 + \frac{6i}{x} - \frac{15}{x^2} - \frac{15i}{x^3}\right) \qquad (16.11)$$

- **$x \ll 1, l$**

For $x \ll 1, l$

$$j_l(x) \to \frac{x^l}{(2l+1)!!}\left(1 - \frac{x^2}{2(2l+3)} + \ldots\right)$$

$$n_l(x) \to -\frac{(2l-1)!!}{x^{l+1}}\left(1 - \frac{x^2}{2(1-2l)} + \ldots\right) \qquad (16.12)$$

where $(2l+1)!! = (2l+1)(2l-1)(2l-3)\ldots 5\cdot3\cdot1$.

- ***x* ≫ *l***

For $x \gg l$

$$j_l(x) \;\to\; \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right)$$

$$n_l(x) \;\to\; -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right)$$

$$h_l^{(1)}(x) \;\to\; (-i)^{l+1}\,\frac{e^{ix}}{x} \tag{16.13}$$

- **Table**

These results can be summarized as

	$x \to 0$	$x \to \infty$
j_l	$\frac{x^l}{(2l+1)!!} \to 0$ for $l \neq 0$	$\frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right) \to 0$
n_l	$\frac{(2l-1)!!}{x^{l+1}} \to -\infty$	$-\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right) \to 0$
$h_l^{(1)}$	divergent	$(-i)^{l+1}\,\frac{e^{ix}}{x} \to 0$
$h_l^{(2)}$	divergent	$(i)^{l+1}\,\frac{e^{-ix}}{x} \to 0$

- **Recursions**

Let $z_l(x)$ be any one of $j_l(x)$, $n_l(x)$, $h_l^{(1)}(x)$, $h_l^{(2)}(x)$.
Some useful recursion formulas are (see Abramowitz for more)

$$\frac{2l+1}{x}\, z_l(x) = z_{l-1}(x) + z_{l+1}(x)$$

$$z_l{}'(x) = \frac{1}{2l+1}\left[\, l\, z_{l-1}(x) - (l+1)\, z_{l+1}(x) \,\right] \tag{16.14}$$

- **Wronskian**

The Wronskians of the various pairs are (see Arfken, 3rd ed., p.631)

$$W(j_l, n_l) = \frac{1}{i}\, W\left(j_l, h_l^{(1)}\right) = -W\left(n_l, h_l^{(1)}\right) = \frac{1}{x^2} \tag{16.15}$$

- **Sturm-Liouville form**

Using

$$\nabla_r^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

(16.5) can be put in the self-adjoint (Sturm-Liouville) form

$$\left[\frac{d}{d r} r^2 \frac{d}{d r} + k^2 r^2 - l\,(l+1)\right] f_{k\,l} = 0$$

The discontinuity of the corresponding green function derivative is therefore $-\frac{1}{r^2}$. (see Arfen p.899)

The proof of this is as follows. The general Sturm-Liouville eq is

$$\left[\frac{d}{d r}\, p\, \frac{d}{d r} + q\right] f = 0$$

where p, q are continuous functions of r.

The corresponding green function is defined by

$$\left[\frac{d}{d r}\, p\, \frac{d}{d r} + q\right] g(r, r') = \alpha\, \delta(r - r')$$

where α is a constant.

Integrating both sides through $\lim_{\epsilon\to 0} \int_{r'-\epsilon}^{r'+\epsilon} d r$ gives

$$\left[p(r)\, \frac{d\, g(r, r')}{d r}\right]_{r'+\epsilon} - \left[p(r)\, \frac{d\, g(r, r')}{d r}\right]_{r'-\epsilon} = \alpha$$

where, assuming $g(r, r')$ to be finite & continuous near $r = r'$,

$$\lim_{\epsilon\to 0} \int_{r'-\epsilon}^{r'+\epsilon} d r\, q(r)\, g(r, r') \simeq \lim_{\epsilon\to 0} 2\,\epsilon\, q(r')\, g(r', r') = 0$$

Hence, the discontinuity of the green function derivative is

$$\left[\frac{d\, g(r, r')}{d r}\right]_{r'+\epsilon} - \left[\frac{d\, g(r, r')}{d r}\right]_{r'-\epsilon} = \frac{\alpha}{p(r)}$$

For our case, $p = r^2$, $\alpha = -1$.

- **Green Function**

The Green function $G(\mathbf{x}, \mathbf{x}')$ for the Helmholtz eq is

$$\left(\nabla^2 + k^2\right) G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \qquad (16.17)$$

For outgoing waves [see chap 6, eq. (6.62)]

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{i k |\mathbf{x}-\mathbf{x}'|}}{4\pi\, |\mathbf{x} - \mathbf{x}'|} \qquad (16.18)$$

The spherical wave expansion for $G(\mathbf{x}, \mathbf{x}')$ is [see chap 3, sec 3.9]

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l\,m} g_l(r, r')\, Y_{l\,m}^*(\theta', \phi')\, Y_{l\,m}(\theta, \phi) \qquad (16.19)$$

To proceed, we turn for guidance to the Green function for the Laplace eq

$$\nabla^2 \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = -4\pi\, \delta(\boldsymbol{x} - \boldsymbol{x}')$$

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \qquad (3.70)$$

Obviously

$$G(\boldsymbol{x}, \boldsymbol{x}') \xrightarrow[k\to 0]{} \frac{1}{4\pi\, |\boldsymbol{x} - \boldsymbol{x}'|} \qquad \text{(static case)}$$

Recalling that $r_<^l$ & $r_>^{l+1}$ appears in (3.70) because they are solutions to the radial part of the Laplace equation, we begin with the ansatz

$$g_l(r, r') = A\, j_l(k r_<)\, h_l^{(1)}(k r_>) \qquad (16.21)$$

where

A is a constant determined by the discontinuity at $r = r'$ or reduction to (3.70) for $k \to 0$.

$j_l(k r_<)$ is to give $r_<^l$ for $k \to 0$.

$h_l^{(1)}(k r_>)$ is to give $r_>^{-l-1}$ for $k r \to 0$,

and $\dfrac{e^{ikr}}{kr}$ for $r \to \infty$ (outgoing wave)

With $A = i k$ (see below), we ahve

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{e^{ik|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi\, |\boldsymbol{x} - \boldsymbol{x}'|}$$

$$= i k \sum_{l=0}^{\infty} j_l(k r_<)\, h_l^{(1)}(k r_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \qquad (16.22)$$

- ***A = i k***

For $k \to 0$,

$$j_l(k r_<) \longrightarrow \frac{(k r_<)^l}{(2l+1)!!}$$

$$h_l^{(1)}(k r_>) \longrightarrow -i \frac{(2l-1)!!}{(k r_>)^{l+1}}$$

so that

$$j_l(k r_<)\, h_l^{(1)}(k r_>) \longrightarrow -\frac{i}{k} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}}$$

Therefore, (16.19) reduces to (3.70) only if

$$A = i k$$

To get A from discontinuity of G', we have [see Arfken p.623]

$$\left(\frac{d g_l(r, r')}{dr}\right)_{r=r_+'} - \left(\frac{d g_l(r, r')}{dr}\right)_{r=r_-'} = -\frac{1}{r^2}$$

where

$r = r_+'$ means $r = r_>$, $r' = r_<$

$r = r_-'$ means $r = r_<$, $r' = r_>$

so that we can also write

$$\left(\frac{d g_l}{d r_>}\right)_{r_>=r} - \left(\frac{d g_l}{d r_<}\right)_{r_<=r} = -\frac{1}{r^2}$$

Hence,

$$k\,A\left[\,j_l\ \ h_l^{(1)\,\prime} - j_l{}'\ h_l^{(1)}\,\right] = -\frac{1}{r^2}$$

The bracket is just the Wronskian $W\left(\,j_l\,,\ h_l^{(1)}\right)$ so that

$$k\,A\,\frac{i}{(k\,r)^2} = \ -\frac{1}{r^2}$$

ie

$$A = i\,k$$

■ *L*

■ Basics

The spherical harmonics $Y_{l\,m}(\,\theta,\,\phi\,)$ (3.53), are solutions of the equation,

$$-\left[\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\,\theta}\left(\sin\theta\,\frac{\partial}{\partial\,\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\,\phi^2}\,\right] Y_{l\,m} = l\,(l+1)\,Y_{l\,m} \qquad (16.23)$$

or

$$\boldsymbol{L}^2\,Y_{l\,m} = l\,(\,l+1\,)\,Y_{l\,m} \qquad (16.24)$$

$$-\boldsymbol{L}^2 = \ \frac{1}{\sin\theta}\,\frac{\partial}{\partial\,\theta}\left(\sin\theta\,\frac{\partial}{\partial\,\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\,\phi^2}$$

The differential operator $\boldsymbol{L}^2 = L_x{}^2 + L_y{}^2 + L_z{}^2$, where

$$\boldsymbol{L} = \ \frac{1}{i}\,(\,\boldsymbol{r}\times\nabla\,) \qquad (16.25)$$

is $\hbar^{-1}$ times the orbital angular- momentum operator of wave mechanics.

The components of L can be written

$$L_x = \ i\left(\sin\phi\,\frac{\partial}{\partial\,\theta} + \cot\theta\,\cos\phi\,\frac{\partial}{\partial\,\phi}\right)$$

$$L_y = i\left(-\cos\phi\,\frac{\partial}{\partial\,\theta} + \cot\theta\,\sin\phi\,\frac{\partial}{\partial\,\phi}\right)$$

$$L_+ = L_x + i\,L_y = e^{i\,\phi}\left(\frac{\partial}{\partial\,\theta} + i\cot\theta\,\frac{\partial}{\partial\,\phi}\right)$$

$$L_- = L_x - i\,L_y = e^{-i\,\phi}\left(-\frac{\partial}{\partial\,\theta} + i\cot\theta\,\frac{\partial}{\partial\,\phi}\right)$$

$$L_z = -i\,\frac{\partial}{\partial\,\phi} \qquad (16.26)$$

L operates only on angular variables and is independent of r. From (16.25)

$$\boldsymbol{r}\cdot\boldsymbol{L} = 0 \qquad (16.27)$$

holds as an operator equation. From the explicit forms (16.26) it is easy to verify that L^2 is equal to the operator on the left side of (16.23).

Using

$$\hat{\boldsymbol{r}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\hat{\boldsymbol{\theta}} = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$$

$$\hat{\boldsymbol{\phi}} = (-\sin\phi,\ \cos\phi,\ 0)$$

we have

$$\begin{aligned} L_r &= \hat{\boldsymbol{r}}\cdot\boldsymbol{L} \\ &= i\Big[\sin\theta\cos\phi\left(\sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi}\right) \\ &\quad +\sin\theta\sin\phi\left(-\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}\right) \\ &\quad -\cos\theta\,\frac{\partial}{\partial\phi}\Big] \\ &= 0 \end{aligned} \qquad (16.27)$$

$$\begin{aligned} L_\theta &= \hat{\boldsymbol{\theta}}\cdot\boldsymbol{L} \\ &= i\Big[\cos\theta\cos\phi\left(\sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi}\right) \\ &\quad +\cos\theta\sin\phi\left(-\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}\right) \\ &\quad +\sin\theta\,\frac{\partial}{\partial\phi}\Big] \\ &= \frac{i}{\sin\theta}\,\frac{\partial}{\partial\phi} \end{aligned}$$

$$\begin{aligned} L_\phi &= \hat{\boldsymbol{\phi}}\cdot\boldsymbol{L} \\ &= i\Big[-\sin\phi\left(\sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi}\right) \\ &\quad +\cos\phi\left(-\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}\right)\Big] \\ &= -i\,\frac{\partial}{\partial\theta} \end{aligned}$$

or

$$\boldsymbol{L} = i\left(\hat{\boldsymbol{\theta}}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi} - \hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\right)$$

$$\hat{\boldsymbol{r}}\times\boldsymbol{L} = i\left(\hat{\boldsymbol{\phi}}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi} + \hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta}\right)$$

From the explicit forms (16.26) and recursion relations for $Y_{l\,m}$ the following useful relations can be established:

$$L_+\,Y_{l\,m} = \sqrt{(l-m)(l+m+1)}\;Y_{l\,m+1}$$

$$L_-\,Y_{l\,m} = \sqrt{(l+m)(l-m+1)}\;Y_{l\,m-1}$$

$$L_z\,Y_{l\,m} = m\,Y_{l\,m} \qquad (16.28)$$

- **Useful Relations**

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\boldsymbol{L}^2}{r^2}$$

$$\boldsymbol{L} = \frac{1}{i}\boldsymbol{r}\times\nabla$$

$$\boldsymbol{L}\times\boldsymbol{L} = i\,\boldsymbol{L} \qquad \text{or} \qquad \left[L_i\,,\,L_j\right] = i\,\epsilon_{i\,j\,k}\,L_k$$

$$\nabla = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} - i\,\frac{1}{r}\hat{\boldsymbol{r}}\times\boldsymbol{L} \qquad \text{or} \qquad i\,\boldsymbol{r}\times\boldsymbol{L} = r^2\left(\hat{\boldsymbol{r}}\frac{\partial}{\partial r} - \nabla\right)$$

$$\boldsymbol{r}\cdot\boldsymbol{L} = 0$$

$$i\,\nabla\times\boldsymbol{L} = \boldsymbol{r}\,\nabla^2 - \nabla\left(1 + r\,\frac{\partial}{\partial r}\right)$$

$$\nabla\cdot\boldsymbol{L} = 0$$

$$L^2 = r\,\frac{\partial^2}{\partial r^2} r - r^2\,\nabla^2$$

$$= \frac{1}{2}\left(L_+\,L_- + L_-\,L_+\right) + L_z^{\,2}$$

$$\left[\boldsymbol{L},\,L^2\right] = 0 \qquad \text{or} \qquad L^2\,\boldsymbol{L} = \boldsymbol{L}\,L^2$$

$$\left[\boldsymbol{L},\,\nabla^2\right] = 0 \qquad \text{or} \qquad L_j\,\nabla^2 = \nabla^2\,L_j$$

Proofs for some of the less familiar relations above are as follows.

$$i\,\boldsymbol{r}\times\boldsymbol{L}\,f = \boldsymbol{r}\times(\boldsymbol{r}\times\nabla\,f)$$
$$= \boldsymbol{r}\,(\boldsymbol{r}\cdot\nabla\,f) - r^2\,\nabla\,f$$
$$= \boldsymbol{r}\,r\,\frac{\partial f}{\partial r} - r^2\,\nabla\,f$$

so that

$$i\,\boldsymbol{r}\times\boldsymbol{L} = r^2\left(\hat{\boldsymbol{r}}\,\frac{\partial}{\partial r} - \nabla\right)$$
$$= -r\left(\hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi}\right)$$

$$\hat{\boldsymbol{r}}\times\boldsymbol{L} = i\left(\hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi}\right)$$

$$\nabla = \hat{\boldsymbol{r}}\,\frac{\partial}{\partial r} - i\,\frac{1}{r^2}\,\boldsymbol{r}\times\boldsymbol{L}$$
$$= \hat{\boldsymbol{r}}\,\frac{\partial}{\partial r} - \frac{1}{r^2}\,\boldsymbol{r}\times(\boldsymbol{r}\times\nabla)$$
$$= \hat{\boldsymbol{r}}\,\frac{\partial}{\partial r} - \hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\nabla\right)$$
$$= \text{radial part + angular part}$$

Using

$$\nabla\times(\psi\,\boldsymbol{a}) = \nabla\,\psi\times\boldsymbol{a} + \psi\,\nabla\times\boldsymbol{a}$$

we have

$$\nabla\times(\boldsymbol{r}\,f\,) = -\boldsymbol{r}\times\nabla\,f + f\,\nabla\times\boldsymbol{r}$$

$$= -\boldsymbol{r}\times\nabla\,f \qquad \text{where } \nabla\times\boldsymbol{r} = 0$$

Hence

$$\nabla\times\boldsymbol{r} = -\boldsymbol{r}\times\nabla$$

if $\nabla\times\boldsymbol{r}$ is treated as an operator.

In operator form: [Put in an arbitrary function *f* if you get lost]

$$i\,\nabla\times\boldsymbol{L} = \nabla\times(\,\boldsymbol{r}\times\nabla\,)$$

$$= \nabla\times(\,\nabla\times\boldsymbol{r}\,)$$

$$= -\nabla\,(\,\nabla\cdot\boldsymbol{r}\,) + \nabla^2\,\boldsymbol{r}$$

Now

$$\nabla\cdot\boldsymbol{r} = \boldsymbol{r}\cdot\nabla + (\,\nabla\cdot\boldsymbol{r}\,) = r\,\frac{\partial}{\partial r} + 3$$

$$\nabla^2\boldsymbol{r} = \partial_j\partial_j\,x_i$$

$$= \partial_j\left(\delta_{i\,j} + x_i\,\partial_j\right)$$

$$= \partial_i + \delta_{i\,j}\,\partial_j + x_i\,\partial_j\partial_j$$

$$= 2\,\nabla + \boldsymbol{r}\,\nabla^2$$

Therefore

$$i\,\nabla\times\boldsymbol{L} = -\nabla\left(r\,\frac{\partial}{\partial r} + 3\right) + 2\,\nabla + \boldsymbol{r}\,\nabla^2$$

$$= -\nabla\left(r\,\frac{\partial}{\partial r} + 1\right) + \boldsymbol{r}\,\nabla^2$$

Note that the operator equation

$$\left[\,\nabla^2,\,\boldsymbol{r}\,\right] = \nabla^2\boldsymbol{r} - \boldsymbol{r}\,\nabla^2 = 2\,\nabla$$

is equivalent to the quantum relation

$$\left[\,\boldsymbol{p}^2,\,\boldsymbol{r}\,\right] = -2\,i\,\hbar\,\boldsymbol{p}$$

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2. Multipole Expansions

■ Vector Formulae

$$\boldsymbol{a}\cdot(\,\boldsymbol{b}\times\boldsymbol{c}\,) = \boldsymbol{b}\cdot(\,\boldsymbol{c}\times\boldsymbol{a}\,) = \boldsymbol{c}\cdot(\,\boldsymbol{a}\times\boldsymbol{b}\,)$$

$$\boldsymbol{a}\times(\,\boldsymbol{b}\times\boldsymbol{c}\,) = (\,\boldsymbol{c}\cdot\boldsymbol{a}\,)\,\boldsymbol{b} - (\,\boldsymbol{a}\cdot\boldsymbol{b}\,)\,\boldsymbol{c}$$

$$(\,\boldsymbol{a}\times\boldsymbol{b}\,)\cdot(\,\boldsymbol{c}\times\boldsymbol{d}\,) = (\,\boldsymbol{a}\cdot\boldsymbol{c}\,)(\,\boldsymbol{b}\cdot\boldsymbol{d}\,) - (\,\boldsymbol{a}\cdot\boldsymbol{d}\,)(\,\boldsymbol{b}\cdot\boldsymbol{c}\,)$$

$$\nabla\times\nabla\,\psi = 0$$

$$\nabla\cdot(\,\nabla\times\boldsymbol{a}\,) = 0$$

$$\nabla\times(\,\nabla\times\boldsymbol{a}\,) = \nabla(\,\nabla\cdot\boldsymbol{a}\,) - \nabla^2\boldsymbol{a}$$

$$\nabla\cdot(\,\psi\,\boldsymbol{a}\,) = \boldsymbol{a}\cdot\nabla\psi + \psi\,\nabla\cdot\boldsymbol{a}$$

$$\nabla\times(\,\psi\,\boldsymbol{a}\,) = \nabla\psi\times\boldsymbol{a} + \psi\,\nabla\times\boldsymbol{a}$$

$$\nabla(\,\boldsymbol{a}\cdot\boldsymbol{b}\,) = (\,\boldsymbol{a}\cdot\nabla\,)\,\boldsymbol{b} + (\,\boldsymbol{b}\cdot\nabla\,)\,\boldsymbol{a} + \boldsymbol{a}\times(\,\nabla\times\boldsymbol{b}\,) + \boldsymbol{b}\times(\,\nabla\times\boldsymbol{a}\,)$$

$$\nabla\cdot(\,\boldsymbol{a}\times\boldsymbol{b}\,) = \boldsymbol{b}\cdot(\,\nabla\times\boldsymbol{a}\,) - \boldsymbol{a}\cdot(\,\nabla\times\boldsymbol{b}\,)$$

$$\nabla\times(\,\boldsymbol{a}\times\boldsymbol{b}\,) = \boldsymbol{a}\,(\,\nabla\cdot\boldsymbol{b}\,) - \boldsymbol{b}\,(\,\nabla\cdot\boldsymbol{a}\,) + (\,\boldsymbol{b}\cdot\nabla\,)\,\boldsymbol{a} - (\,\boldsymbol{a}\cdot\nabla\,)\,\boldsymbol{b}$$

Let

$$\boldsymbol{x} = r\,\boldsymbol{n} \qquad r = |\,\boldsymbol{x}\,|$$

then

$$\nabla\cdot\boldsymbol{x} = 3 \qquad \nabla\times\boldsymbol{x} = 0$$

$$\nabla\cdot\boldsymbol{n} = \frac{2}{r} \qquad \nabla\times\boldsymbol{n} = 0$$

$$(\boldsymbol{a}\cdot\nabla)\,\boldsymbol{n} = \frac{1}{r}\,[\,\boldsymbol{a} - \boldsymbol{n}\,(\boldsymbol{a}\cdot\boldsymbol{n})\,] = \frac{1}{r}\,\boldsymbol{a}_\perp$$

■ Multipole Fields

Maxwell eqs for source- free region for a single Fourier (ω) field component are

$$\nabla\times\boldsymbol{E} = i\,k\,\boldsymbol{B} \qquad \nabla\cdot\boldsymbol{E} = 0$$

$$\nabla\times\boldsymbol{B} = -i\,k\,\boldsymbol{E} \qquad \nabla\cdot\boldsymbol{B} = 0$$

Therefore

$$\boldsymbol{B} = \frac{1}{i\,k}\,\nabla\times\boldsymbol{E}$$

$$\nabla\times\boldsymbol{B} = \frac{1}{i\,k}\,\nabla\times(\nabla\times\boldsymbol{E})$$

$$= \frac{1}{i\,k}\left[\,\nabla(\nabla\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E}\,\right]$$

$$= -\frac{1}{i\,k}\,\nabla^2\boldsymbol{E} = -i\,k\,\boldsymbol{E}$$

and

$$\boldsymbol{E} = -\frac{1}{i\,k}\,\nabla\times\boldsymbol{B}$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{i\,k}\,\nabla\times(\nabla\times\boldsymbol{B})$$

$$= -\frac{1}{i\,k}\left[\,\nabla(\nabla\cdot\boldsymbol{B}) - \nabla^2\boldsymbol{B}\,\right]$$

$$= \frac{1}{i\,k}\,\nabla^2\boldsymbol{B} = i\,k\,\boldsymbol{B}$$

Actually, these 2 sets of eqs are related by the transform $\boldsymbol{E}\to\boldsymbol{B}$, $\boldsymbol{B}\to-\boldsymbol{E}$ so only 1 of them need be calculated in detail.

Hence

$$\left(\nabla^2 + k^2\right)\boldsymbol{E} = 0$$

$$\left(\nabla^2 + k^2\right)\boldsymbol{B} = 0$$

Let E_i and B_i be the cartesian components of the fields. An ansatz in spherical coordinates is

$$E_i = z_l^i(\,k\,r\,)\,Y_{l\,m}(\,\Omega\,)$$

and similarly for B_i.

In order to satisfy $\nabla\cdot\boldsymbol{E} = 0$, z_l^i will become m dependent. Furthermore, the calculation of $\frac{\partial}{\partial x^i}\left(z_l^i\,Y_{l\,m}\right)$ is not trivial. [See C.J.Bouwkamp, H.B.G.Casmir, Physica 20, 539 (54) for detailed discussions]

Here, as suggested by Bouwkamp & Casimir, we'll made use of the vector harmonics instead.

Digression

The idea of multipole expansion of the radiation fields outside the sources is an obvious generalization of the same technique used so successfully in the static case. (see Chap 4) Preliminary study in the cases of dipole & quadrupole radiations was already discussed in Chap 9. What we do here is merely to make things more systematic. (with the help of more powerful mathematical technique, of course)

Our aim is to find a set of orthogonal, complete functions that are solutions to the source- free Maxwell equations. Any particular solutions (for specific boundary conditions and / or sources) can then be reduced to the (relatively) simple calculations of expansion coefficients.

In terms of spherical coordinates, experience from the static case tells us that these functions will be multipole fields (angular part proportional to $Y_{l\,m}$)

Since we are most interested in radiation fields which by definition are transverse in the far zone, the transverse fields are of paramount importance. However, in the near zone, the fields are not transverse (Chap 9). This then leads to the idea of the transverse electric (TE) and transverse magnetic (TM) modes as discussed in Chap 8. The virtue of these modes is that they are orthogonal, ie.

$$\int d\,\Omega\; \boldsymbol{E}_a^{(\alpha)} \cdot \boldsymbol{E}_b^{(\beta)} \propto \delta_{\alpha\,\beta}\;\delta_{a\,b}$$
$$\int d\,\Omega\; \boldsymbol{B}_a^{(\alpha)} \cdot \boldsymbol{B}_b^{(\beta)} \propto \delta_{\alpha\,\beta}\;\delta_{a\,b}$$
$$\int d\,\Omega\; \boldsymbol{n} \cdot \left[\boldsymbol{E}_a^{(\alpha)} \times \boldsymbol{B}_b^{(\beta)} \right] \propto \delta_{\alpha\,\beta}\;\delta_{a\,b}$$

where α, β denote the modes and a, b the order of the fields.

The completeness of these fields will be assumed without proof.

A general field can then be expanded as

$$\boldsymbol{E} = \sum_{\alpha\,a} C_{\alpha\,a}\, \boldsymbol{E}_a^{(\alpha)}$$

with

$$C_{\alpha\,a} = \frac{1}{N_{\alpha\,a}} \int d\,\Omega\, \boldsymbol{E} \cdot \boldsymbol{E}_a^{(\alpha)}$$

where $N_{\alpha\,a}$ is a normalization factor. Similarly for $\boldsymbol{B}$.

Now, what is a, the order of the fields? Obviously, it means l, m for multipole fields.
If we were dealing with scalar fields, the orthogonal conditions can be satisfied if the modes are proportional to $Y_{l\,m}$. The following pages are devoted to show you that proper generalization to vector fields is the vector harmonics $\boldsymbol{L}\, Y_{l\,m}$ and their partners $\nabla \times \boldsymbol{L}\, Y_{l\,m}$.

■ Multipole Fields Resumed

Consider

$$\nabla^2 (\,\boldsymbol{r} \cdot \boldsymbol{A}\,) = \nabla \cdot [\, \nabla (\,\boldsymbol{r} \cdot \boldsymbol{A}\,)\,]$$

Now

$$\nabla (\,\boldsymbol{r} \cdot \boldsymbol{A}\,) = (\,\boldsymbol{r} \cdot \nabla\,)\,\boldsymbol{A} + (\boldsymbol{A} \cdot \nabla\,)\,\boldsymbol{r} + \boldsymbol{r} \times (\nabla \times \boldsymbol{A}\,) + \boldsymbol{A} \times (\nabla \times \boldsymbol{r}\,)$$

Since

$$\nabla \times \boldsymbol{r} = 0$$
$$(\boldsymbol{A} \cdot \nabla\,)\,\boldsymbol{r} = A_j\, \partial_j\, x_i = A_j\, \delta_{j\,i} = A_i = \boldsymbol{A}$$

we have

$$\nabla (\,\boldsymbol{r} \cdot \boldsymbol{A}\,) = (\,\boldsymbol{r} \cdot \nabla\,)\,\boldsymbol{A} + \boldsymbol{A} + \boldsymbol{r} \times (\nabla \times \boldsymbol{A}\,)$$

and

$$\nabla^2 (\,\boldsymbol{r} \cdot \boldsymbol{A}\,) = \nabla \cdot [\,(\,\boldsymbol{r} \cdot \nabla\,)\,\boldsymbol{A}\,] + \nabla \cdot \boldsymbol{A} + \nabla \cdot [\,\boldsymbol{r} \times (\nabla \times \boldsymbol{A}\,)]$$

Now

$$\nabla\cdot[\,\boldsymbol{r}\times(\nabla\times\boldsymbol{A})] = (\nabla\times\boldsymbol{A})\cdot(\nabla\times\boldsymbol{r}) - \boldsymbol{r}\cdot[\nabla\times(\nabla\times\boldsymbol{A})]$$

$$= -\boldsymbol{r}\cdot\left[\nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}\right]$$

$$= -(\boldsymbol{r}\cdot\nabla)(\nabla\cdot\boldsymbol{A}) + \boldsymbol{r}\cdot\left(\nabla^2\boldsymbol{A}\right)$$

$$\nabla\cdot[(\boldsymbol{r}\cdot\nabla)\boldsymbol{A}] = \partial_i\left(x_j\,\partial_j\,A_i\right)$$

$$= \delta_{ij}\,\partial_j\,A_i + x_j\,\partial_j\,\partial_i A_i$$

$$= \nabla\cdot\boldsymbol{A} + (\boldsymbol{r}\cdot\nabla)(\nabla\cdot\boldsymbol{A})$$

Hence

$$\nabla^2(\boldsymbol{r}\cdot\boldsymbol{A}) = 2\,\nabla\cdot\boldsymbol{A} + \boldsymbol{r}\cdot\left(\nabla^2\boldsymbol{A}\right) \qquad (16.34)$$

and

$$\nabla^2(\boldsymbol{r}\cdot\boldsymbol{A}) = \boldsymbol{r}\cdot\left(\nabla^2\boldsymbol{A}\right) \qquad \text{if} \qquad \nabla\cdot\boldsymbol{A} = 0$$

The foregoing proof for (16.34) is actually unnecessarily complicated. A simpler way is as follows

$$\nabla^2(\boldsymbol{r}\cdot\boldsymbol{A}) = \partial_i\partial_i\left(r_j\,A_j\right)$$

$$= \partial_i\left[\delta_{ij}\,A_j + r_j\,\partial_i\,A_j\right]$$

$$= \partial_i A_i + \delta_{ij}\,\partial_i\,A_j + r_j\,\partial_i\partial_i\,A_j$$

$$= 2\,\nabla\cdot\boldsymbol{A} + \boldsymbol{r}\cdot\left(\nabla^2\boldsymbol{A}\right)$$

The Helmholtz equation

$$\left(\nabla^2 + k^2\right)\boldsymbol{A} = 0 \qquad [\,\boldsymbol{A} = \boldsymbol{E} \text{ or } \boldsymbol{B}\,]$$

thus becomes

$$\boldsymbol{r}\cdot\left[\left(\nabla^2 + k^2\right)\boldsymbol{A}\right] = 0$$

$$\left(\nabla^2 + k^2\right)(\boldsymbol{r}\cdot\boldsymbol{A}) = 0 \qquad \text{since} \quad \nabla\cdot\boldsymbol{A} = 0 \qquad (16.35)$$

with solution given by (16.16):

$$\boldsymbol{r}\cdot\boldsymbol{A} = z_l(k\,r)\,Y_{lm}(\Omega)$$

As discussed in Chap 8 (§ 8.2, 11), we can define 2 sets of independent modes:

TE:	$\boldsymbol{r}\cdot\boldsymbol{E}_{lm}^{(\mathrm{TE})} = 0 = \boldsymbol{r}\cdot\boldsymbol{E}_{lm}^{(M)}$	(Magnetic multipole fields)
TM:	$\boldsymbol{r}\cdot\boldsymbol{B}_{lm}^{(\mathrm{TM})} = 0 = \boldsymbol{r}\cdot\boldsymbol{B}_{lm}^{(E)}$	(Electric multipole fields)

Obviously, these modes satisfy the helmholtz eqs trivially,

$$\left(\nabla^2 + k^2\right)\boldsymbol{r}\cdot\boldsymbol{E}_{lm}^{(M)} = 0$$

$$\left(\nabla^2 + k^2\right)\boldsymbol{r}\cdot\boldsymbol{B}_{lm}^{(E)} = 0$$

while their dual fields can be written as

$$\boldsymbol{r}\cdot\boldsymbol{B}_{lm}^{(M)} = z_l(k\,r)\,Y_{lm}(\Omega) \qquad (16.36)$$

$$\boldsymbol{r}\cdot\boldsymbol{E}_{lm}^{(E)} = \tilde{z}_l(k\,r)\,Y_{lm}(\Omega) \qquad (16.41)$$

Furthermore

$$\boldsymbol{B}_{l\,m}^{(M)} = \frac{1}{i\,k}\,\nabla\times\boldsymbol{E}_{l\,m}^{(M)} \qquad (16.40)$$

$$\boldsymbol{E}_{l\,m}^{(E)} = -\frac{1}{i\,k}\,\nabla\times\boldsymbol{B}_{l\,m}^{(E)} \qquad (16.42)$$

which means

$$\begin{aligned}\boldsymbol{r}\cdot\boldsymbol{B}_{l\,m}^{(M)} &= \frac{1}{i\,k}\,\boldsymbol{r}\cdot\left[\nabla\times\boldsymbol{E}_{l\,m}^{(M)}\right]\\ &= \frac{1}{i\,k}\,(\boldsymbol{r}\times\nabla)\cdot\boldsymbol{E}_{l\,m}^{(M)}\\ &= \frac{1}{k}\,\boldsymbol{L}\cdot\boldsymbol{E}_{l\,m}^{(M)} \qquad (16.38)\end{aligned}$$

and

$$\boldsymbol{r}\cdot\boldsymbol{E}_{l\,m}^{(E)} = -\frac{1}{k}\,\boldsymbol{L}\cdot\boldsymbol{B}_{l\,m}^{(E)} \qquad [\text{ by } \boldsymbol{E}\to\boldsymbol{B},\ \boldsymbol{B}\to-\boldsymbol{E}\ ]$$

To summarize:

$$\boldsymbol{L}\cdot\boldsymbol{E}_{l\,m}^{(M)} = k\,z_l(k\,r)\,Y_{l\,m}(\Omega)$$

$$\boldsymbol{r}\cdot\boldsymbol{E}_{l\,m}^{(M)} = 0$$

and

$$\boldsymbol{L}\cdot\boldsymbol{B}_{l\,m}^{(E)} = -k\,\tilde{z}_l(k\,r)\,Y_{l\,m}(\Omega)$$

$$\boldsymbol{r}\cdot\boldsymbol{B}_{l\,m}^{(E)} = 0$$

Using

$$\boldsymbol{r}\cdot\boldsymbol{L} = 0$$

$$\boldsymbol{L}^2\,Y_{l\,m} = l\,(l+1)\,Y_{l\,m}$$

we see that these criteria can be satisfied by

$$\boldsymbol{E}_{l\,m}^{(M)} = \frac{k}{l\,(l+1)}\,z_l\,\boldsymbol{L}\,Y_{l\,m} \qquad (16.40)$$

$$\boldsymbol{B}_{l\,m}^{(E)} = -\frac{k}{l\,(l+1)}\,\tilde{z}_l\,\boldsymbol{L}\,Y_{l\,m} \qquad (16.42)$$

so that

$$\begin{aligned}\boldsymbol{B}_{l\,m}^{(M)} &= \frac{1}{i\,k}\,\nabla\times\boldsymbol{E}_{l\,m}^{(M)}\\ &= \frac{1}{i\,l\,(l+1)}\,\nabla\times[\,z_l\,\boldsymbol{L}\,Y_{l\,m}\,]\\ &= \frac{1}{i\,l\,(l+1)}\,\nabla\times\boldsymbol{L}\,(z_l\,Y_{l\,m}) \qquad (\boldsymbol{L}\,z_l = 0)\end{aligned}$$

$$\begin{aligned}\boldsymbol{E}_{l\,m}^{(E)} &= -\frac{1}{i\,k}\,\nabla\times\boldsymbol{B}_{l\,m}^{(E)}\\ &= -\frac{1}{i\,l\,(l+1)}\,\nabla\times[\,\tilde{z}_l\,\boldsymbol{L}\,Y_{l\,m}\,]\\ &= -\frac{1}{i\,l\,(l+1)}\,\nabla\times\boldsymbol{L}\,(\tilde{z}_l\,Y_{l\,m})\end{aligned}$$

To conform with Jackson's notations, we set

$$g_l = \frac{k}{l(l+1)} z_l$$

$$f_l = -\frac{k}{l(l+1)} \tilde{z}_l$$

so that

$$\boldsymbol{E}_{lm}^{(M)} = g_l \, \boldsymbol{L} \, Y_{lm} \qquad (16.40)$$

$$\boldsymbol{B}_{lm}^{(E)} = f_l \, \boldsymbol{L} \, Y_{lm} \qquad (16.42)$$

Consider

$$I = \int d\Omega \, (\boldsymbol{L} Y_{lm})^* \cdot (\boldsymbol{L} Y_{l'm'})$$

Since

$$\boldsymbol{L} = \frac{1}{i} \boldsymbol{r} \times \nabla$$

we have

$$\boldsymbol{L}^* = -\boldsymbol{L} \qquad \boldsymbol{L}^* \cdot \boldsymbol{L} = -\boldsymbol{L}^2$$

so that

$$\boldsymbol{L}^* \cdot (Y_{lm}^* \, \boldsymbol{L} \, Y_{l'm'}) = (\boldsymbol{L} Y_{lm})^* \cdot (\boldsymbol{L} Y_{l'm'}) - Y_{lm}^* \, \boldsymbol{L}^2 \, Y_{l'm'} \qquad (1)$$

also

$$\begin{aligned} \boldsymbol{L}^* \cdot (Y_{lm}^* \, \boldsymbol{L} \, Y_{l'm'}) &= \boldsymbol{L}^* \cdot [\, \boldsymbol{L} (Y_{lm}^* Y_{l'm'}) - Y_{l'm'} \, \boldsymbol{L} \, Y_{lm}^* \,] \\ &= -\boldsymbol{L}^2 (Y_{lm}^* Y_{l'm'}) - \boldsymbol{L} \cdot (Y_{l'm'} \, \boldsymbol{L}^* \, Y_{lm}^*) \end{aligned}$$

Using

$$\boldsymbol{L} \cdot (Y_{l'm'} \, \boldsymbol{L}^* \, Y_{lm}^*) = (\boldsymbol{L} Y_{l'm'}) \cdot (\boldsymbol{L} Y_{lm})^* - Y_{l'm'} \, \boldsymbol{L}^2 \, Y_{lm}^*$$

we have

$$\boldsymbol{L}^* \cdot (Y_{lm}^* \, \boldsymbol{L} \, Y_{l'm'}) = -\boldsymbol{L}^2 (Y_{lm}^* Y_{l'm'}) - (\boldsymbol{L} Y_{l'm'}) \cdot (\boldsymbol{L} Y_{lm})^* + Y_{l'm'} \, \boldsymbol{L}^2 \, Y_{lm}^*$$

Together with (1), we have

$$(\boldsymbol{L} Y_{lm})^* \cdot (\boldsymbol{L} Y_{l'm'}) = \frac{1}{2} \left\{ -\boldsymbol{L}^2 (Y_{lm}^* Y_{l'm'}) + Y_{lm}^* \, \boldsymbol{L}^2 \, Y_{l'm'} + Y_{l'm'} \, \boldsymbol{L}^2 \, Y_{lm}^* \right\}$$

Hence

$$I = \int d\Omega \, \frac{1}{2} \left\{ -\boldsymbol{L}^2 (Y_{lm}^* Y_{l'm'}) + Y_{lm}^* \, \boldsymbol{L}^2 \, Y_{l'm'} + Y_{l'm'} \, \boldsymbol{L}^2 \, Y_{lm}^* \right\}$$

Now,

$$-\boldsymbol{L}^2 = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}$$

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

so that

$$\begin{aligned} \int d\Omega \, \boldsymbol{L}^2 f &= -\int d\phi \int_0^{\pi} d\theta \, \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial f}{\partial\theta} \right) - \int \frac{1}{\sin\theta} d\theta \int_0^{2\pi} d\phi \, \frac{\partial}{\partial\phi} \left(\frac{\partial f}{\partial\phi} \right) \\ &= 0 \qquad \text{if } \frac{\partial f}{\partial\phi} \text{ is single- valued } \left[\left(\frac{\partial f}{\partial\phi} \right)_{\phi=0} = \left(\frac{\partial f}{\partial\phi} \right)_{\phi=2\pi} \right] \end{aligned}$$

Therefore

$$I = \int d\,\Omega\, \frac{1}{2} \left\{ Y_{l\,m}{}^{*}\, \boldsymbol{L}^2\, Y_{l'\,m'} + Y_{l'\,m'}\, \boldsymbol{L}^2\, Y_{l\,m}{}^{*} \right\}$$

$$= \frac{1}{2} [\, l'\,(\,l'+1) + l\,(\,l+1)\,] \int d\,\Omega\, Y_{l\,m}{}^{*}\; Y_{l'\,m'}$$

$$= l\,(\,l+1)\, \delta_{l\,l'}\, \delta_{m\,m'}$$

The normalized vector spherical harmonics are defined as

$$\boldsymbol{X}_{l\,m} = \frac{1}{\sqrt{l\,(\,l\,+\,1\,)}}\, \boldsymbol{L}\, Y_{l\,m} \qquad (16.43)$$

Note: It is to be understood that the operator $\boldsymbol{L}$ in $\boldsymbol{X}_{l\,m}$ does not operate further than $Y_{l\,m}$. ie.

$$\boldsymbol{X}_{l\,m}\; f \equiv (\, \boldsymbol{X}_{l\,m}\,)\; f = f\; \boldsymbol{X}_{l\,m}$$

for any arbitrary function f.

Obviously,

$$\boldsymbol{r} \cdot \boldsymbol{X}_{l\,m} = \frac{1}{\sqrt{l\,(\,l\,+\,1\,)}}\, \boldsymbol{r} \cdot \boldsymbol{L}\, Y_{l\,m} = 0$$

$$\int d\,\Omega\;\; \boldsymbol{X}_{l\,m}{}^{*} \cdot \boldsymbol{X}_{l'\,m'} = \frac{1}{\sqrt{l\,(\,l\,+\,1\,)\,l'\,(\,l'+\,1\,)}} \int d\,\Omega\;\, (\,\boldsymbol{L}\, Y_{l\,m}\,)^{*} \cdot (\,\boldsymbol{L}\, Y_{l'\,m'}\,)$$

$$= \frac{1}{\sqrt{l\,(\,l\,+\,1\,)\,l'\,(\,l'+\,1\,)}}\, l\,(\,l+1)\, \delta_{l\,l'}\, \delta_{m\,m'}$$

$$= \delta_{l\,l'}\, \delta_{m\,m'} \qquad (16.44)$$

Consider

$$\int d\,\Omega\, \left(\hat{\boldsymbol{r}} \times \boldsymbol{X}_{l'\,m'} \right) \cdot \boldsymbol{X}_{l\,m}{}^{*}$$

$$= \frac{1}{\sqrt{l\,(\,l\,+\,1\,)\,l'\,(\,l'+\,1\,)}} \int d\,\Omega\, \left(\hat{\boldsymbol{r}} \times \boldsymbol{L}\, Y_{l'\,m'} \right) \cdot\, (\boldsymbol{L}\, Y_{l\,m})^{*}$$

Now

$$\hat{\boldsymbol{r}} \times \boldsymbol{L} = \; i \left(\hat{\boldsymbol{\theta}}\, \frac{\partial}{\partial\,\theta} + \hat{\boldsymbol{\phi}}\, \frac{1}{\sin\theta}\, \frac{\partial}{\partial\,\phi} \right)$$

$$\boldsymbol{L} = i \left(\hat{\boldsymbol{\theta}}\; \frac{1}{\sin\theta}\, \frac{\partial}{\partial\,\phi} - \hat{\boldsymbol{\phi}}\, \frac{\partial}{\partial\,\theta} \right)$$

so that

$$\left(\hat{\boldsymbol{r}} \times \boldsymbol{L}\; f \,\right) \cdot \boldsymbol{L}^{*}\, g = \frac{\partial\, f}{\partial\,\theta}\, \frac{1}{\sin\theta}\, \frac{\partial\, g}{\partial\,\phi} - \frac{1}{\sin\theta}\, \frac{\partial\, f}{\partial\,\phi}\, \frac{\partial\, g}{\partial\,\theta}$$

$$\int d\,\Omega\, \left(\hat{\boldsymbol{r}} \times \boldsymbol{L}\; f \,\right) \cdot \boldsymbol{L}^{*}\, g = \int_0^{2\pi} d\,\phi \int_0^{\pi} d\,\theta \left[\frac{\partial\, f}{\partial\,\theta}\, \frac{\partial\, g}{\partial\,\phi} - \frac{\partial\, f}{\partial\,\phi}\, \frac{\partial\, g}{\partial\,\theta} \right]$$

$$= \; 0 \qquad \text{provided } \frac{\partial\, g}{\partial\,\phi} \text{ and } \frac{\partial\, f}{\partial\,\phi} \text{ are single valued.}$$

Therefore

$$\int d\,\Omega\, \left(\hat{\boldsymbol{r}} \times \boldsymbol{X}_{l'\,m'} \right) \cdot \boldsymbol{X}_{l\,m}{}^{*} = 0$$

Next

$$\int d\,\Omega\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\right)^*\cdot\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\right)$$

$$= \int d\,\Omega\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}{}^*\right)\cdot\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\right)$$

$$= \int d\,\Omega\,\left[\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}{}^*\right)\times\hat{\boldsymbol{r}}\right]\cdot\boldsymbol{X}_{l'\,m'}$$

$$= \int d\,\Omega\,\left[-\hat{\boldsymbol{r}}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{X}_{l\,m}{}^*\right)+\ \boldsymbol{X}_{l\,m}{}^*\right]\cdot\boldsymbol{X}_{l'\,m'}$$

$$= \int d\,\Omega\,\boldsymbol{X}_{l\,m}{}^*\cdot\boldsymbol{X}_{l'\,m'}$$

$$= \delta_{l\,l'}\,\delta_{m\,m'}$$

Hence, $\left\{\boldsymbol{X}_{l\,m}\,,\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\right\}$ is an orthonormal basis on the surface of the sphere. (completeness of this basis will be assumed without proof)

Setting

$$\boldsymbol{E}^{(M)}_{l\,m}=g_l\,\boldsymbol{X}_{l\,m} \qquad \boldsymbol{B}^{(M)}_{l\,m}=\frac{1}{i\,k}\,\nabla\times(\,g_l\,\boldsymbol{X}_{l\,m}\,)$$

$$\boldsymbol{B}^{(E)}_{l\,m}=\,f_l\,\boldsymbol{X}_{l\,m} \qquad \boldsymbol{E}^{(E)}_{l\,m}=-\frac{1}{i\,k}\,\nabla\times(\,f_l\,\boldsymbol{X}_{l\,m})$$

an arbitrary field can be expanded as

$$\boldsymbol{E}=\sum_{\alpha\,l\,m}a_\alpha(\,l\,m\,)\,\boldsymbol{E}^{(\alpha)}_{l\,m}$$

$$=\sum_{l\,m}\left[\,a_M(\,l\,m\,)\,g_l\,\boldsymbol{X}_{l\,m}-\frac{1}{i\,k}\,a_E(\,l\,m\,)\,\nabla\times(\,f_l\,\boldsymbol{X}_{l\,m})\right]$$

$$\boldsymbol{B}=\sum_{\alpha\,l\,m}a_\alpha(\,l\,m\,)\,\boldsymbol{B}^{(\alpha)}_{l\,m} \tag{16.46}$$

$$=\sum_{l\,m}\left[\,\frac{1}{i\,k}\,a_M(\,l\,m\,)\,\nabla\times(\,g_l\,\boldsymbol{X}_{l\,m}\,)+a_E(\,l\,m\,)\,f_l\,\boldsymbol{X}_{l\,m}\right]$$

Note that the same a's appear in both $\boldsymbol{E}$ & $\boldsymbol{B}$ due to requirement

$$\boldsymbol{B}=\frac{1}{i\,k}\,\nabla\times\boldsymbol{E} \qquad \boldsymbol{E}=-\frac{1}{i\,k}\,\nabla\times\boldsymbol{B}$$

This is easily checked using, for example, the relation,

$$\nabla\times[\nabla\times(\,f_l\,\boldsymbol{X}_{l\,m})]=\nabla\,[\,\nabla\cdot(\,f_l\,\boldsymbol{X}_{l\,m})\,]-\nabla^2\,(\,f_l\,\boldsymbol{X}_{l\,m})$$

$$=\ k^2\,f_l\,\boldsymbol{X}_{l\,m}$$

where, by definition, $\nabla\cdot\boldsymbol{B}^{(E)}_{l\,m}=0$ and $\left[\,\nabla^2+k^2\,\right]\boldsymbol{B}^{(E)}_{l\,m}=0$.

$$\nabla\times(\,f_l\,\boldsymbol{X}_{l\,m})=(\,\nabla\,f_l\,)\times\boldsymbol{X}_{l\,m}+\,f_l\,\nabla\times\boldsymbol{X}_{l\,m}$$

$$=\ \frac{d\,f_l}{d\,r}\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}+\frac{1}{\sqrt{l\,(\,l\,+\,1\,)}}\,f_l\,\nabla\times\boldsymbol{L}\,Y_{l\,m}$$

$$\nabla\times\boldsymbol{L}\,Y_{l\,m}=\frac{1}{i}\left[\,\boldsymbol{r}\,\nabla^2-\nabla\left(\,1+r\,\frac{\partial}{\partial\,r}\,\right)\right]Y_{l\,m}$$

$$=\ \frac{1}{i}\left[\,\boldsymbol{r}\,\nabla^2-\nabla\,\right]Y_{l\,m}$$

$$=\ i\left[\ \frac{l\,(\,l+1\,)}{r^2}\,\boldsymbol{r}\ +\nabla\,\right]Y_{l\,m} \qquad \left(\,\nabla^2=\ \nabla_r{}^2-\frac{1}{r^2}\,\boldsymbol{L}^2\,\right)$$

In order for these expansions to be useful, $\boldsymbol{E}_{l\,m}^{(\alpha)}$ and $\boldsymbol{B}_{l\,m}^{(\alpha)}$ must be orthogonal in the sense discussed in the Digression. This we shall prove in the following.

Since

$$\nabla = \hat{\boldsymbol{r}} \frac{\partial}{\partial r} - i \frac{1}{r} \hat{\boldsymbol{r}} \times \boldsymbol{L}$$

we have

$$\nabla \times \boldsymbol{L}\, Y_{l\,m} = i\left[\frac{l(l+1)}{r^2} \boldsymbol{r} - i \frac{1}{r} \hat{\boldsymbol{r}} \times \boldsymbol{L} \right] Y_{l\,m}$$

$$= i \frac{l(l+1)}{r^2} \boldsymbol{r}\, Y_{l\,m} + \frac{\sqrt{l(l+1)}}{r} \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l\,m}$$

and

$$\nabla \times (f_l \boldsymbol{X}_{l\,m}) = \left(\frac{d f_l}{d r} + \frac{f_l}{r} \right) \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l\,m} + i \frac{\sqrt{l(l+1)}}{r} \hat{\boldsymbol{r}}\, f_l\, Y_{l\,m}$$

$$= \frac{1}{r} \frac{d\, r f_l}{d r} \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l\,m} + i \frac{\sqrt{l(l+1)}}{r} \hat{\boldsymbol{r}}\, f_l\, Y_{l\,m}$$

$$= \text{angular part} + \text{radial part}$$

Hence

$$\int d\Omega\; g_{l'}\, \boldsymbol{X}_{l'\,m'}^{*} \cdot \nabla \times (f_l\, \boldsymbol{X}_{l\,m}) = 0 \qquad \forall\; f_l\,,\, g_l$$

$$\int d\Omega\; \nabla \times (g_{l'}\, \boldsymbol{X}_{l'\,m'})^{*} \cdot \nabla \times (f_l\, \boldsymbol{X}_{l\,m}) \propto \delta_{l\,l'}\, \delta_{m\,m'}$$

Note: the angular & radial parts of $\nabla \times (f_l\, \boldsymbol{X}_{l\,m})$ are orthogonal. (prove it!)

Alternatively,

$$\nabla \times (f_l\, \boldsymbol{X}_{l\,m}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times \boldsymbol{L}\; (f_l\, Y_{l\,m})$$

$$= \frac{1}{i\sqrt{l(l+1)}} \left[\boldsymbol{r}\, \nabla^2 - \nabla \left(1 + r \frac{\partial}{\partial r} \right) \right] (f_l\, Y_{l\,m})$$

$$= \frac{1}{i\sqrt{l(l+1)}} \left\{ -\boldsymbol{r}\, k^2\, f_l\, Y_{l\,m} - \nabla \left[\left(f_l + r \frac{d f_l}{d r} \right) Y_{l\,m} \right] \right\}$$

Using

$$\nabla = \hat{\boldsymbol{r}} \frac{\partial}{\partial r} - i \frac{1}{r} \hat{\boldsymbol{r}} \times \boldsymbol{L}$$

we have

$$\nabla \left[\left(f_l + r \frac{d f_l}{d r} \right) Y_{l\,m} \right] = \hat{\boldsymbol{r}} \left[\frac{d}{d r} \left(1 + r \frac{d}{d r} \right) f_l \right] Y_{l\,m} - i \frac{1}{r} \left(f_l + r \frac{d f_l}{d r} \right) \hat{\boldsymbol{r}} \times \boldsymbol{L}\, Y_{l\,m}$$

Now

$$\frac{d}{d r} \left(1 + r \frac{d}{d r} \right) f_l = r \left[\frac{d^2 f_l}{d r^2} + \frac{2}{r} \frac{d f_l}{d r} \right]$$

$$= r \left[-k^2 + \frac{l(l+1)}{r^2} \right] f_l \qquad [\; f_l \text{ combination of Bessel functions }]$$

Hence

$$\nabla\times(f_l\,\boldsymbol{X}_{l\,m}) = \frac{1}{i\sqrt{l(l+1)}}\left\{-\boldsymbol{r}\,\frac{l(l+1)}{r^2}\,f_l\,Y_{l\,m} + i\,\frac{1}{r}\left(f_l + r\,\frac{d\,f_l}{d\,r}\right)\hat{\boldsymbol{r}}\times\boldsymbol{L}\,Y_{l\,m}\right\}$$

$$= i\,\frac{\sqrt{l(l+1)}}{r}\,\hat{\boldsymbol{r}}\,f_l\,Y_{l\,m} + \left(\frac{d\,f_l}{d\,r} + \frac{f_l}{r}\right)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}$$

To summarize, the electric multipole (TM mode) consists of

a transverse magnetic field $\boldsymbol{B}^{(E)}_{l\,m} = f_l\,\boldsymbol{X}_{l\,m} = f_l\,\boldsymbol{L}\,Y_{l\,m}$

& a electric field $\boldsymbol{E}^{(E)}_{l\,m}$

with transverse component $-\frac{1}{i\,k\,r}\left(\frac{d}{d\,r}\,r\,f_l\right)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}$

& radial component $-\frac{\sqrt{l(l+1)}}{k\,r}\,\hat{\boldsymbol{r}}\,f_l\,Y_{l\,m}$

The 3 directions $\boldsymbol{L}$, $\hat{\boldsymbol{r}}\times\boldsymbol{L}$, and $\hat{\boldsymbol{r}}$ are of course mutually orthogonal.

Similarly, the magnetic multipole (TE mode) consists of

a transverse electric field $\boldsymbol{E}^{(M)}_{l\,m} = g_l\,\boldsymbol{X}_{l\,m} = g_l\,\boldsymbol{L}\,Y_{l\,m}$

& a magnetic field $\boldsymbol{B}^{(M)}_{l\,m}$

with transverse component $\frac{1}{i\,k\,r}\left(\frac{d}{d\,r}\,r\,g_l\right)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}$

& radial component $\frac{\sqrt{l(l+1)}}{k\,r}\,\hat{\boldsymbol{r}}\,g_l\,Y_{l\,m}$

The 3 directions $\boldsymbol{L}$, $\hat{\boldsymbol{r}}\times\boldsymbol{L}$, and $\hat{\boldsymbol{r}}$ are of course mutually orthogonal.

Now that the orthogonality of the fields is established, the expansion co-efficients $a(l\,m)$ of an arbitrary field are readily obtained. For example, using

$$\int d\,\Omega\,\boldsymbol{X}_{l\,m}{}^{*}\cdot\boldsymbol{X}_{l'\,m'} = \delta_{l\,l'}\,\delta_{m\,m'}$$

on

$$\boldsymbol{E} = \sum_{l\,m}\left[a_M(l\,m)\,g_l\,\boldsymbol{X}_{l\,m} - \frac{1}{i\,k}\,a_E(l\,m)\,\nabla\times(f_l\,\boldsymbol{X}_{l\,m})\right]$$

we have

$$a_M(l\,m)\,g_l = \int d\,\Omega\,\boldsymbol{X}_{l\,m}{}^{*}\cdot\boldsymbol{E}$$

$$= \frac{1}{\sqrt{l(l+1)}}\int d\,\Omega\;i\,(\,\boldsymbol{r}\times\nabla Y_{l\,m}{}^{*}\,)\cdot\boldsymbol{E}$$

$$= \frac{i}{\sqrt{l(l+1)}}\,\boldsymbol{r}\cdot\int d\,\Omega\,(\nabla Y_{l\,m}{}^{*})\times\boldsymbol{E}$$

Now

$$\nabla\times(Y_{lm}{}^{*}\boldsymbol{E}) = (\nabla Y_{lm}{}^{*})\times\boldsymbol{E} + Y_{lm}{}^{*}\nabla\times\boldsymbol{E}$$

$$\nabla\times\boldsymbol{E} = i\,k\,\boldsymbol{B}$$

so that

$$a_M(l\,m)\,g_l = \frac{k}{\sqrt{l(l+1)}}\int d\,\Omega\;Y_{lm}{}^{*}\,\boldsymbol{r}\cdot\boldsymbol{B} \qquad (16.47)$$

where we've used

$$\int d\,\Omega\;\nabla\times(Y_{lm}{}^{*}\boldsymbol{E}) = 0$$

Similarly,

$$a_E(l\,m)\,f_l = -\frac{k}{\sqrt{l(l+1)}}\int d\,\Omega\;Y_{lm}{}^{*}\,\boldsymbol{r}\cdot\boldsymbol{E}$$

The same results can of course be obtained using the orthogonality of $\hat{\boldsymbol{r}}\times\boldsymbol{X}_{lm}$. It'll be left as an exercise.

Yet another way to do it is as follows (see Jackson)

Since

$$\boldsymbol{r}\cdot\boldsymbol{X}_{lm} = 0$$

we have

$$\boldsymbol{r}\cdot\boldsymbol{B} = \sum_{lm}\frac{1}{i\,k}\,a_M(l\,m)\;\boldsymbol{r}\cdot[\nabla\times(g_l\,\boldsymbol{X}_{lm})]$$

Now

$$\begin{aligned}
\boldsymbol{r}\cdot[\nabla\times(g_l\,\boldsymbol{X}_{lm})] &= \frac{1}{\sqrt{l(l+1)}}\,\boldsymbol{r}\cdot[\nabla\times(g_l\,\boldsymbol{L}\,Y_{lm})] \\
&= \frac{1}{\sqrt{l(l+1)}}\,(\boldsymbol{r}\times\nabla)\cdot(g_l\,\boldsymbol{L}\,Y_{lm}) \\
&= \frac{i}{\sqrt{l(l+1)}}\,\boldsymbol{L}\cdot(g_l\,\boldsymbol{L}\,Y_{lm}) \\
&= \frac{i}{\sqrt{l(l+1)}}\,g_l\,\boldsymbol{L}^2\,Y_{lm} \\
&= i\sqrt{l(l+1)}\;g_l\;Y_{lm}
\end{aligned}$$

Hence

$$\boldsymbol{r}\cdot\boldsymbol{B} = \sum_{lm}\frac{\sqrt{l(l+1)}}{k}\,a_M(l\,m)\;g_l\;Y_{lm}$$

so that

$$a_M(l\,m)\;g_l = \frac{k}{\sqrt{l(l+1)}}\int d\,\Omega\;Y_{lm}{}^{*}\,\boldsymbol{r}\cdot\boldsymbol{B}$$

Query: What happens if

$$\boldsymbol{r}\cdot\boldsymbol{B} = \boldsymbol{r}\cdot\boldsymbol{E} = 0$$

3. Properties

The electric multipole fields (TM modes) are

$$\boldsymbol{B}_{l\,m}^{(E)} = f_l\, \boldsymbol{X}_{l\,m} \qquad\qquad \boldsymbol{r}\cdot\boldsymbol{B}_{l\,m}^{(E)} = 0$$

$$\boldsymbol{E}_{l\,m}^{(E)} = -\frac{1}{i\,k}\, \nabla\times(f_l\, \boldsymbol{X}_{l\,m})$$

$$= -\frac{1}{i\,k\sqrt{l\,(l+1)}}\, \nabla\times\boldsymbol{L}\,(f_l\, Y_{l\,m})$$

$$= -\frac{1}{i\,k\,r}\left(\frac{d}{d\,r}\, r\, f_l\right)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m} - \frac{\sqrt{l\,(l+1)}}{k\,r}\, \hat{\boldsymbol{r}}\, f_l\, Y_{l\,m}$$

The magnetic multipole fields (TE modes) are

$$\boldsymbol{E}_{l\,m}^{(M)} = g_l\, \boldsymbol{X}_{l\,m} \qquad\qquad \boldsymbol{r}\cdot\boldsymbol{E}_{l\,m}^{(M)} = 0$$

$$\boldsymbol{B}_{l\,m}^{(M)} = \frac{1}{i\,k}\, \nabla\times(g_l\, \boldsymbol{X}_{l\,m})$$

$$= \frac{1}{i\,k\sqrt{l\,(l+1)}}\, \nabla\times\boldsymbol{L}\,(g_l\, Y_{l\,m})$$

$$= \frac{1}{i\,k\,r}\left(\frac{d}{d\,r}\, r\, g_l\right)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m} - \frac{\sqrt{l\,(l+1)}}{k\,r}\, \hat{\boldsymbol{r}}\, g_l\, Y_{l\,m}$$

■ Long- Wavelength Limit

If we set

$$f_l = \alpha\, n_l \qquad\qquad (\alpha = \text{const})$$

then for $k\,r \to 0$,

$$f_l \longrightarrow -\alpha\, \frac{(2\,l-1)!!}{(k\,r)^{l+1}} \tag{16.12}$$

$$\boldsymbol{E}_{l\,m}^{(E)} = -\frac{1}{k\sqrt{l\,(l+1)}}\left\{\boldsymbol{r}\,k^2\, f_l\, Y_{l\,m} + \nabla\left[\left(f_l + r\,\frac{d\, f_l}{d\,r}\right) Y_{l\,m}\right]\right\}$$

The 1st term goes as $(k\,r)^{-l}$ while the 2nd, $(k\,r)^{-(l+1)}$. The former can therefore be dropped so that

$$\boldsymbol{E}_{l\,m}^{(E)} \xrightarrow[k\,r\to 0]{} \alpha\, \frac{(2\,l-1)!!}{k\sqrt{l\,(l+1)}}\, \nabla\left\{\left[\frac{1}{(k\,r)^{l+1}} + r\,\frac{d}{d\,r}\left(\frac{1}{(k\,r)^{l+1}}\right)\right] Y_{l\,m}\right\}$$

Now

$$r\,\frac{d}{d\,r}\left(\frac{1}{(k\,r)^{l+1}}\right) = -\frac{l+1}{(k\,r)^{l+1}}$$

so that

$$\boldsymbol{E}_{l\,m}^{(E)} \xrightarrow[k\,r\to 0]{} -\alpha\, \frac{(2\,l-1)!!}{k^{l+2}}\sqrt{\frac{l}{l+1}}\; \nabla\left(\frac{Y_{l\,m}}{r^{l+1}}\right)$$

Setting

$$\alpha = \sqrt{\frac{l+1}{l}}\ \frac{k^{l+2}}{(2\,l\,-\,1)!!}$$

gives

$$f_l \xrightarrow[k\,r \to 0]{} -\sqrt{\frac{l+1}{l}}\ \frac{k}{r^{l+1}}$$

$$\boldsymbol{E}^{(E)}_{l\,m} \xrightarrow[k\,r \to 0]{} -\nabla\left(\frac{Y_{l\,m}}{r^{l+1}}\right) \qquad (16.51)$$

which is simply the static electric multipole field of order (l, m).

Also

$$\begin{aligned} \boldsymbol{B}^{(E)}_{l\,m} &= f_l\, \boldsymbol{X}_{l\,m} \\ &\xrightarrow[k\,r \to 0]{} -\sqrt{\frac{l+1}{l}}\ \frac{k}{r^{l+1}}\ \boldsymbol{X}_{l\,m} \\ &= \ -\frac{k}{l}\ \frac{1}{r^{l+1}}\ \boldsymbol{L}\, Y_{l\,m} \qquad (16.48) \end{aligned}$$

■ Radiation Zone

If we set

$$f_l = \alpha\, h^{(1)}_l \qquad (\,\alpha = \text{const}\,)$$

then for $k\,r \to \infty$,

$$f_l \longrightarrow \alpha\,(-i)^{l+1}\ \frac{e^{i\,k\,r}}{k\,r} \qquad (16.13)$$

$$\boldsymbol{E}^{(E)}_{l\,m} = -\frac{\sqrt{l\,(\,l\,+\,1\,)}}{k\,r}\ \hat{\boldsymbol{r}}\ f_l\, Y_{l\,m} - \frac{1}{i\,k\,r}\ \frac{d\,r\,f_l}{d\,r}\ \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}$$

The 1st term goes as $(k\,r)^{-2}$ and can be dropped since the 2nd term goes as $(k\,r\,)^{-1}$

$$\begin{aligned} \boldsymbol{E}^{(E)}_{l\,m} &\xrightarrow[k\,r \to \infty]{} -\alpha\,(-i)^{l+1}\ \frac{e^{i\,k\,r}}{k\,r}\ \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m} \\ &= \ -\frac{\alpha}{\sqrt{l\,(\,l\,+\,1\,)}}\ (-i)^{l+1}\ \frac{e^{i\,k\,r}}{k\,r}\ \hat{\boldsymbol{r}}\times\boldsymbol{L}\, Y_{l\,m} \\ &= \ -\frac{\alpha}{\sqrt{l\,(\,l\,+\,1\,)}}\ h^{(1)}_l\, \hat{\boldsymbol{r}}\times\boldsymbol{L}\, Y_{l\,m} \\ &= \ -\alpha\, h^{(1)}_l\, \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m} \end{aligned}$$

Also

$$\begin{aligned} \boldsymbol{B}^{(E)}_{l\,m} = f_l\, \boldsymbol{X}_{l\,m} &\xrightarrow[k\,r \to \infty]{} \frac{\alpha}{\sqrt{l\,(\,l\,+\,1\,)}}\ (-i)^{l+1}\ \frac{e^{i\,k\,r}}{k\,r}\ \boldsymbol{L}\, Y_{l\,m} \\ &= \ \frac{\alpha}{\sqrt{l\,(\,l\,+\,1\,)}}\ h^{(1)}_l\, \boldsymbol{L}\, Y_{l\,m} \\ &= \ \alpha\, h^{(1)}_l\, \boldsymbol{X}_{l\,m} \end{aligned}$$

so that

$$\boldsymbol{E}^{(E)}_{l\,m} = \boldsymbol{B}^{(E)}_{l\,m}\times\hat{\boldsymbol{r}}$$

Similarly, with $g_l = \alpha\, h_l^{(1)}$,

$$\boldsymbol{B}_{l\,m}^{(M)} \xrightarrow[k\,r\to\infty]{} \alpha\,(-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}$$

$$= \frac{\alpha}{\sqrt{l\,(l+1)}}\,(-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}\,\hat{\boldsymbol{r}}\times\boldsymbol{L}\,Y_{l\,m}$$

$$= \frac{\alpha}{\sqrt{l\,(l+1)}}\,h_l^{(1)}\,\hat{\boldsymbol{r}}\times\boldsymbol{L}\,Y_{l\,m}$$

$$= \alpha\,h_l^{(1)}\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}$$

$$\boldsymbol{E}_{l\,m}^{(M)} = g_l\,\boldsymbol{X}_{l\,m} \xrightarrow[k\,r\to\infty]{} \frac{\alpha}{\sqrt{l\,(l+1)}}\,(-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}\,\boldsymbol{L}\,Y_{l\,m}$$

$$= \frac{\alpha}{\sqrt{l\,(l+1)}}\,h_l^{(1)}\,\boldsymbol{L}\,Y_{l\,m}$$

$$= \alpha\,h_l^{(1)}\,\boldsymbol{X}_{l\,m}$$

so that

$$\boldsymbol{B}_{l\,m}^{(M)} = \hat{\boldsymbol{r}}\times\boldsymbol{E}_{l\,m}^{(M)}$$

For a general field

$$\boldsymbol{E} = \sum_{l\,m}\left[\,a_E(\,l\,m\,)\,\boldsymbol{E}_{l\,m}^{(E)} + a_M(\,l\,m\,)\,\boldsymbol{E}_{l\,m}^{(M)}\,\right]$$

$$\boldsymbol{B} = \sum_{l\,m}\left[\,a_E(\,l\,m\,)\,\boldsymbol{B}_{l\,m}^{(E)} + a_M(\,l\,m\,)\,\boldsymbol{B}_{l\,m}^{(M)}\,\right]$$

In the radiation zone:

$$\boldsymbol{E} = \sum_{l\,m} h_l^{(1)}\left[-\,a_E(\,l\,m\,)\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m} + a_M(\,l\,m\,)\,\boldsymbol{X}_{l\,m}\,\right] \qquad (\,\alpha \text{ set to } 1\,)$$

$$\boldsymbol{B} = \sum_{l\,m} h_l^{(1)}\left[\,a_E(\,l\,m\,)\,\boldsymbol{X}_{l\,m} + a_M(\,l\,m\,)\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,\right]$$

The next logical move is to see how these fields are related to the sources. However, following the lead of Jackson, we'll make a digression to the study of energy flux & angular distribution first. You can skip to §16.5 (multipole moments) if you like.

■ Energy Density (Radiation Zone)

Let

$$\boldsymbol{B}_l^{(E)} = \sum_m a_E(\,l\,m\,)\,\boldsymbol{B}_{l\,m}^{(E)}$$

= most general electric multipole field of order l

$$= \sum_m a_E(\,l\,m\,)\,h_l^{(1)}\,\boldsymbol{X}_{l\,m}\,e^{-i\,\omega\,t} \qquad (\text{ Radiation zone })$$

$$\boldsymbol{E}_l^{(E)} = -\frac{1}{i\,k}\,\nabla\times\boldsymbol{B}_l^{(E)}$$

The time averaged energy density is

$$u = \frac{1}{16\,\pi}\left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right)$$

The time averaged energy is

$$U = \int d^3 x\, u$$
$$= \int r^2\, d r \int d\,\Omega\; u$$

Hence, the energy in a spherical shell between r and $r + d\, r$ is

$$d\,U = r^2\, d r \int d\,\Omega\; u$$
$$= r^2\, d r \int d\,\Omega\; \frac{2}{16\pi} \sum_{m\,m'} a_E(l\,m)^* \, a_E(l\,m')\, \boldsymbol{B}^{(E)*}_{l\,m} \cdot \boldsymbol{B}^{(E)}_{l\,m'}$$

where we've used

$$\boldsymbol{E}^{(E)}_{l\,m} = \boldsymbol{B}^{(E)}_{l\,m} \times \hat{\boldsymbol{r}}$$

so that

$$\boldsymbol{E}^{(E)*}_{l\,m} \cdot \boldsymbol{E}^{(E)}_{l\,m'} = \boldsymbol{B}^{(E)*}_{l\,m} \cdot \boldsymbol{B}^{(E)}_{l\,m'}$$

In the radiation zone, $k\,r \gg 1$, so that

$$h_l^{(1)} \longrightarrow (-i)^{l+1} \frac{e^{i k r}}{k\,r}$$
$$\boldsymbol{B}^{(E)*}_{l\,m} \cdot \boldsymbol{B}^{(E)}_{l\,m'} \longrightarrow \frac{1}{(k\,r)^2} \boldsymbol{X}_{l\,m}^* \cdot \boldsymbol{X}_{l\,m'}$$
$$\int d\,\Omega\; \boldsymbol{B}^{(E)*}_{l\,m} \cdot \boldsymbol{B}^{(E)}_{l\,m'} \longrightarrow \frac{1}{(k\,r)^2} \delta_{m\,m'}$$

Hence

$$d\,U \longrightarrow d\,r\; \frac{1}{8\pi k^2} \sum_m |\, a_E(l\,m)|^2$$

or

$$\frac{d\,U}{d\,r} \simeq \frac{1}{8\pi k^2} \sum_m |\, a_E(l\,m)|^2 \qquad (16.60)$$

For a general radiation of multipole of order l

$$\boldsymbol{B}_l = \sum_m \left[a_E(l\,m)\, \boldsymbol{B}^{(E)}_{l\,m} + a_M(l\,m)\, \boldsymbol{B}^{(M)}_{l\,m} \right]$$

Since the fields $\boldsymbol{B}^{(E)}_{l\,m}$ & $\boldsymbol{E}^{(E)}_{l'\,m'}$ are orthogonal, we have

$$\frac{d\,U}{d\,r} \simeq \frac{1}{8\pi k^2} \sum_{l\,m} \left[|\, a_E(l\,m)|^2 + |\, a_M(l\,m)|^2 \right]$$

= Incoherent sum of all multipole fields.

(each (unit) multipole contributes the same amount $\frac{1}{8\pi k^2}$)

▪ Angular Momentum Density

Time averaged angular momentum density is (see eqs. 6.158-9)

$$\boldsymbol{M} = \frac{1}{2} \cdot \frac{1}{4\pi c} \text{Re}\left[\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}^*) \right]$$

Note: For transverse fields

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{B} = \hat{\boldsymbol{r}}\cdot\boldsymbol{E} = 0 \qquad\qquad \boldsymbol{E} = \boldsymbol{B}\times\hat{\boldsymbol{r}}$$

and

$$\begin{aligned}\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}^*) = \boldsymbol{r}\times\left[\left(\boldsymbol{B}\times\hat{\boldsymbol{r}}\right)\times\boldsymbol{B}^*\right] \\ &= \boldsymbol{r}\times\left[\hat{\boldsymbol{r}}\,|\boldsymbol{B}|^2 - \boldsymbol{B}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{B}^*\right)\right] \\ &= 0\end{aligned}$$

so that

$$\boldsymbol{M} = 0$$

For electric multipole fields

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{B}^{(E)} = 0$$

so that

$$\begin{aligned}\boldsymbol{r}\times\left(\boldsymbol{E}^{(E)}\times\boldsymbol{B}^{(E)*}\right) &= -\boldsymbol{B}^{(E)*}\left(\boldsymbol{r}\cdot\boldsymbol{E}^{(E)}\right) \\ &= \frac{1}{i\,k}\boldsymbol{B}^{(E)*}\,\boldsymbol{r}\cdot\left(\nabla\times\boldsymbol{B}^{(E)}\right) \\ &= \frac{1}{k}\boldsymbol{B}^{(E)*}\left(\frac{1}{i}\boldsymbol{r}\times\nabla\right)\cdot\boldsymbol{B}^{(E)} \\ &= \frac{1}{k}\boldsymbol{B}^{(E)*}\left(\boldsymbol{L}\cdot\boldsymbol{B}^{(E)}\right)\end{aligned}$$

Hence

$$\boldsymbol{M}^{(E)} = \frac{1}{8\pi c\,k}\,\mathrm{Re}\left[\boldsymbol{B}^{(E)*}\left(\boldsymbol{L}\cdot\boldsymbol{B}^{(E)}\right)\right] \qquad (16.62)$$

For magnetic multipole fields

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{E}^{(M)} = 0$$

so that

$$\begin{aligned}\boldsymbol{r}\times\left(\boldsymbol{E}^{(M)}\times\boldsymbol{B}^{(M)*}\right) &= \boldsymbol{E}^{(M)}\left(\boldsymbol{r}\cdot\boldsymbol{B}^{(M)*}\right) \\ &= -\frac{1}{i\,k}\boldsymbol{E}^{(M)}\,\boldsymbol{r}\cdot\left(\nabla\times\boldsymbol{E}^{(M)*}\right) \\ &= -\frac{1}{k}\boldsymbol{E}^{(M)}\left(\frac{1}{i}\boldsymbol{r}\times\nabla\right)\cdot\boldsymbol{E}^{(M)*} \\ &= -\frac{1}{k}\boldsymbol{E}^{(M)}\,\boldsymbol{L}\cdot\boldsymbol{E}^{(M)*} \\ &= \frac{1}{k}\boldsymbol{E}^{(M)}\left(\boldsymbol{L}\cdot\boldsymbol{E}^{(M)}\right)^*\end{aligned}$$

Hence

$$\boldsymbol{M}^{(M)} = \frac{1}{8\pi c\,k}\,\mathrm{Re}\left[\boldsymbol{E}^{(M)}\left(\boldsymbol{L}\cdot\boldsymbol{E}^{(M)}\right)^*\right]$$

■ Electric Multipole Fields (Radiation Zone)

$$\boldsymbol{B}_{l\,m}^{(E)} = \boldsymbol{X}_{l\,m}\; h_l^{(1)}\, e^{-i\,\omega\,t}$$
$$= \frac{1}{\sqrt{l\,(l\,+1)}}\,(\boldsymbol{L}\,Y_{l\,m})\; h_l^{(1)}\, e^{-i\,\omega\,t}$$
$$\boldsymbol{E}_{l\,m}^{(E)} = -\frac{1}{i\,k}\,\nabla\times\boldsymbol{B}_{l\,m}^{(E)}$$
$$= -\left[\frac{\sqrt{l\,(l\,+\,1)}}{k\,r}\,\hat{\boldsymbol{r}}\;\; h_l^{(1)}\,Y_{l\,m} + \frac{1}{i\,k\,r}\,\frac{d\,r\,h_l^{(1)}}{d\,r}\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\right] e^{-i\,\omega\,t}$$
$$= \text{radial part} + \text{transverse part}$$

Now

$$h_l^{(1)} \to\; (-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}$$
$$\frac{d\,r\,h_l^{(1)}}{d\,r}\;\to\; i\,(-i)^{l+1}\,e^{i\,k\,r} = (-i)^l\,e^{i\,k\,r}$$
$$\frac{1}{i\,k\,r}\,\frac{d\,r\,h_l^{(1)}}{d\,r}\;\to (-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}$$

The radial term is $\frac{1}{k\,r}$ smaller than the transverse term. Usually, it's dropped.

However, since $|\,\boldsymbol{M}\,| \propto r$, it'll give a term to $\frac{d\,\boldsymbol{M}}{d\,r}$ that is r independent. Since the transverse part does not contribute, it's also the lowest order term.

Thus

$$\boldsymbol{B}_{l\,m}^{(E)} \simeq \frac{1}{\sqrt{l\,(l\,+1)}}\,(\boldsymbol{L}\,Y_{l\,m})\;\;(-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}\,e^{-i\,\omega\,t}$$
$$= (-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}\;\boldsymbol{X}_{l\,m}\,e^{-i\,\omega\,t}$$
$$\boldsymbol{E}_{l\,m}^{(E)} \simeq -\left[\frac{\sqrt{l\,(l\,+\,1)}}{k\,r}\;\hat{\boldsymbol{r}}\,Y_{l\,m} + \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\right](-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}\,e^{-i\,\omega\,t}$$

Using

$$\boldsymbol{E} = -\frac{1}{i\,k}\,\nabla\times\boldsymbol{B}$$

we have

$$\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}^*) = -\frac{1}{i\,k}\boldsymbol{r}\times[\,(\nabla\times\boldsymbol{B})\times\boldsymbol{B}^*\,)]$$
$$= -\frac{1}{i\,k}\left[(\nabla\times\boldsymbol{B})\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{B}^*\right) - \boldsymbol{B}^*\,\boldsymbol{r}\cdot(\nabla\times\boldsymbol{B})\;\right]$$

■ Angular Momentum on the Shell (Electric Multipoles)

$$\frac{d\,\boldsymbol{M}}{d\,r} = r^2\int d\,\Omega\,\frac{1}{8\,\pi\,\omega}\,\text{Re}\,[\,\boldsymbol{B}^*\,(\boldsymbol{L}\cdot\boldsymbol{B}\;)\,] \qquad\qquad \omega = c\,k$$

With

$$\boldsymbol{B}_l^{(E)} = \sum_m a_E(l\,m)\,(-i)^{l+1}\,\frac{e^{i k r}}{k r}\,\boldsymbol{X}_{l\,m}\,e^{-i\omega t}$$

$$= \sum_m a_E(l\,m)\,(-i)^{l+1}\,\frac{e^{i k r}}{k r}\,\frac{1}{\sqrt{l(l+1)}}\,\boldsymbol{L}\,Y_{l\,m}\,e^{-i\omega t}$$

$$\boldsymbol{L}\cdot\boldsymbol{B}_l^{(E)} = \sum_m a_E(l\,m)\,(-i)^{l+1}\,\frac{e^{i k r}}{k r}\,\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m}\,e^{-i\omega t}$$

$$= \sum_m a_E(l\,m)\,(-i)^{l+1}\,\frac{e^{i k r}}{k r}\,\sqrt{l(l+1)}\,Y_{l\,m}\,e^{-i\omega t}$$

$$\boldsymbol{B}_l^{(E)*}\left[\boldsymbol{L}\cdot\boldsymbol{B}_l^{(E)}\right] = \frac{1}{k^2 r^2}\sum_{m\,m'} a_E(l\,m)^*\,a_E(l\,m')\,\boldsymbol{X}_{l\,m}{}^*\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m'})$$

$$= \frac{1}{k^2 r^2}\sum_{m\,m'} a_E(l\,m)^*\,a_E(l\,m')\,(\boldsymbol{L}\,Y_{l\,m})^*\,Y_{l\,m'}$$

we have

$$\frac{d\boldsymbol{M}}{d r} = \frac{1}{8\pi\omega k^2}\,\mathrm{Re}\sum_{m\,m'} a_E(l\,m)^*\,a_E(l\,m')\int d\Omega\;\boldsymbol{X}_{l\,m}{}^*\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m'})$$

$$= \frac{1}{8\pi\omega k^2}\,\mathrm{Re}\sum_{m\,m'} a_E(l\,m)^*\,a_E(l\,m')\int d\Omega\,(-\boldsymbol{L}\,Y_{l\,m}{}^*)\,Y_{l\,m'}$$

[Note that the same result can be obtained by using only the radial part of $\boldsymbol{E}_l$]

Thus

$$\boldsymbol{r}\times\left(\boldsymbol{E}_{l\,m'}^{(E)}\times\boldsymbol{B}_{l\,m}^{(E)*}\right) = \boldsymbol{r}\times\left\{\left(-\frac{\sqrt{l(l+1)}}{k r}\,\hat{\boldsymbol{r}}\;h_l^{(1)}\,Y_{l\,m'}\right)\times\left(\frac{1}{\sqrt{l(l+1)}}\,(-\boldsymbol{L}\,Y_{l\,m}{}^*)\;h_l^{(1)*}\right)\right\}$$

$$= \frac{1}{(k r)^3}\,Y_{l\,m'}\,\boldsymbol{r}\times\left(\hat{\boldsymbol{r}}\times\boldsymbol{L}\,Y_{l\,m}{}^*\right)$$

$$= \frac{1}{(k r)^3}\,Y_{l\,m'}\left[\hat{\boldsymbol{r}}\,(\boldsymbol{r}\cdot\boldsymbol{L}\,Y_{l\,m}{}^*) - \boldsymbol{L}\,Y_{l\,m}{}^*\,r\right]$$

$$= -\frac{1}{k^3 r^2}\,Y_{l\,m'}\,\boldsymbol{L}\,Y_{l\,m}{}^*$$

$$= \frac{1}{k^3 r^2}\,Y_{l\,m'}\,(\boldsymbol{L}\,Y_{l\,m})^*$$

Now

$$\mathrm{Re}\,[-\alpha\,Y_{l\,m'}\,\boldsymbol{L}\,Y_{l\,m}{}^*\,] = \mathrm{Re}\,[\alpha\,Y_{l\,m'}\,(\boldsymbol{L}\,Y_{l\,m})^*\,]$$

$$= \mathrm{Re}\,[\alpha^*\,Y_{l\,m'}{}^*\,\boldsymbol{L}\,Y_{l\,m}\,]$$

where we've used

$$\mathrm{Re}(a+b\,i) = a = \mathrm{Re}(a-b\,i) = \mathrm{Re}[\,(a+b\,i)^*\,] \qquad (a,\,b\text{ real})$$

ie

$$\mathrm{Re}\,z = \mathrm{Re}\,z^*$$

Hence

$$\frac{d\boldsymbol{M}}{dr} = \frac{1}{8\pi\omega k^2}\,\mathrm{Re}\sum_{m\,m'} a_E(l\,m)\,a_E(l\,m')^* \int d\Omega\; \boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m'})^* \tag{16.63}$$

$$= \frac{1}{8\pi\omega k^2}\,\mathrm{Re}\sum_{m\,m'} a_E(l\,m)\,a_E(l\,m')^* \int d\Omega\,(\boldsymbol{L}\,Y_{l\,m})\,Y_{l\,m'}{}^* \tag{16.64}$$

Now

$$\boldsymbol{L} = \hat{\boldsymbol{x}}\,L_x + \hat{\boldsymbol{y}}\,L_y + \hat{\boldsymbol{z}}\,L_z$$

$$= \hat{\boldsymbol{x}}\,\frac{1}{2}(L_+ + L_-) + \hat{\boldsymbol{y}}\,\frac{1}{2i}(L_+ - L_-) + \hat{\boldsymbol{z}}\,L_z$$

$$L_\pm\,Y_{l\,m} = \sqrt{(l\mp m)(l\pm m+1)}\;Y_{l\,m\pm1}$$

Therefore

$$\sum_{m'}\int d\Omega\; Y_{l\,m'}{}^*\,L_x\,Y_{l\,m}\,a_E(l\,m)^*$$

$$= \frac{1}{2}\left\{\sqrt{(l-m)(l+m+1)}\;a_E(l\,m+1)^* + \sqrt{(l+m)(l-m+1)}\;a_E(l\,m-1)^*\right\}$$

And

$$\frac{dM_x}{dr} = \frac{1}{16\pi\omega k^2}\,\mathrm{Re}\sum_m \left\{\sqrt{(l-m)(l+m+1)}\;a_E(l\,m+1)^* + \sqrt{(l+m)(l-m+1)}\;a_E(l\,m-1)^*\right\} a_E(l\,m) \tag{16.65}$$

Using

$$\mathrm{Re}\left[\frac{1}{i}(a+b\,i)\right] = b = \mathrm{Im}(a+b\,i)$$

or

$$\mathrm{Re}\left[\frac{1}{i}\,z\right] = \mathrm{Im}\,z$$

we have

$$\frac{dM_y}{dr} = \frac{1}{16\pi\omega k^2}\,\mathrm{Im}\sum_m \left\{\sqrt{(l-m)(l+m+1)}\;a_E(l\,m+1)^* - \sqrt{(l+m)(l-m+1)}\;a_E(l\,m-1)^*\right\} a_E(l\,m) \tag{16.66}$$

$$\frac{dM_z}{dr} = \frac{1}{8\pi\omega k^2}\sum_m m\left|a_E(l\,m)\right|^2 \tag{16.67}$$

For a single electric multipole field of order $(l\,m)$

$$\frac{dM_x}{dr} = \frac{dM_y}{dr} = 0$$

$$\frac{dM_z}{dr} = \frac{1}{8\pi\omega k^2}\;m\left|a_E(l\,m)\right|^2$$

Since

$$\frac{dU}{dr} \simeq \frac{1}{8\pi k^2} \sum_m |a_E(l\,m)|^2 \qquad (16.60)$$

we have

$$\frac{dM_z}{dr} = \frac{m}{\omega} \frac{dU}{dr} \qquad (16.68)$$

or

$$\frac{1}{m\hbar} \Delta M_z = \frac{1}{\hbar\omega} \Delta U$$

We may say that a change of energy by $\hbar\,\omega$ is equivalent to a change of the z component of the angular momentum by $\hbar\,m$ for an electric multipole of order ($l\,m$). [Compare this with the quantum mechanical selection rule]

For the case with other multipoles (same l, different m) present, $\frac{dM_x}{dr}$ and $\frac{dM_y}{dr}$ can be finite. $\frac{d\boldsymbol{M}}{dr}$ is then a coherent sum of them.

Angular Momentum: General Fields

For a general field

$$\boldsymbol{E} = \sum_{l\,m} \left[a_E(l\,m)\, \boldsymbol{E}_{l\,m}^{(E)} + a_M(l\,m)\, \boldsymbol{E}_{l\,m}^{(M)} \right]$$

$$\boldsymbol{B} = \sum_{l\,m} \left[a_E(l\,m)\, \boldsymbol{B}_{l\,m}^{(E)} + a_M(l\,m)\, \boldsymbol{B}_{l\,m}^{(M)} \right]$$

$$\boldsymbol{E}\times\boldsymbol{B}^* = \sum_{l\,m\,l'\,m'} \Big[a_E(l\,m)\, a_E(l'\,m')^*\, \boldsymbol{E}_{l\,m}^{(E)}\times\boldsymbol{B}_{l'\,m'}^{(E)\,*} + a_M(l\,m)\, a_M(l'\,m')^*\, \boldsymbol{E}_{l\,m}^{(M)}\times\boldsymbol{B}_{l'\,m'}^{(M)\,*}$$
$$+ a_E(l\,m)\, a_M(l'\,m')^*\, \boldsymbol{E}_{l\,m}^{(E)}\times\boldsymbol{B}_{l'\,m'}^{(M)\,*} + a_M(l\,m)\, a_E(l'\,m')^*\, \boldsymbol{E}_{l\,m}^{(M)}\times\boldsymbol{B}_{l'\,m'}^{(E)\,*} \Big]$$

As already shown before

$$\boldsymbol{r}\times\left(\boldsymbol{E}^{(E)}\times\boldsymbol{B}^{(E)*}\right) = \frac{1}{k}\boldsymbol{B}^{(E)*}\left(\boldsymbol{L}\cdot\boldsymbol{B}^{(E)}\right)$$

$$\boldsymbol{r}\times\left(\boldsymbol{E}^{(M)}\times\boldsymbol{B}^{(M)*}\right) = \frac{1}{k}\boldsymbol{E}^{(M)}\left(\boldsymbol{L}\cdot\boldsymbol{E}^{(M)}\right)^*$$

Similarly, we see that

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{E}^{(M)} = 0 \qquad \& \qquad \hat{\boldsymbol{r}}\cdot\boldsymbol{B}^{(E)} = 0$$

gives

$$\boldsymbol{r}\times\left(\boldsymbol{E}^{(M)}\times\boldsymbol{B}^{(E)*}\right) = 0$$

$$\boldsymbol{r}\times\left(\boldsymbol{E}^{(E)}\times\boldsymbol{B}^{(M)*}\right) = \boldsymbol{E}^{(E)}\left(\boldsymbol{r}\cdot\boldsymbol{B}^{(M)*}\right) - \boldsymbol{B}^{(M)*}\left(\boldsymbol{r}\cdot\boldsymbol{E}^{(E)}\right)$$
$$= -\frac{1}{i\,k}\left[\boldsymbol{E}^{(E)}\,\boldsymbol{r}\cdot\left(\nabla\times\boldsymbol{E}^{(M)*}\right) - \boldsymbol{B}^{(M)*}\,\boldsymbol{r}\cdot\left(\nabla\times\boldsymbol{B}^{(E)}\right)\right]$$
$$= -\frac{1}{k}\left[\boldsymbol{E}^{(E)}\left(\boldsymbol{L}\cdot\boldsymbol{E}^{(M)*}\right) - \boldsymbol{B}^{(M)*}\left(\boldsymbol{L}\cdot\boldsymbol{B}^{(E)}\right)\right]$$
$$= \frac{1}{k}\left[\boldsymbol{E}^{(E)}\left(\boldsymbol{L}\cdot\boldsymbol{E}^{(M)}\right)^* + \boldsymbol{B}^{(M)*}\left(\boldsymbol{L}\cdot\boldsymbol{B}^{(E)}\right)\right]$$

Hence

$$\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}^*) = \frac{1}{k}\sum_{l\,m\,l'\,m'}\Big[\, a_E(l\,m)\, a_E(l'\,m')^*\, \boldsymbol{B}^{(E)\,*}_{l'\,m'}\big(\boldsymbol{L}\cdot\boldsymbol{B}^{(E)}_{l\,m}\big)$$

$$+a_M(l\,m)\, a_M(l'\,m')^*\, \boldsymbol{E}^{(M)}_{l\,m}\big(\boldsymbol{L}\cdot\boldsymbol{E}^{(M)}_{l'\,m'}\big)^*$$

$$+\,a_E(l\,m)\, a_M(l'\,m')^*\Big[\, \boldsymbol{E}^{(E)}_{l\,m}\big(\boldsymbol{L}\cdot\boldsymbol{E}^{(M)}_{l'\,m'}\big)^* + \boldsymbol{B}^{(M)\,*}_{l'\,m'}\big(\boldsymbol{L}\cdot\boldsymbol{B}^{(E)}_{l\,m}\big)\Big]$$

In the radiation zone,

$$\boldsymbol{E}^{(M)}_{l\,m} = (-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}\,\boldsymbol{X}_{l\,m}$$

$$\boldsymbol{B}^{(E)}_{l\,m} = (-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}\,\boldsymbol{X}_{l\,m}$$

and, dropping the radial parts,

$$\boldsymbol{B}^{(M)}_{l\,m} \simeq \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}$$

$$\boldsymbol{E}^{(E)}_{l\,m} \simeq -\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(-i)^{l+1}\,\frac{e^{i\,k\,r}}{k\,r}$$

Therefore

$$\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}^*) = \frac{1}{k^3\,r^2}\sum_{l\,m\,l'\,m'} i^{\,l'+1}\,(-i)^{l+1}\,[\, a_E(l\,m)\, a_E(l'\,m')^*\, \boldsymbol{X}_{l'\,m'}{}^*(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})$$

$$+a_M(l\,m)\, a_M(l'\,m')^*\, \boldsymbol{X}_{l\,m}(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*$$

$$+\,a_E(l\,m)\, a_M(l'\,m')^*\Big[-\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^* + \big(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\big)^*(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})\Big]$$

Putting everything together

$$\frac{d\,\boldsymbol{M}}{d\,r} = \frac{1}{8\,\pi\,\omega\,k^2}\,\mathrm{Re}\int d\,\Omega\sum_{l\,m\,l'\,m'} i^{\,l'-l}\,[\, a_E(l\,m)\, a_E(l'\,m')^*\, \boldsymbol{X}_{l'\,m'}{}^*(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})$$

$$+a_M(l\,m)\, a_M(l'\,m')^*\, \boldsymbol{X}_{l\,m}(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*$$

$$+\,a_E(l\,m)\, a_M(l'\,m')^*\Big[-\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^* + \big(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\big)^*(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})\Big]$$

Now

$$\mathrm{Re}\sum_{l\,m\,l'\,m'} i^{\,l'-l}\, a_E(l\,m)\, a_E(l'\,m')^*\, \boldsymbol{X}_{l'\,m'}{}^*(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})$$

$$= \mathrm{Re}\sum_{l\,m\,l'\,m'} i^{\,l-l'}\, a_E(l'\,m')\, a_E(l\,m)^*\, \boldsymbol{X}_{l\,m}{}^*(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})$$

$$= \mathrm{Re}\sum_{l\,m\,l'\,m'} (-i)^{\,l-l'}\, a_E(l'\,m')^*\, a_E(l\,m)\, \boldsymbol{X}_{l\,m}(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*$$

Since

$$\int d\,\Omega\,\boldsymbol{X}_{l\,m}(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^* \propto \delta_{l\,l'}$$

we can set

$$\mathrm{Re}\int d\,\Omega\sum_{l\,m\,l'\,m'} i^{\,l'-l}\, a_E(l\,m)\, a_E(l'\,m')^*\, \boldsymbol{X}_{l'\,m'}{}^*(\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})$$

$$= \mathrm{Re}\sum_{l\,m\,l'\,m'} a_E(l'\,m')^*\, a_E(l\,m)\, \boldsymbol{X}_{l\,m}(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*$$

Next

$$\operatorname{Re}\sum_{l\,m\,l'\,m'} i^{\,l'-l}\, a_E(l\,m)\, a_M(l'\,m')^* \left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\right)^* (\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})$$

$$= \operatorname{Re}\sum_{l\,m\,l'\,m'} i^{\,l-l'}\, a_E(l'\,m')\, a_M(l\,m)^* \left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\right)^* (\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})$$

$$= \operatorname{Re}\sum_{l\,m\,l'\,m'} i^{\,-l+l'}\, a_E(l'\,m')^*\, a_M(l\,m) \left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\right) (\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*$$

so that

$$\operatorname{Re}\sum_{l\,m\,l'\,m'} i^{\,l'-l}\, a_E(l\,m)\, a_M(l'\,m')^* \left[-\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^* + \left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\right)^* (\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})\right]$$

$$= \operatorname{Re}\sum_{l\,m\,l'\,m'} i^{\,-l+l'}\left[-a_E(l\,m)\, a_M(l'\,m')^* + a_E(l'\,m')^*\, a_M(l\,m)\right] \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*$$

Since

$$\int d\,\Omega\; \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^* \propto \delta_{l\,l'\pm 1}$$

we can set

$$\operatorname{Re}\int d\,\Omega \sum_{l\,m\,l'\,m'} i^{\,l'-l}\, a_E(l\,m)\, a_M(l'\,m')^* \left[-\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^* + \left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\right)^* (\boldsymbol{L}\cdot\boldsymbol{X}_{l\,m})\right]$$

$$= \operatorname{Re}\int d\,\Omega \sum_{l\,m\,l'\,m'} i\left[-a_E(l\,m)\, a_M(l'\,m')^* + a_E(l'\,m')^*\, a_M(l\,m)\right] \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*$$

Hence

$$\frac{d\,\boldsymbol{M}}{d\,r} = \frac{1}{8\,\pi\,\omega\,k^2}\operatorname{Re}\int d\,\Omega$$
$$\times \sum_{l\,m\,l'\,m'} \Big\{\left[a_E(l'\,m')^*\, a_E(l\,m) + a_M(l\,m)\, a_M(l'\,m')^*\right] \boldsymbol{X}_{l\,m}(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*$$
$$+ i\left[-a_E(l\,m)\, a_M(l'\,m')^* + a_E(l'\,m')^*\, a_M(l\,m)\right] \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\,(\boldsymbol{L}\cdot\boldsymbol{X}_{l'\,m'})^*\Big\} \qquad (16.69)$$

The cross term is not quite the same as (16.69) given by Jackson.

■ Parity

$$H_{\text{int}} = \frac{q}{c}\,\boldsymbol{v}\cdot\boldsymbol{A}_{l\,m}$$

where $\boldsymbol{A}_{l\,m}$ is the vector potential of the $(l\,m)$ multipole field.

Transition amplitude $\propto \langle f\,|H_{\text{int}}|\,i\rangle$

Let the angular dependence be

$$|\,i\rangle \propto |\,J\,M\rangle$$
$$|\,f\rangle \propto |\,J'\,M'\rangle$$
$$H_{\text{int}} \propto (l\,m)$$

The Wigner- Eckert theorem then gives the selection rules

$$|J-l| \le J' \le J+l$$
$$M' = M - m$$

or

$$|J-J'| \le l \le J+J'$$
$$m = M - M'$$

Parity is conserved for EM interaction.
Hence

Products of the parities of $|f\rangle$, $|i\rangle$ and L_{int} is even

ie

$$P(H_{\text{int}}) = P(|f\rangle)\, P(|i\rangle)$$

Now

$$P(\boldsymbol{v}) = -1$$
$$P(\boldsymbol{B}) = P(\nabla\times\boldsymbol{A}) = -P(\boldsymbol{A})$$
$$P(H_{\text{int}}) = P(\boldsymbol{v})\, P(\boldsymbol{A}) = P(\boldsymbol{B})$$

ie, parity of multipole transition is the same as $\boldsymbol{B}_{l\,m}$.

Now

$$\boldsymbol{B}_{l\,m}^{(E)} \propto \boldsymbol{L}\, Y_{l\,m} \qquad \text{(radial part has even parity)}$$
$$P(\boldsymbol{L}) = 1$$
$$P(Y_{l\,m}) = (-)^l$$

Therefore

$$P\left(\boldsymbol{B}_{l\,m}^{(E)}\right) = (-)^l$$

Similarly

$$\boldsymbol{B}_{l\,m}^{(E)} \propto \nabla\times(\boldsymbol{L}\, Y_{l\,m})$$

gives

$$P\left(\boldsymbol{B}_{l\,m}^{(M)}\right) = (-)^{l+1}$$

Eg. $\quad J = \frac{1}{2} \longrightarrow \quad J' = \frac{3}{2}$

Therefore

$$|J - J'| = 1 \le l \le J + J' = 2$$

Thus

$$l = 1,\ 2$$

Now, if parities of $|i\rangle$ & $|f\rangle$ are the same.
Thus

l = even for electric multipole transition $\longrightarrow$ $l = 2$ (quadrupole)
l = odd for magnetic multipole transition $\longrightarrow$ $l = 1$ (dipole)

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4. Angular Distribution

Time averaged power radiated into $d\,\Omega$

$$d\,P = \boldsymbol{S}\cdot\hat{\boldsymbol{r}}\, r^2\, d\,\Omega$$
$$= \frac{c}{8\pi}\,(\boldsymbol{E}\times\boldsymbol{B}^*)\cdot\hat{\boldsymbol{r}}\, r^2\, d\,\Omega$$

In the radiation zone

$$\boldsymbol{B} \simeq \sum_{l\,m} h_l^{(1)} \left[a_E(l\,m)\, \boldsymbol{X}_{l\,m} + a_M(l\,m)\, \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m} \right]$$
$$\simeq \frac{e^{i\,k\,r - i\,\omega\,t}}{k\,r} \sum_{l\,m} (-i)^{l+1} \left[a_E(l\,m)\, \boldsymbol{X}_{l\,m} + a_M(l\,m)\, \hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m} \right]$$
$$\boldsymbol{E} = \boldsymbol{B}\times\hat{\boldsymbol{r}}$$

Using

$$\hat{\boldsymbol{r}}\cdot(\boldsymbol{E}\times\boldsymbol{B}^*) = \hat{\boldsymbol{r}}\cdot\left[\left(\boldsymbol{B}\times\hat{\boldsymbol{r}}\right)\times\boldsymbol{B}^*\right]$$
$$= \hat{\boldsymbol{r}}\cdot\left[\hat{\boldsymbol{r}}\,|\boldsymbol{B}|^2 - \boldsymbol{B}\left(\hat{\boldsymbol{r}}\cdot\boldsymbol{B}^*\right)\right]$$
$$= |\boldsymbol{B}|^2$$
$$= \left|\hat{\boldsymbol{r}}\times\boldsymbol{E}\right|^2$$
$$= |\boldsymbol{E}|^2$$

We have

$$\frac{dP}{d\Omega} = \frac{c}{8\pi k^2}\left|\sum_{lm}(-i)^{l+1}\left[a_E(l\,m)\,\boldsymbol{X}_{lm} + a_M(l\,m)\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{lm}\right]\right|^2$$
$$= \frac{c}{8\pi k^2}\left|\sum_{lm}(-i)^{l+1}\left[a_E(l\,m)\,\boldsymbol{X}_{lm}\times\hat{\boldsymbol{r}} + a_M(l\,m)\,\boldsymbol{X}_{lm}\right]\right|^2$$

Thus, magnetic & electric multipoles have the same angular dependence.
To distinguish between them, one must use polarization measurements.

For a single multipole field of order ($l\,m$),

$$\frac{dP(l\,m)}{d\Omega} = \frac{c}{8\pi k^2}|a(l\,m)|^2\,|\boldsymbol{X}_{lm}|^2$$
$$|\boldsymbol{X}_{lm}|^2 = \frac{1}{l(l+1)}|\boldsymbol{L}\,Y_{lm}|^2$$
$$|\boldsymbol{L}\,Y_{lm}|^2 = (\boldsymbol{L}\,Y_{lm})\cdot(\boldsymbol{L}\,Y_{lm})^*$$
$$= |L_x Y_{lm}|^2 + \left|L_y Y_{lm}\right|^2 + |L_z Y_{lm}|^2$$
$$= \frac{1}{4}|(L_+ + i L_-)Y_{lm}|^2 + \frac{1}{4}|(L_+ - i L_-)Y_{lm}|^2 + |L_z Y_{lm}|^2$$
$$= \frac{1}{2}\left|\,|L_+ Y_{lm}|^2 + \frac{1}{2}|L_- Y_{lm}|^2 + |L_z Y_{lm}|^2\right.$$
$$= \frac{1}{2}(l-m)(l+m+1)|Y_{l\,m+1}|^2 + \frac{1}{2}(l+m)(l-m+1)|Y_{l\,m-1}|^2 + m^2|Y_{lm}|^2$$

$$\frac{dP(l\,m)}{d\Omega} = \frac{c}{8\pi k^2}\left|a(l\,m)\right|^2\left[\frac{1}{2}(l-m)(l+m+1)|Y_{l\,m+1}|^2 + \frac{1}{2}(l+m)(l-m+1)|Y_{l\,m-1}|^2 + m^2|Y_{lm}|^2\right] \qquad (16.76)$$

The values of Y_{lm} and $\boldsymbol{X}_{lm}$ are listed below for $l = 0,\ 1,\ 2$.

	$m = 0$	$m = \pm 1$	$m = \pm 2$
$l = 0$	$Y_{00} = \frac{1}{\sqrt{4\pi}}$		
$l = 1$	$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$ $\|X_{10}\|^2 = \frac{3}{8\pi}\sin^2\theta$	$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{\pm i\phi}$ $\left\|X_{1,\pm1}\right\|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta\right)$	
$l = 2$	$Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$ $\|X_{20}\|^2 = \frac{15}{8\pi}\sin^2\theta\cos^2\theta$	$Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{\pm i\phi}$ $\left\|X_{2,\pm1}\right\|^2 = \frac{5}{16\pi}\left(1-3\cos^2\theta + 4\cos^4\theta\right)$	$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{\pm 2i\phi}$ $\left\|X_{2,\pm2}\right\|^2 = \frac{5}{16\pi}\left(1-\cos^4\theta\right)$

For example

$$|X_{10}|^2 = \frac{1}{2}\left\{\frac{1}{2}\cdot 1\cdot 2\,|Y_{11}|^2 + \frac{1}{2}\cdot 1\cdot 2\,|Y_{1\,-1}|^2 + 0\right\}$$

$$= \frac{3}{8\pi}\sin^2\theta$$

$$\left|X_{1,\pm 1}\right|^2 = \frac{1}{2}\left\{\frac{1}{2}\cdot 2\cdot 1\,|Y_{10}|^2 + |Y_{1\times 1}|^2\right\}$$

$$= \frac{1}{2}\left(\frac{3}{4\pi}\cos^2\theta + \frac{3}{8\pi}\sin^2\theta\right)$$

$$= \frac{3}{16\pi}\left(1+\cos^2\theta\right)$$

etc

From the sum rule for $Y_{l\,m}$, ie

$$\sum_m |Y_{l\,m}|^2 = \frac{2\,l+1}{4\pi} \qquad (3.69)$$

we have the sum rule for $\boldsymbol{X}_{l\,m}$

$$\sum_m |\boldsymbol{X}_{l\,m}|^2 = \frac{2\,l+1}{4\pi}$$

The proof is as follows

$$|\boldsymbol{X}_{l\,m}|^2 = \frac{1}{l\,(l+1)}|\boldsymbol{L}\,Y_{l\,m}|^2$$

$$= -\frac{1}{l\,(l+1)}(\boldsymbol{L}\,Y_{l\,m})\cdot(\boldsymbol{L}\,Y_{l\,m}^*)$$

Consider

$$\boldsymbol{L}\cdot[\,Y_{l\,m}\,\boldsymbol{L}\,Y_{l\,m}^*\,] = (\boldsymbol{L}\,Y_{l\,m})\cdot(\boldsymbol{L}\,Y_{l\,m}^*) + Y_{l\,m}\,\boldsymbol{L}^2\,Y_{l\,m}^*$$

$$\boldsymbol{L}\cdot[\,Y_{l\,m}^*\,\boldsymbol{L}\,Y_{l\,m}\,] = (\boldsymbol{L}\,Y_{l\,m}^*)\cdot(\boldsymbol{L}\,Y_{l\,m}) + Y_{l\,m}^*\,\boldsymbol{L}^2\,Y_{l\,m}$$

Hence

$$(\boldsymbol{L}\,Y_{l\,m})\cdot(\boldsymbol{L}\,Y_{l\,m}^*) = \frac{1}{2}\,\boldsymbol{L}\cdot[\,Y_{l\,m}\,\boldsymbol{L}\,Y_{l\,m}^* + Y_{l\,m}^*\,\boldsymbol{L}\,Y_{l\,m}\,] - l\,(l+1)\,|Y_{l\,m}|^2$$

so that

$$|\boldsymbol{X}_{l\,m}|^2 = |Y_{l\,m}|^2 + A_{l\,m}$$

where

$$A_{l\,m} = -\frac{1}{2\,l\,(l+1)}\,\boldsymbol{L}\cdot[\,Y_{l\,m}\,\boldsymbol{L}\,Y_{l\,m}^* + Y_{l\,m}^*\,\boldsymbol{L}\,Y_{l\,m}\,]$$

Therefore, the proof is done if we can show

$$\sum_m A_{l\,m} = 0$$

Now

$$\boldsymbol{L}\cdot[\,Y_{lm}\,\boldsymbol{L}\,Y_{lm}{}^{*}+Y_{lm}{}^{*}\,\boldsymbol{L}\,Y_{lm}\,]=\boldsymbol{L}\cdot\boldsymbol{L}\,(\,Y_{lm}{}^{*}\,Y_{lm}\,)=\boldsymbol{L}^2\,|\,Y_{lm}|^2$$

so that

$$\sum_m A_{lm} = -\frac{1}{2\,l\,(\,l\,+\,1\,)}\,\boldsymbol{L}^2\sum_m |\,Y_{lm}|^2$$

$$= -\frac{1}{2\,l\,(\,l\,+\,1\,)}\,\boldsymbol{L}^2\left(\frac{2\,l\,+\,1}{4\,\pi}\right)$$

$$= 0 \qquad\qquad (\,\boldsymbol{L}^2 \text{ is a differential operator }).$$

Total radiated power of a single multipole is

$$P(\,l\,m\,) = \int d\,\Omega\;\frac{d\,P(\,l\,m\,)}{d\,\Omega}$$

$$= \frac{c}{8\,\pi\,k^2}|\,a(\,l\,m\,)\,|^2\int d\,\Omega\;|\,\boldsymbol{X}_{lm}|^2$$

$$= \frac{c}{8\,\pi\,k^2}|\,a(\,l\,m\,)\,|^2$$

Total radiated power of general multipoles is

$$P = \int d\,\Omega\;\frac{d\,P}{d\,\Omega}$$

$$= \frac{c}{8\,\pi\,k^2}\int d\,\Omega\left|\sum_{lm}(-i)^{l+1}\left[\,a_E(\,l\,m\,)\,\boldsymbol{X}_{lm}\times\hat{\boldsymbol{r}}+a_M(\,l\,m\,)\,\boldsymbol{X}_{lm}\,\right]\right|^2$$

$$= \frac{c}{8\,\pi\,k^2}\sum_{lm}\left[\,|\,a_E(\,l\,m\,)|^2+|\,a_M(\,l\,m\,)|^2\,\right]$$

---

5. Multipole Moments

Consider harmonically varying sources (or 1 Fourier component):

$$\rho,\;\boldsymbol{J},\;\boldsymbol{M}\propto e^{-i\,\omega\,t}$$

eg.

$$\rho(\,\boldsymbol{x},\;t\,)=\rho(\,\boldsymbol{x},\;\omega\,)\,e^{-i\,\omega\,t} \qquad \text{etc}$$

Maxwell's eqs become

$$\nabla\cdot\boldsymbol{B}=0 \qquad\qquad \nabla\cdot\boldsymbol{E}=4\,\pi\,\rho$$

$$\nabla\times\boldsymbol{B}=\frac{4\,\pi}{c}\,(\,\boldsymbol{J}+c\,\nabla\times\boldsymbol{M}\,)-i\,k\,\boldsymbol{E} \qquad\qquad \nabla\times\boldsymbol{E}=i\,k\,\boldsymbol{B}$$

where

$$k=\frac{\omega}{c}$$

$$\boldsymbol{H}=\boldsymbol{B}-4\,\pi\,\boldsymbol{M}$$

and all charges are treated as true so that $\boldsymbol{E}=\boldsymbol{D}$.

The equation of continuity is

$$-i\,\omega\;\rho+\nabla\cdot\boldsymbol{J}=0$$

Hence

$$\nabla \cdot \boldsymbol{E} = \frac{4\pi}{i\omega} \nabla \cdot \boldsymbol{J}$$

or

$$\nabla \cdot \left(\boldsymbol{E} - \frac{4\pi}{i\omega} \boldsymbol{J} \right) = 0 = \nabla \cdot \boldsymbol{E}'$$

where

$$\boldsymbol{E}' = \boldsymbol{E} - \frac{4\pi}{i\omega} \boldsymbol{J}$$

is divergenceless. Note that $\boldsymbol{E}' = \boldsymbol{E}$ outside sources ($\boldsymbol{J} = 0$).

The Maxwell's eqs can be written as

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{E}' = 0$$

$$\nabla \times \boldsymbol{B} = 4\pi \nabla \times \boldsymbol{M} - i k \boldsymbol{E}' \qquad \nabla \times \boldsymbol{E}' = i k \boldsymbol{B} - \frac{4\pi}{i\omega} \nabla \times \boldsymbol{J}$$

Thus

$$\boldsymbol{B} = \frac{1}{i k} \nabla \times \left(\boldsymbol{E}' + \frac{4\pi}{i\omega} \boldsymbol{J} \right)$$

$$\boldsymbol{E}' = -\frac{1}{i k} \nabla \times (\boldsymbol{B} - 4\pi \boldsymbol{M})$$

so that

$$\begin{aligned}
\nabla \times \boldsymbol{B} &= \frac{1}{i k} \nabla \times \left[\nabla \times \left(\boldsymbol{E}' + \frac{4\pi}{i\omega} \boldsymbol{J} \right) \right] \\
&= \frac{1}{i k} \left\{ \nabla \left[\nabla \cdot \left(\boldsymbol{E}' + \frac{4\pi}{i\omega} \boldsymbol{J} \right) \right] - \nabla^2 \left(\boldsymbol{E}' + \frac{4\pi}{i\omega} \boldsymbol{J} \right) \right\} \\
&= \frac{1}{i k} \left\{ \frac{4\pi}{i\omega} \left[\nabla (\nabla \cdot \boldsymbol{J}) - \nabla^2 \boldsymbol{J} \right] - \nabla^2 \boldsymbol{E}' \right\} \\
&= \frac{1}{i k} \left\{ \frac{4\pi}{i\omega} \nabla \times (\nabla \times \boldsymbol{J}) - \nabla^2 \boldsymbol{E}' \right\} \\
&= 4\pi \nabla \times \boldsymbol{M} - i k \boldsymbol{E}'
\end{aligned}$$

ie

$$\frac{4\pi}{i\omega} \nabla \times (\nabla \times \boldsymbol{J}) - \nabla^2 \boldsymbol{E}' = i k 4\pi \nabla \times \boldsymbol{M} + k^2 \boldsymbol{E}'$$

or

$$\begin{aligned}
\left(\nabla^2 + k^2 \right) \boldsymbol{E}' &= \frac{4\pi}{i\omega} \nabla \times (\nabla \times \boldsymbol{J}) - i k 4\pi \nabla \times \boldsymbol{M} \\
&= -4\pi i k \nabla \times \left[\boldsymbol{M} + \frac{1}{c k^2} \nabla \times \boldsymbol{J} \right] \qquad (16.85)
\end{aligned}$$

Similarly

$$\nabla\times \boldsymbol{E}\,' = -\frac{1}{i\,k}\nabla\times[\nabla\times(\boldsymbol{B} - 4\pi\boldsymbol{M})]$$

$$= -\frac{1}{i\,k}\left\{\nabla\,[\nabla\cdot(\boldsymbol{B} - 4\pi\boldsymbol{M})] - \nabla^2(\boldsymbol{B} - 4\pi\boldsymbol{M})\right\}$$

$$= \frac{1}{i\,k}\left\{4\pi\nabla\times(\nabla\times\boldsymbol{M}) + \nabla^2\,\boldsymbol{B}\right\}$$

$$= i\,k\,\boldsymbol{B} - \frac{4\pi}{i\,\omega}\nabla\times\boldsymbol{J}$$

so that

$$\left(\nabla^2 + k^2\right)\boldsymbol{B} = -\frac{4\pi}{c}\nabla\times[\boldsymbol{J} + c\,\nabla\times\boldsymbol{M}] \qquad (16.85)$$

Now, what about the substitution rules?
To derive them, we write the Maxwell eqs as

$$\nabla\cdot\boldsymbol{B} = 0 \qquad \nabla\cdot\boldsymbol{E}\,' = 0$$

$$\nabla\times\boldsymbol{H} = -i\,k\,\boldsymbol{E}\,' \qquad \nabla\times\boldsymbol{E} = i\,k\,\boldsymbol{B}$$

These eqs are invariant under the substitutions

$$\boldsymbol{E}\,' \longrightarrow \boldsymbol{B} \qquad \boldsymbol{E} \longrightarrow \boldsymbol{H}$$

$$\boldsymbol{B} \longrightarrow -\boldsymbol{E}\,' \qquad \boldsymbol{H} \longrightarrow -\boldsymbol{E}$$

Since

$$\boldsymbol{E}\,' = \boldsymbol{E} - \frac{4\pi}{i\,\omega}\boldsymbol{J}$$

$$\boldsymbol{H} = \boldsymbol{B} - 4\pi\boldsymbol{M}$$

applying the new rules to the 1st eq gives

$$\boldsymbol{B} = \boldsymbol{H} - \frac{4\pi}{i\,\omega}\boldsymbol{X}$$

where $\boldsymbol{X}$ is what $\boldsymbol{J}$ should become. Comparing this with the 2nd eq gives

$$\boldsymbol{X} = -i\,\omega\,\boldsymbol{M}$$

Similarly, applying the new rules to the 2nd eq gives

$$-\boldsymbol{E} = -\boldsymbol{E}\,' - 4\pi\boldsymbol{Y}$$

where $\boldsymbol{Y}$ is what $\boldsymbol{M}$ should become. Comparing this with the 1st eq gives

$$\boldsymbol{Y} = \frac{1}{i\,\omega}\boldsymbol{J}$$

To summarize, the new rules are

$$\boldsymbol{E}\,' \longrightarrow \boldsymbol{B} \qquad \boldsymbol{E} \longrightarrow \boldsymbol{H} \qquad \boldsymbol{J} \longrightarrow -i\,\omega\,\boldsymbol{M}$$

$$\boldsymbol{B} \longrightarrow -\boldsymbol{E}\,' \qquad \boldsymbol{H} \longrightarrow -\boldsymbol{E} \qquad \boldsymbol{M} \longrightarrow \frac{1}{i\,\omega}\boldsymbol{J}$$

and

$$\left(\nabla^2 + k^2\right)\boldsymbol{E}\,' = -4\pi\,i\,k\,\nabla\times\left[\boldsymbol{M} + \frac{1}{c\,k^2}\nabla\times\boldsymbol{J}\right]$$

$$\left(\nabla^2 + k^2\right)\boldsymbol{B} = -\frac{4\pi}{c}\nabla\times[\boldsymbol{J} + c\,\nabla\times\boldsymbol{M}]$$

Now, since

$$\nabla \cdot \boldsymbol{E}' = 0$$

we have

$$\boldsymbol{r}\cdot\left(\nabla^2 + k^2\right)\boldsymbol{E}' = \left(\nabla^2 + k^2\right)\boldsymbol{r}\cdot\boldsymbol{E}'$$

$$= -4\pi i k\, \boldsymbol{r}\cdot\left[\nabla\times\left(\boldsymbol{M} + \frac{1}{c k^2}\nabla\times\boldsymbol{J}\right)\right]$$

$$= -4\pi i k\, (\boldsymbol{r}\times\nabla)\cdot\left(\boldsymbol{M} + \frac{1}{c k^2}\nabla\times\boldsymbol{J}\right)$$

$$= 4\pi k\, \boldsymbol{L}\cdot\left(\boldsymbol{M} + \frac{1}{c k^2}\nabla\times\boldsymbol{J}\right)$$

Using the substitution rules:

$$\left(\nabla^2 + k^2\right)\boldsymbol{r}\cdot\boldsymbol{B} = 4\pi k\, \boldsymbol{L}\cdot\left[\frac{1}{i\omega}\boldsymbol{J} + \frac{1}{c k^2}(-i\omega)\nabla\times\boldsymbol{M}\right]$$

$$= -\frac{4\pi}{c} i\, \boldsymbol{L}\cdot(\boldsymbol{J} + c\nabla\times\boldsymbol{M})$$

Using

$$\left(\nabla^2 + k^2\right)G(\boldsymbol{x}, \boldsymbol{x}') = -\delta(\boldsymbol{x} - \boldsymbol{x}')$$

where

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{e^{i k |\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi\, |\boldsymbol{x}-\boldsymbol{x}'|}$$

we have

$$\boldsymbol{r}\cdot\boldsymbol{E}' = -\int d^3x'\; 4\pi k\left[\boldsymbol{L}'\cdot\left(\boldsymbol{M} + \frac{1}{c k^2}\nabla'\times\boldsymbol{J}\right)\right]G(\boldsymbol{x}, \boldsymbol{x}')$$

$$= -k\int d^3x'\; \frac{e^{i k |\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}\boldsymbol{L}'\cdot\left(\boldsymbol{M} + \frac{1}{c k^2}\nabla'\times\boldsymbol{J}\right)$$

where, ' denotes operators that act on the source points $\boldsymbol{x}'$. The sources $\boldsymbol{M}$ & $\boldsymbol{J}$ are of course functions of $\boldsymbol{x}'$.

Similarly

$$\boldsymbol{r}\cdot\boldsymbol{B} = \frac{i}{c}\int d^3x'\; \frac{e^{i k |\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}\boldsymbol{L}'\cdot(\boldsymbol{J} + c\nabla'\times\boldsymbol{M}) \qquad (16.87)$$

From eq.16.47, we have

$$a_E(l\, m)\, f_l = -\frac{k}{\sqrt{l(l+1)}}\int d\Omega\; Y_{l m}(\Omega)^*\; \boldsymbol{r}\cdot\boldsymbol{E}'$$

$$= \frac{k^2}{\sqrt{l(l+1)}}\int d\Omega\; Y_{l m}(\Omega)^*\int d^3x'\; \frac{e^{i k |\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}\boldsymbol{L}'\cdot\left(\boldsymbol{M} + \frac{1}{c k^2}\nabla'\times\boldsymbol{J}\right)$$

Now

$$\frac{e^{i k |\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|} = \sum_{l' m'} 4\pi i k\, j_{l'}(k r_<)\, h_{l'}^{(1)}(k r_>)\, Y_{l' m'}(\Omega)\; Y_{l' m'}(\Omega')^*$$

Since

$$\int d\,\Omega\; Y_{l\,m}\,(\Omega)^*\; Y_{l'\,m'}(\Omega)\; = \delta_{l\,l'}\,\delta_{m\,m'}$$

only the $(\,l\,m\,)$ term survives

$$a_E(\,l\,m\,)\,f_l\; = \frac{4\,\pi\,i\,k^3}{\sqrt{l\,(\,l\,+\,1\,)}}\int d^3\,x'\;\; j_l(\,k\,r\,'\,)\,h_l^{(1)}(\,k\,r\,)\,Y_{l\,m}(\Omega'\,)^*\,\boldsymbol{L}'\cdot\left(\boldsymbol{M}+\frac{1}{c\,k^2}\,\nabla'\times\boldsymbol{J}\right)$$

where we've set $r_< = r\,'$ and $r_> = r$ since we're interested only in the fields outside the sources.

Far away from the sources,

$$f_l\; = \alpha\;\, h_l^{(1)}$$

so that

$$a_E(\,l\,m\,)\;\to\;\frac{4\,\pi\,i\,k^3}{\alpha\;\sqrt{l\,(\,l\,+\,1\,)}}\int d^3\,x\;j_l\,Y_{l\,m}{}^*\,\boldsymbol{L}\cdot\left(\boldsymbol{M}+\frac{1}{c\,k^2}\,\nabla\times\boldsymbol{J}\right) \qquad (16.89)$$

where ' can be suppressed since it involves dummy variables.

To obtain $a_M(\,l\,m\,)$, we need only replace

$$\boldsymbol{M}+\frac{1}{c\,k^2}\,\nabla\times\boldsymbol{J}$$

with

$$\frac{i}{k\,c}[\,\boldsymbol{J}+c\,\nabla\times\,\boldsymbol{M}\,]$$

so that

$$a_M(\,l\,m\,)\;\to -\frac{4\,\pi\,k^2}{\alpha\,c\,\sqrt{l\,(\,l\,+\,1\,)}}\int d^3\,x\;\; j_l\,Y_{l\,m}{}^*\,\boldsymbol{L}\cdot(\;\boldsymbol{J}+c\,\nabla\times\,\boldsymbol{M}\,)$$

Now

$$\begin{aligned}\boldsymbol{L}\cdot\boldsymbol{A} &= \;\frac{1}{i}\,(\,\boldsymbol{r}\times\nabla\,)\cdot\boldsymbol{A}\\ &=\;\frac{1}{i}\,\boldsymbol{r}\cdot(\,\nabla\,\times\boldsymbol{A}\,)\end{aligned}$$

$$\begin{aligned}\nabla\cdot(\;\boldsymbol{r}\;\times\boldsymbol{A}\,) &= \;-\frac{1}{i}\,\boldsymbol{r}\cdot(\,\nabla\,\times\boldsymbol{A}\,)+\boldsymbol{A}\cdot(\,\nabla\,\times\boldsymbol{r}\,)\\ &=\;-\frac{1}{i}\,\boldsymbol{r}\cdot(\,\nabla\,\times\boldsymbol{A}\,)\end{aligned}$$

Hence

$$\boldsymbol{L}\cdot\boldsymbol{A}=i\,\nabla\cdot(\;\boldsymbol{r}\;\times\boldsymbol{A}\,) \qquad (16.90)$$

Now

$$\begin{aligned}\boldsymbol{L}\cdot(\,\nabla\,\times\boldsymbol{A}\,) &= -i\,(\;\boldsymbol{r}\;\times\nabla\,)\cdot(\;\boldsymbol{r}\;\times\boldsymbol{A}\,)\\ &=\;-i\;\boldsymbol{r}\;\cdot[\,\nabla\,\times(\;\nabla\;\times\boldsymbol{A}\,)\,]\\ &=\;-i\;\boldsymbol{r}\;\cdot\left[\,\nabla\,(\;\nabla\;\cdot\boldsymbol{A}\,)-\nabla^2\,\boldsymbol{A}\,\right]\\ &=\;-i\;\left[\,r\,\frac{\partial}{\partial\,r}\,(\;\nabla\;\cdot\boldsymbol{A}\,)-\nabla^2(\,\boldsymbol{r}\;\cdot\boldsymbol{A}\,)+2\,\nabla\cdot\boldsymbol{A}\,\right]\\ &=\;-i\;\left[\,\frac{1}{r}\;\frac{\partial}{\partial\,r}\left(\,r^2\,\nabla\cdot\boldsymbol{A}\right)-\nabla^2(\,\boldsymbol{r}\cdot\boldsymbol{A}\,)\right]\\ &=\;i\left[\,\nabla^2(\,\boldsymbol{r}\;\cdot\boldsymbol{A}\,)-\;\frac{1}{r}\;\frac{\partial}{\partial\,r}\left(\,r^2\,\nabla\;\cdot\boldsymbol{A}\right)\right] \qquad (16.90)\end{aligned}$$

Hence

$$\boldsymbol{L}\cdot(\boldsymbol{J}+c\nabla\times\boldsymbol{M}) = i\nabla\cdot(\boldsymbol{r}\times\boldsymbol{J}) + ic\left[\nabla^2(\boldsymbol{r}\cdot\boldsymbol{M}) - \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\nabla\cdot\boldsymbol{M}\right)\right]$$

And

$$a_M(l\,m) \to -\frac{4\pi k^2 i}{\alpha c\sqrt{l(l+1)}}\int d^3x\; j_l(kr)\,Y_{l\,m}(\Omega)^*$$
$$\times\left\{\nabla\cdot(\boldsymbol{r}\times\boldsymbol{J}) + c\left[\nabla^2(\boldsymbol{r}\cdot\boldsymbol{M}) - \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\nabla\cdot\boldsymbol{M}\right)\right]\right\}$$

Consider

$$I = \int d^3x\, j_l\, Y_{l\,m}{}^*\;\nabla^2(\boldsymbol{r}\cdot\boldsymbol{M})$$

Since

$$\nabla\cdot[Y\,j\,\nabla(\boldsymbol{r}\cdot\boldsymbol{M})] = \nabla(Y\,j)\cdot\nabla(\boldsymbol{r}\cdot\boldsymbol{M}) + Y\,j\,\nabla^2(\boldsymbol{r}\cdot\boldsymbol{M})$$

we have

$$I = -\int d^3x\;\nabla[j_l\,Y_{l\,m}{}^*]\cdot\nabla(\boldsymbol{r}\cdot\boldsymbol{M})$$

Next, using

$$\nabla\cdot[\boldsymbol{r}\cdot\boldsymbol{M}\;\nabla(Y\,j)] = \nabla(Y\,j)\cdot\nabla(\boldsymbol{r}\cdot\boldsymbol{M}) + \boldsymbol{r}\cdot\boldsymbol{M}\,\nabla^2(Y\,j)$$

we have

$$I = \int d^3x\,\boldsymbol{r}\cdot\boldsymbol{M}\,\nabla^2[j_l\,Y_{l\,m}{}^*]$$

$$= k^2\int d^3x\;\boldsymbol{r}\cdot\boldsymbol{M}\;\; j_l\,Y_{l\,m}{}^* \qquad \text{(from the Helmholtz eq with } j_l \text{ real)}$$

Next, consider (' suppressed for simplicity)

$$I = \int r^2\,dr\,j_l\,\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\nabla\cdot\boldsymbol{M}\right) \qquad \text{[radial integral of 3rd term in } a_M(l\,m) \text{]}$$

$$= \left[r\,j_l\,r^2\,\nabla\cdot M\right]_0^\infty - \int dr\;\frac{d\,r\,j_l}{dr}\;r^2\,\nabla\cdot\boldsymbol{M}$$

$$= -\int r^2\,dr\;\frac{d\,r\,j_l}{dr}\;\nabla\cdot\boldsymbol{M}$$

Hence

$$a_M(l\,m) = -\frac{4\pi k^2 i}{\alpha\sqrt{l(l+1)}}\int d^3x\;Y_{l\,m}{}^*\left\{\frac{1}{c}\,j_l\;\nabla\cdot(\boldsymbol{r}\times\boldsymbol{J}) - k^2\,j_l\,\boldsymbol{r}\cdot\boldsymbol{M} + \frac{d\,r\,j_l}{dr}\nabla\cdot\boldsymbol{M}\right\} \qquad (16.92)$$

We can go through similar procedure to get $a_E(l\,m)$.
However, it's easier to invoke the substitution rules, which should be supplemented by

$$a_M \longrightarrow a_E$$
$$a_E \longrightarrow -a_M$$

so that

$$a_E(l\,m) = -\frac{4\pi k^2 i}{\alpha\sqrt{l(l+1)}}\int d^3x\, Y_{l\,m}{}^*\left\{-i\,k\,j_l\,\nabla\cdot(\boldsymbol{r}\times\boldsymbol{M}) - k^2\,j_l\,\frac{1}{i\,k\,c}\,\boldsymbol{r}\cdot\boldsymbol{J} + \frac{1}{i\,k\,c}\,\frac{d\,r\,j_l}{d\,r}\,\nabla\cdot\boldsymbol{J}\right\}$$

$$= -\frac{4\pi k^2 i}{\alpha\sqrt{l(l+1)}}\int d^3x\, Y_{l\,m}{}^*\left\{-i\,k\,j_l\,\nabla\cdot(\boldsymbol{r}\times\boldsymbol{M}) + \frac{i\,k}{c}\,j_l\,\boldsymbol{r}\cdot\boldsymbol{J} + \rho\left(\frac{d\,r\,j_l}{d\,r}\right)\right\} \qquad (16.91)$$

where

$$\rho = \frac{1}{i\,\omega}\nabla\cdot\boldsymbol{J}$$

In the long wavelength limit

$$k\,r_{\max} \ll 1$$

where sources are confined to $r \leq r_{\max}$.

$$j_l(k\,r) \simeq \frac{(k\,r)^l}{(2\,l+1)!!}$$
$$\frac{d\,r\,j_l}{d\,r} \simeq \frac{k^l}{(2\,l+1)!!}\,\frac{d\,r^{l+1}}{d\,r}$$
$$\simeq \frac{(l+1)(k\,r)^l}{(2\,l+1)!!}$$

With $\alpha = 1$,

$$a_E(l\,m) \simeq \frac{4\pi k^2}{i\sqrt{l(l+1)}}\int d^3x\, Y_{l\,m}{}^*\,\frac{(k\,r)^l}{(2\,l+1)!!}\left\{(l+1)\,\rho + \frac{i\,k}{c}\,\boldsymbol{r}\cdot\boldsymbol{J} - i\,k\,\nabla\cdot(\boldsymbol{r}\times\boldsymbol{M})\right\}$$

$$= \frac{4\pi k^{l+2}}{i\,(2\,l+1)!!}\sqrt{\frac{l+1}{l}}\int d^3x\, Y_{l\,m}{}^*\,r^l\left\{\rho + \frac{i\,k}{c\,(l+1)}\,\boldsymbol{r}\cdot\boldsymbol{J} - \frac{i\,k}{l+1}\,\nabla\cdot(\boldsymbol{r}\times\boldsymbol{M})\right\}$$

$$= \frac{4\pi k^{l+2}}{i\,(2\,l+1)!!}\sqrt{\frac{l+1}{l}}\,[\,Q_{l\,m} + Q_{l\,m}'\,]$$

where

$$Q_{l\,m} = \int d^3x\, Y_{l\,m}{}^*\,r^l\,\rho = \quad \text{electric multipoles}$$

$$Q_{l\,m}' = -\frac{i\,k}{l+1}\int d^3x\, Y_{l\,m}{}^*\,r^l\,\nabla\cdot(\boldsymbol{r}\times\boldsymbol{M})$$

and the term with $\boldsymbol{r}\cdot\boldsymbol{J}$ is dropped since it's proportional to $(k\,r)^{l+1}$

Since

$$\nabla\cdot(\boldsymbol{r}\times\boldsymbol{M}) = \boldsymbol{M}\cdot(\nabla\times\boldsymbol{r}) - \boldsymbol{r}\cdot(\nabla\times\boldsymbol{M})$$
$$= -\boldsymbol{r}\cdot(\nabla\times\boldsymbol{M})$$

$Q_{l\,m}'$ is also $k\,r$ times smaller than $Q_{l\,m}$ if $\nabla\times\boldsymbol{M}$ is independent of r. This is the case for atomic & nuclear problems.

Similarly

$$a_M(l\,m) \simeq -\frac{4\pi k^2 i}{\sqrt{l(l+1)}}\frac{k^l}{(2l+1)!!}\int d^3x\, Y_{l\,m}{}^* r^l \left\{\frac{1}{c}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{J}) - k^2\,\boldsymbol{r}\cdot\boldsymbol{M} + (l+1)\nabla\cdot\boldsymbol{M}\right\}$$

$$= \frac{4\pi k^{l+2} i}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\,[M_{l\,m} + M_{l\,m}']$$

where

$$M_{l\,m} = -\frac{1}{l+1}\int d^3x\, Y_{l\,m}{}^* r^l \frac{1}{c}\nabla\cdot(\boldsymbol{r}\times\boldsymbol{J})$$

$$M_{l\,m}' = -\int d^3x\, Y_{l\,m}{}^* r^l\,\nabla\cdot\boldsymbol{M} = \quad \text{magnetic multipoles}$$

and the term with $k^2\ \boldsymbol{r}\cdot\boldsymbol{M}$ is dropped. note also the minus sign in the definitions of $M_{l\,m}$ & $M_{l\,m}'$.

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6. Nuclear Systems

Radiated power of a multipole is given by

$$P(l\,m) = \frac{c}{8\pi k^2}\,|\,a(l\,m)|^2 \qquad (16.78)$$

$$a_E(l\,m) = \frac{4\pi k^{l+2}}{i\,(2l+1)!!}\sqrt{\frac{l+1}{l}}\,[Q_{l\,m} + Q_{l\,m}']$$

$$a_M(l\,m) = \frac{4\pi k^{l+2} i}{(2l+1)!!}\sqrt{\frac{l+1}{l}}\,[M_{l\,m} + M_{l\,m}']$$

$$\Longrightarrow \qquad P_E(l\,m) = \frac{c}{8\pi k^2}\left(\frac{4\pi k^{l+2}}{(2l+1)!!}\right)^2 \frac{l+1}{l}|\,Q_{l\,m} + Q_{l\,m}'\,|^2$$

$$= \frac{2\pi c\,k^{2l+2}}{[(2l+1)!!]^2}\frac{l+1}{l}|\,Q_{l\,m} + Q_{l\,m}'\,|^2$$

$$P_E(l\,m) = \frac{2\pi c\,k^{2l+2}}{[(2l+1)!!]^2}\frac{l+1}{l}|\,M_{l\,m} + M_{l\,m}'\,|^2$$

In quantum mechanical terms, the radiation energy is carried away by a photon with energy $\hbar\,\omega$. ie.

$$P\,\tau = \hbar\,\omega$$

where $P = d\,E/d\,t$ is the power, τ the mean life- time of the radiative transition.

Hence, the transition probability is

$$\frac{1}{\tau} = \frac{P}{\hbar\,\omega}$$

Let

$$\rho = \begin{cases} \frac{3e}{a^3}\,Y_{l\,m} & \text{for} \quad r < a \\ 0 & \quad r > a \end{cases}$$

The choice of the coefficient $3e/a^3$ is to make

$$\int_0^\infty r^2\, dr\, \rho(\boldsymbol{x}) = \int_0^a r^2\, dr\, \frac{3e}{a^3}\, Y_{lm} = e\, Y_{lm} \equiv \rho(\Omega)$$

so that the total charge in the double cone of between solid angle Ω and $\Omega + d\,\Omega$ is $e\, Y_{lm}$.

The presence of Y_{lm} is just to simplify the calculations.
Since $|Y_{lm}|^2 \lesssim 1$, the total charge is of order e.

$$\begin{aligned} Q_{l'm'} &= \int d^3x\, r^l\, Y_{l'm'}^*\, \rho \\ &= \int_0^a r^2\, dr\, \frac{3e}{a^3}\, r^l \int d\,\Omega\; Y_{l'm'}^*\, Y_{lm} \\ &= \frac{3e}{l+3}\, a^l\, \delta_{l'l}\, \delta_{m'm} \end{aligned}$$

Thus, ρ represents a single electric multipole of order $(l\, m)$, ie

$$Q_{lm} = \frac{3e}{l+3}\, a^l$$

We'll neglect Q_{lm}' since it's usually smaller than Q_{lm} (see below).

M_{lm} & M_{lm}' are usually of the same order of magnitude so we set

$$\frac{1}{c\,(l+1)}\, \nabla\cdot(\boldsymbol{r}\times\boldsymbol{J}) + \nabla\cdot\boldsymbol{M} = \begin{cases} \frac{2g}{a^3}\, Y_{lm}\, \frac{e\hbar}{mc}\, \frac{1}{r} & \text{for } r < a \\ 0 & r > a \end{cases}$$

Once again, the choice of the coefficient is to give the total magnetic moment in the double cone a value of

$$g\, \frac{e\hbar}{mc}\, Y_{lm}$$

Note: $g\, \frac{e\hbar}{mc}$ times $\boldsymbol{S}$ or $\boldsymbol{L}$ is the magnetic moment of the particle.

$$\begin{aligned} M_{l'm'} + M_{l'm'}' &= -\int d^3x\; Y_{lm}^*\, r^l \left[\frac{1}{c\,(l+1)}\, \nabla\cdot(\boldsymbol{r}\times\boldsymbol{J}) + \nabla\cdot\boldsymbol{M} \right] \\ &= -\int_0^a r^2\, dr\, \frac{2g}{a^3}\, \frac{e\hbar}{mc}\, r^{l-1} \int d\,\Omega\; Y_{l'm'}^*\, Y_{lm} \\ &= -g\, \frac{e\hbar}{mc}\, \frac{2}{l+2}\, a^{l-1}\, \delta_{l'l}\, \delta_{m'm} \end{aligned}$$

or

$$\begin{aligned} M_{lm} + M_{lm}' &= -g\, \frac{e\hbar}{mc}\, \frac{2}{l+2}\, a^{l-1} \\ &\simeq -g\, \frac{\hbar}{mca}\, Q_{lm} \end{aligned}$$

Note:
Assuming

$$\boldsymbol{M} \simeq g \frac{e \hbar}{m c} \boldsymbol{L} Y_{l m} \qquad \rho \simeq e Y_{l m}$$

then

$$\begin{aligned} Q_{l m}' &= -\frac{i k}{l+1} \int d^3 x \, Y_{l m}^* r^l \, \nabla \cdot (\boldsymbol{r} \times \boldsymbol{M}) \\ &= \frac{k}{l+1} \int d^3 x \, Y_{l m}^* r^l \, \boldsymbol{L} \cdot \boldsymbol{M} \\ &\simeq \frac{k}{l+1} g \frac{e \hbar}{m c} l (l+1) \int_0^a r^2 d r \int d \Omega \, Y_{l m}^* Y_{l m} r^l \\ &\simeq g \frac{\hbar \omega}{m c^2} l Q_{l m} \end{aligned}$$

Since $\hbar \omega \ll m c^2$ for the usual atomic or nuclear transitions, $Q_{l m}' \ll Q_{l m}$.

Thus, with

$$Q_{l m} = \frac{3 e}{l+3} a^l$$

we have

$$\begin{aligned} \frac{1}{\tau_E(l)} &= \frac{P_E}{\hbar \omega} \\ &\simeq \frac{2 \pi c k^{2l+2}}{\hbar \omega [(2 l + 1) !!]^2} \frac{l+1}{l} \left(\frac{3 e}{l+3} \right)^2 a^{2 l} \\ &= \left(\frac{e^2}{\hbar c} \right) \frac{2 \pi}{[(2 l + 1) !!]^2} \frac{l+1}{l} \left(\frac{3}{l+3} \right)^2 \left[\frac{c^2 k^{2l+2}}{\omega} a^{2 l} \right] \\ &= \alpha \frac{2 \pi}{[(2 l + 1) !!]^2} \frac{l+1}{l} \left(\frac{3}{l+3} \right)^2 (k a)^{2 l} \omega \end{aligned}$$

Hence

$$\frac{1}{\tau_E(l)} \propto (k a)^{2 l} \omega$$

In the long wavelength limit, $k a \ll 1$ so that

$$\frac{1}{\tau_E(l)} \longrightarrow 0 \qquad \text{for large } l$$

Therefore, only the lowest multipoles are important.

Since

$$M_{l m} + M_{l m}' \simeq -g \frac{\hbar}{m c a} Q_{l m}$$

$$\frac{1}{\tau_M(l)} \simeq \left(g \frac{\hbar}{m c a} \right)^2 \frac{1}{\tau_E(l)}$$

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7. Center- Fed Antenna

■ Results from § 9.3

$$I(z,t) = I(z)\, e^{-i\omega t}$$

$$I(z) = \left(1 - 2\frac{|z|}{d}\right) I_0\ \theta\!\left(\frac{d}{2} - |z|\right) \qquad (9.25)$$

so that

$$I(0) = I_0$$

$$I\left(\pm\frac{d}{2}\right) = 0$$

The eq. of continuity

$$-\frac{\partial\rho}{\partial t} = i\,\omega\,\rho = \nabla\cdot\boldsymbol{J}$$

gives a linear charge density

$$\rho' = \int d\,\sigma\,\rho = \frac{1}{i\,\omega}\int d\,\sigma\,\nabla\cdot\boldsymbol{J}$$

where $d\,\sigma$ is a surface element normal to a linear direction ($\hat{\boldsymbol{z}}$ in our case)

For the antenna

$$\boldsymbol{J} = J_z\,\hat{\boldsymbol{z}}$$

$$I(z) = \int\!\!\int d\,x\,d\,y\,J_z(\boldsymbol{x}) = \left(1 - 2\frac{|z|}{d}\right) I_0\ \theta\!\left(\frac{d}{2} - |z|\right)$$

means that

$$J_z(\boldsymbol{x}) = I(z)\,\delta(x)\,\delta(y)\,\theta\!\left(\frac{d}{2} - |z|\right)$$

$$= \left(1 - 2\frac{|z|}{d}\right) I_0\,\delta(x)\,\delta(y)\,\theta\!\left(\frac{d}{2} - |z|\right)$$

and

$$\nabla\cdot\boldsymbol{J} = \frac{\partial J_z}{\partial z} = \mp\frac{2}{d}\,I_0\,\delta(x)\,\delta(y)\,\theta\!\left(\frac{d}{2} - |z|\right) \qquad \text{for } z \gtrless 0$$

$$\rho = \pm\, i\,\frac{2}{\omega\,d}\,I_0\,\delta(x)\,\delta(y)\,\theta\!\left(\frac{d}{2} - |z|\right) \qquad \text{for } z \gtrless 0$$

With $d\,\sigma = d\,x\,d\,y$, we have

$$\rho' = \pm\, i\,\frac{2}{\omega\,d}\,I_0\,\theta\!\left(\frac{d}{2} - |z|\right) \qquad \text{for } z \gtrless 0 \qquad (9.26)$$

The dipole moment is defined as

$$\boldsymbol{p} = \int d^3 x' \; \boldsymbol{x}' \rho(\boldsymbol{x}') \qquad (9.17)$$

For the antenna

$$\boldsymbol{p} = i \frac{2}{\omega d} I_0 \int d x' \int d y' \left[\int_0^{d/2} d z' - \int_{-d/2}^{0} d z' \right] \boldsymbol{x}' \; \delta(x') \, \delta(y')$$

so that

$$p_x = p_y = 0$$

$$p_z = i \frac{2}{\omega d} I_0 \left[\int_0^{d/2} d z' \, z' - \int_{-d/2}^{0} d z' \, z' \right]$$

$$= i \frac{2}{\omega d} I_0 \left(\frac{d^2}{4} \right)$$

$$= i \frac{d}{2 \omega} I_0 \qquad (9.27)$$

In the radiation zone

$$\boldsymbol{B} = k^2 \frac{e^{i k r}}{r} \boldsymbol{n} \times \boldsymbol{p}$$

$$\boldsymbol{E} = \boldsymbol{B} \times \boldsymbol{n} = k^2 \frac{e^{i k r}}{r} (\boldsymbol{n} \times \boldsymbol{p}) \times \boldsymbol{n}$$

so that time- averaged radiated power per solid angle is

$$\frac{d P}{d \Omega} = r^2 \hat{\boldsymbol{n}} \cdot \boldsymbol{S} \qquad \boldsymbol{S} = \frac{1}{2} \cdot \frac{1}{4 \pi} \boldsymbol{E} \times \boldsymbol{B}^*$$

$$= \frac{c}{8 \pi} k^4 \left| (\boldsymbol{n} \times \boldsymbol{p}) \times \boldsymbol{n} \right|^2$$

$$= \frac{c}{8 \pi} k^4 \left| \boldsymbol{p} - \boldsymbol{n} (\boldsymbol{p} \cdot \boldsymbol{n}) \right|^2$$

$$= \frac{c}{8 \pi} k^4 \left| \boldsymbol{p} \right|^2 \sin^2 \theta \qquad \text{where} \quad \boldsymbol{p} \cdot \boldsymbol{n} = p \cos \theta$$

$$= \frac{c}{32 \pi} k^4 \frac{d^2}{\omega^2} I_0^2 \sin^2 \theta$$

$$= \frac{1}{32 \pi c} (k d)^2 I_0^2 \sin^2 \theta \qquad [\, \omega = k c \,] \qquad (9.28)$$

The total radiated power is

$$P = \frac{1}{32 \pi c} (k d)^2 I_0^2 \; 2 \pi \int_{-1}^{1} d x \left(1 - x^2 \right) \qquad [\, x = \cos \theta \,]$$

$$= \frac{1}{12 c} (k d)^2 I_0^2 \qquad (9.29)$$

where

$$\int_{-1}^{1} d\,x\left(1-x^2\right) = 2\int_{0}^{1} d\,x\left(1-x^2\right)$$
$$= \; 2\left[1-\frac{1}{3}\right]$$
$$= \; \frac{4}{3}$$

■ Results From § 9.4

■ Sinusoidal Current

$$I(z) = \theta\left(\frac{d}{2} - |\,z\,|\right)\; I\sin\left(\frac{k\,d}{2} - k\,|\,z\,|\right)$$

where I is the "peak" current at $\frac{k\,d}{2} - k\left|z\right| = \frac{\pi}{2}$. Note that since $|\,z\,| \;\leq\; \frac{d}{2}$, this "peak" is actually realized only if $k\,d \geq \pi$.

Also

$$I_0 = I(0) = \;\; I\sin\left(\frac{k\,d}{2}\right)$$
$$J_z(\boldsymbol{x}) = \;\; I\sin\left(\frac{k\,d}{2} - k\,|\,z|\right)\;\delta(x)\,\delta(y)\,\theta\left(\frac{d}{2} - |\,z\,|\right) \qquad (9.53)$$

In the radiation zone, the vector potential at $\boldsymbol{x} = r\,\boldsymbol{n} = (\,r,\,\theta,\,\phi\,)_{\text{sph}}$ is

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{e^{i\,k\,r}}{c\,r}\int d^3\,x'\,\boldsymbol{J}(\boldsymbol{x}')\,e^{-i\,k\,\boldsymbol{n}\cdot\boldsymbol{x}'} \qquad (9.8)$$
$$= \;\hat{\boldsymbol{z}}\,\frac{e^{i\,k\,r}}{c\,r}\,I\,\left[\int_{-d/2}^{0} d\,z'\sin\left(\frac{k\,d}{2} + k\,z'\right)e^{-i\,k\,z'\cos\theta} + \int_{0}^{d/2} d\,z'\sin\left(\frac{k\,d}{2} - k\,z'\right)e^{-i\,k\,z'\cos\theta}\right]$$

Now

$$\int_{-d/2}^{0} d\,z'\sin\left(\frac{k\,d}{2} + k\,z'\right)e^{-i\,k\,z'\cos\theta} \underset{z'\to -z'}{=} -\int_{d/2}^{0} d\,z'\sin\left(\frac{k\,d}{2} - k\,z'\right)e^{i\,k\,z'\cos\theta}$$

Hence

$$\boldsymbol{A}(\boldsymbol{x}) = \hat{\boldsymbol{z}}\,\frac{e^{i\,k\,r}}{c\,r}\,I\int_{0}^{d/2} d\,z'\;2\sin\left(\frac{k\,d}{2} - k\,z'\right)\cos(\,k\,z'\cos\theta\,)$$

To use

$$\sin a + \sin b = 2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)$$

we need

$$\frac{1}{2}(a+b) = \frac{k\,d}{2} - k\,z'$$

$$\frac{1}{2}(a-b) = k\,z'\cos\theta$$

so that

$$a = \frac{k\,d}{2} - k\,z'(1-\cos\theta)$$

$$b = \frac{k\,d}{2} - k\,z'(1+\cos\theta)$$

and

$$\boldsymbol{A}(\boldsymbol{x}) = \hat{\boldsymbol{z}}\,\frac{e^{i k r}}{c\,r}\,I \int_0^{d/2} d\,z' \left\{ \sin\left[\frac{k\,d}{2} - k\,z'(1-\cos\theta)\right] + \sin\left[\frac{k\,d}{2} - k\,z'(1+\cos\theta)\right]\right\}$$

$$= \hat{\boldsymbol{z}}\,\frac{e^{i k r}}{c\,r}\,I \left\{ \frac{1}{k(1-\cos\theta)}\left[\cos\left(\frac{k\,d}{2}\cos\theta\right) - \cos\frac{k\,d}{2}\right] + \frac{1}{k(1+\cos\theta)}\left[\cos\left(\frac{k\,d}{2}\cos\theta\right) - \cos\frac{k\,d}{2}\right]\right\}$$

$$= \hat{\boldsymbol{z}}\,\frac{e^{i k r}}{c\,r}\,I \left[\cos\left(\frac{k\,d}{2}\cos\theta\right) - \cos\frac{k\,d}{2}\right] \frac{2}{k\left(1-\cos^2\theta\right)}$$

$$= \hat{\boldsymbol{z}}\,\frac{e^{i k r}}{c\,k\,r}\,2\,I \left[\frac{\cos\left(\frac{k\,d}{2}\cos\theta\right) - \cos\frac{k\,d}{2}}{\sin^2\theta}\right] \qquad (9.55)$$

Now, in the radiation zone,

$$\boldsymbol{B} = i\,k\,\hat{\boldsymbol{n}}\times\boldsymbol{A}$$

Hence

$$|\boldsymbol{B}| = k\,|\boldsymbol{A}|\sin\theta$$

$$= \frac{1}{c\;r}\,2\,I\left[\frac{\cos\left(\frac{k\,d}{2}\cos\theta\right) - \cos\frac{k\,d}{2}}{\sin\theta}\right]$$

$$= |\boldsymbol{E}|$$

And

$$\frac{d\,P}{d\,\Omega} = \frac{c}{8\,\pi}|\,r\,\boldsymbol{E}\,|^2$$

$$= \frac{I^2}{2\,\pi\,c}\left[\frac{\cos\left(\frac{k\,d}{2}\cos\theta\right) - \cos\frac{k\,d}{2}}{\sin\theta}\right]^2 \qquad (9.56)$$

For $k\,d \ll 1$,

$$\frac{\cos\left(\frac{k\,d}{2}\cos\theta\right) - \cos\frac{k\,d}{2}}{\sin\theta} \simeq \frac{\frac{1}{2}\left(\frac{k\,d}{2}\right)^2\left[\cos^2\theta - 1\right]}{\sin\theta}$$

$$= \frac{1}{2}\left(\frac{k\,d}{2}\right)^2 \sin\theta$$

and

$$\frac{d\,P}{d\,\Omega} \simeq \frac{I^2}{128\,\pi\,c}(k\,d)^4 \sin^2\theta$$

Since, in this limit,

$$I_0 \simeq I \frac{k d}{2}$$

we have

$$\frac{d P}{d \Omega} \simeq \frac{{I_0}^2}{32 \pi c} (k d)^2 \sin^2 \theta$$

which is just the dipole result.

For $k d = \pi$.

$$\frac{d P}{d \Omega} = \frac{I^2}{2 \pi c} \left[\frac{\cos\left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta} \right]^2 \qquad \text{(half- wave)}$$

For $k d = 2 \pi$.

$$\frac{d P}{d \Omega} = \frac{I^2}{2 \pi c} \left[\frac{\cos(\pi \cos \theta) + 1}{\sin \theta} \right]^2$$

$$= \frac{I^2}{2 \pi c} \left[\frac{2 \cos^2\left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta} \right]^2 \qquad \text{(full- wave)}$$

- **Present Case**

$$I(z, t) = I(| z|) \theta\left(\frac{d}{2} - | z | \right) e^{-i \omega t} \qquad (16.112)$$

$$I\left(| z| = \frac{d}{2}\right) = 0$$

$$\boldsymbol{J} = J_z \hat{\boldsymbol{z}}$$

$$I(z) = \int \int d x\, d y\, J_z(\boldsymbol{x}) = I(| z|)\ \theta\left(\frac{d}{2} - | z | \right)$$

means that

$$J_z(\boldsymbol{x}) = I(| z|) \delta(x) \delta(y)\ \theta\left(\frac{d}{2} - | z | \right)$$

and

$$\nabla \cdot \boldsymbol{J} = \frac{\partial J_z}{\partial z}$$

$$= \frac{\partial I(| z|)}{\partial z} \delta(x) \delta(y) + I(| z|) \delta(x) \delta(y) \delta\left(\frac{d}{2} - | z | \right) \operatorname{sgn} z$$

$$= \frac{\partial I(| z|)}{\partial z} \delta(x) \delta(y)$$

$$\rho = - \frac{i}{\omega} \frac{\partial I(| z|)}{\partial z} \delta(x) \delta(y)$$

With $d \sigma = d x\, d y$, we have

$$\rho' = - \frac{i}{\omega} \frac{\partial I(| z|)}{\partial z}$$

In a multipole expansion, we need to work in the spherical coordinates. Writing

$$I(z') = \int\int\int d\,x\,d\,y\,d\,z\,\delta(z' - z)\,J_z(\boldsymbol{x})$$
$$= \int\int\int r^2\,d\,r\,d\cos\theta\,d\,\phi\;\delta(z' - r\cos\theta)\,J_z(\boldsymbol{x})$$
$$= \begin{cases} \theta\left(\frac{d}{2} - r'\right) I(r') & \text{for} \quad \theta' = 0, \pi \\ 0 & \text{otherwise} \end{cases}$$

where $I(r) \equiv I(\,|\,z\,| = r\,)$, we have

$$J_z(\boldsymbol{x}) = \theta\left(\frac{d}{2} - r\right) \frac{1}{2\pi r^2} I(r)\;\delta(\,|\cos\theta| - 1\,)$$
$$= \theta\left(\frac{d}{2} - r\right) \frac{1}{2\pi r^2} I(r)\;[\,\delta(\cos\theta - 1\,) + \delta(\cos\theta + 1\,)\,]$$

Note that J_z has the same sign for both $\cos\theta = \pm 1$.

Using

$$\hat{\boldsymbol{r}} = \pm\hat{\boldsymbol{z}} \qquad \text{for} \qquad \cos\theta = \pm 1$$

we can also write

$$\boldsymbol{J} = J_r\,\hat{\boldsymbol{r}}$$

with

$$J_r(\boldsymbol{x}) = \theta\left(\frac{d}{2} - r\right) \frac{1}{2\pi r^2} I(r)\;[\,\delta(\cos\theta - 1\,) - \delta(\cos\theta + 1\,)\,] \qquad (16.113)$$

Note the "−" sign in front of $\delta(\cos\theta + 1\,)$ which comes from $\hat{\boldsymbol{r}} = -\hat{\boldsymbol{z}}$ when $\cos\theta = -1$.

Now

$$\nabla\cdot\boldsymbol{J} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 J_r\right)$$
$$= \frac{1}{2\pi r^2}\left\{\frac{d}{d\,r}\left[\theta\left(\frac{d}{2} - r\right) I(r)\right]\right\} \times [\,\delta(\cos\theta - 1\,) - \delta(\cos\theta + 1\,)\,]$$
$$= \frac{1}{2\pi r^2}\,\theta\left(\frac{d}{2} - r\right) \frac{d\,I(r)}{d\,r}\;[\,\delta(\cos\theta - 1\,) - \delta(\cos\theta + 1\,)\,]$$

where we've used

$$\delta\left(\frac{d}{2} - r\right) I(r) = \delta\left(\frac{d}{2} - r\right) I\left(\frac{d}{2}\right) = 0$$

Thus

$$\rho(\boldsymbol{x}) = \frac{1}{i\,\omega}\,\nabla\cdot\boldsymbol{J}$$
$$= \frac{1}{2\pi\,i\,\omega\,r^2}\,\theta\left(\frac{d}{2} - r\right) \frac{d\,I(r)}{d\,r}\;[\,\delta(\cos\theta - 1\,) - \delta(\cos\theta + 1\,)\,] \qquad (16.114)$$

Since $\boldsymbol{M}=0$, eq(16.91) becomes

$$a_E(l\,m) = -\frac{4\pi k^2 i}{\sqrt{l(l+1)}} \int d^3x\, Y_{lm}{}^* \left\{ \frac{ik}{c}\, j_l\, \boldsymbol{r}\cdot\boldsymbol{J} + \rho\, \frac{d\, r\, j_l}{d\, r} \right\}$$

$$= \quad -\frac{4\pi k^2 i}{\sqrt{l(l+1)}} \int_0^{d/2} r^2\, dr \int d\Omega\, Y_{lm}{}^*$$

$$\times \left\{ \frac{ik}{c}\, j_l\, \frac{1}{2\pi r}\, I(r)\, [\, \delta(\cos\theta - 1) - \delta(\cos\theta + 1)\,] \right.$$

$$\left. + \frac{d\, r\, j_l}{d\, r}\, \frac{1}{2\pi i \omega r^2}\, \frac{d\, I(r)}{d\, r}\, [\, \delta(\cos\theta - 1) - \delta(\cos\theta + 1)\,] \right\}$$

$$= \quad \frac{2k^2}{\sqrt{l(l+1)}} \int d\Omega\, Y_{lm}{}^*\, [\, \delta(\cos\theta - 1) - \delta(\cos\theta + 1)\,]$$

$$\times \int_0^{d/2} dr \left[\frac{k}{c}\, r\, j_l\, I(r) - \frac{d\, r\, j_l}{d\, r}\, \frac{1}{\omega}\, \frac{d\, I(r)}{d\, r} \right] \qquad (16.115)$$

Now

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi\, Y_{lm}{}^*\, [\, \delta(\cos\theta - 1) - \delta(\cos\theta + 1)\,]$$

$$= \int_0^{2\pi} d\phi\, [\, Y_{lm}{}^*(0, \phi) - Y_{lm}{}^*(\pi, \phi)\,]$$

From

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\, \frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\, e^{im\phi}$$

we see that

$$\int_0^{2\pi} d\phi\, Y_{lm}{}^*(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\, \frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta) \int_0^{2\pi} d\phi\, e^{-im\phi}$$

$$= \sqrt{\frac{2l+1}{4\pi}\, \frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\; 2\pi\, \delta_{m0}$$

$$= \; 2\pi\, \delta_{m0} \sqrt{\frac{2l+1}{4\pi}}\; P_l(\cos\theta)$$

$$= \; 2\pi\, \delta_{m0}\, Y_{l0}(\theta)$$

Hence

$$\int d\Omega\; Y_{lm}{}^*\, [\, \delta(\cos\theta - 1) - \delta(\cos\theta + 1)\,]$$

$$= \; 2\pi\, \delta_{m0}\, [\, Y_{l0}(0) - Y_{l0}(\pi)\,] \qquad (16.116)$$

$$= \; 2\pi\, \delta_{m0} \sqrt{\frac{2l+1}{4\pi}}\, [\, P_l(1) - P_l(-1)\,]$$

$$= \; 2\pi\, \delta_{m0} \sqrt{\frac{2l+1}{4\pi}}\, \left[1 - (-)^l \right]$$

And

$$a_E(l\,m) = \delta_{m\,0}\,\left[1-(-)^l\right]\,k^2\,\sqrt{\frac{4\pi\,(2\,l+1)}{l\,(l+1)}}\int_0^{d/2} d\,r\left[\frac{k}{c}\,r\,j_l\;I(r) - \frac{d\,r\,j_l}{d\,r}\,\frac{1}{\omega}\,\frac{d\,I(r)}{d\,r}\right]$$

Thus $a_E \neq 0$ only for $m = 0$ and $l =$ odd, in which case

$$a_E(l\,0) = 2\,k^2\,\sqrt{\frac{4\pi\,(2\,l+1)}{l\,(l+1)}}\int_0^{d/2} d\,r\left[\frac{k}{c}\,r\,j_l\;I(r) - \frac{d\,r\,j_l}{d\,r}\,\frac{1}{\omega}\,\frac{d\,I(r)}{d\,r}\right]$$

$$= \frac{2\,k}{c}\,\sqrt{\frac{4\pi\,(2\,l+1)}{l\,(l+1)}}\int_0^{d/2} d\,r\left[k^2\,r\,j_l\;I(r) - \frac{d\,r\,j_l}{d\,r}\,\frac{d\,I(r)}{d\,r}\right]$$

Now

$$\frac{d}{d\,r}\left[r\,j_l\,\frac{d\,I}{d\,r}\right] = \frac{d\,r\,j_l}{d\,r}\,\frac{d\,I}{d\,r} + r\,j_l\,\frac{d^2\,I}{d\,r^2}$$

so that

$$a_E(l\,0) = 2\,\frac{k}{c}\,\sqrt{\frac{4\pi\,(2\,l+1)}{l\,(l+1)}}\int_0^{d/2} d\,r\left\{r\,j_l\left[k^2\,I(r) + \frac{d^2\,I}{d\,r^2}\right] - \frac{d}{d\,r}\left[r\,j_l\,\frac{d\,I}{d\,r}\right]\right\} \qquad (16.118)$$

For [see § 9.4]

$$I(z) = \theta\left(\frac{d}{2} - |z|\right)\,I\,\sin\left(\frac{k\,d}{2} - k\,|z|\right) \qquad (16.119)$$

we have

$$I(r) = \theta\left(\frac{d}{2} - r\right)\,I\,\sin\left(\frac{k\,d}{2} - k\,r\right)$$

$$\frac{d\,I}{d\,r} = \delta\left(\frac{d}{2} - r\right)\,I\,\sin\left(\frac{k\,d}{2} - k\,r\right) - k\,\theta\left(\frac{d}{2} - r\right)\,I\,\cos\left(\frac{k\,d}{2} - k\,r\right)$$

$$= -k\,\theta\left(\frac{d}{2} - r\right)\,I\,\cos\left(\frac{k\,d}{2} - k\,r\right)$$

$$\frac{d^2\,I}{d\,r^2} = -k^2\,\theta\left(\frac{d}{2} - r\right)\,I\,\sin\left(\frac{k\,d}{2} - k\,r\right)$$

so that

$$k^2\,I(r) + \frac{d^2\,I}{d\,r^2} = 0$$

And, for l odd,

$$a_E(l\,0) = -2\,\frac{k}{c}\,\sqrt{\frac{4\pi\,(2\,l+1)}{l\,(l+1)}}\left(r\,j_l\,\frac{d\,I}{d\,r}\right)_0^{d/2}$$

$$= \frac{2\,k}{c}\,\sqrt{\frac{4\pi\,(2\,l+1)}{l\,(l+1)}}\;I\left[\frac{k\,d}{2}\,j_l\!\left(\frac{k\,d}{2}\right)\right]$$

$$= \frac{4\,I}{c\,d}\,\sqrt{\frac{4\pi\,(2\,l+1)}{l\,(l+1)}}\left[\left(\frac{k\,d}{2}\right)^2\,j_l\!\left(\frac{k\,d}{2}\right)\right] \qquad (16.120)$$

For electric multipoles, (16.74) becomes

$$\frac{dP}{d\Omega} = \frac{c}{8\pi k^2}\left|\sum_{lm}(-)^{l+1}\,a_E(l\,m)\,\boldsymbol{X}_{lm}\times\hat{\boldsymbol{r}}\right|^2$$

Now

$$\begin{aligned}\left|\boldsymbol{X}\times\hat{\boldsymbol{r}}\right|^2 &= (\boldsymbol{X}\times\hat{\boldsymbol{r}})\cdot(\boldsymbol{X}^*\times\hat{\boldsymbol{r}})\\ &= [(\boldsymbol{X}\times\hat{\boldsymbol{r}})\times\boldsymbol{X}^*]\cdot\hat{\boldsymbol{r}}\\ &= \boldsymbol{X}\cdot\boldsymbol{X}^* \qquad\qquad (\hat{\boldsymbol{r}}\cdot\boldsymbol{X}=0)\\ &= |\boldsymbol{X}|^2\end{aligned}$$

so that

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{c}{8\pi k^2}\left|\sum_{lm}(-i)^{l+1}\,a_E(l\,m)\,\boldsymbol{X}_{lm}\right|^2\\ &= \frac{c}{8\pi k^2}\left|\sum_{lm}(-i)^{l+1}\,a_E(l\,m)\,\frac{1}{\sqrt{l(l+1)}}\,\boldsymbol{L}\,Y_{lm}\right|^2\end{aligned}$$

For the antenna with only $l = 1, 3$, we have

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{c}{8\pi k^2}\left|-a_E(10)\,\frac{1}{\sqrt{2}}\,\boldsymbol{L}\,Y_{10} + a_E(30)\,\frac{1}{\sqrt{12}}\,\boldsymbol{L}\,Y_{30}\right|^2\\ &= \frac{c}{16\pi k^2}\,|a_E(10)|^2\left|\boldsymbol{L}\,Y_{10} - \frac{a_E(30)}{a_E(10)}\,\frac{1}{\sqrt{6}}\,\boldsymbol{L}\,Y_{30}\right|^2 \qquad (16.121)\\ &= \frac{c}{16\pi k^2}\,|a_E(10)|^2\left\{|\boldsymbol{L}\,Y_{10}|^2 + \frac{1}{6}\left|\frac{a_E(30)}{a_E(10)}\right|^2 |\boldsymbol{L}\,Y_{30}|^2\right.\\ &\quad\left. - \frac{1}{\sqrt{6}}\left[\left(\frac{a_E(30)}{a_E(10)}\right)^*(\boldsymbol{L}\,Y_{10})\cdot(\boldsymbol{L}\,Y_{30})^* + \frac{a_E(30)}{a_E(10)}\,(\boldsymbol{L}\,Y_{10})^*\cdot(\boldsymbol{L}\,Y_{30})\right]\right\}\end{aligned}$$

Using (16.120), we have

$$a_E(10) = 4\,\frac{I}{c\,d}\,\sqrt{6\pi}\,\left[\left(\frac{k\,d}{2}\right)^2 j_1\!\left(\frac{k\,d}{2}\right)\right]$$

$$a_E(30) = 4\,\frac{I}{c\,d}\,\sqrt{\frac{7}{3}\pi}\,\left[\left(\frac{k\,d}{2}\right)^2 j_3\!\left(\frac{k\,d}{2}\right)\right]$$

$$\frac{a_E(30)}{a_E(10)} = \sqrt{\frac{7}{18}}\;\frac{j_3\!\left(\frac{k\,d}{2}\right)}{j_1\!\left(\frac{k\,d}{2}\right)}$$

which are all real so that

$$\frac{c}{16\pi k^2} |a_E(10)|^2 = \frac{3}{2}\frac{I^2}{c}\left(\frac{kd}{2}\right)^2 \left[j_1\left(\frac{kd}{2}\right)\right]^2$$

and

$$\frac{dP}{d\Omega} = \frac{3}{2}\frac{I^2}{c}\left(\frac{kd}{2}\right)^2 \left[j_1\left(\frac{kd}{2}\right)\right]^2 \left| \boldsymbol{L}\, Y_{10} - \frac{a_E(30)}{a_E(10)}\frac{1}{\sqrt{6}}\boldsymbol{L}\, Y_{30}\right|^2$$

$$= \frac{3}{2}\frac{I^2}{c}\left(\frac{kd}{2}\right)^2 \left[j_1\left(\frac{kd}{2}\right)\right]^2 \left\{ |\boldsymbol{L}\, Y_{10}|^2 + \frac{1}{6}\left(\frac{a_E(30)}{a_E(10)}\right)^2 |\boldsymbol{L}\, Y_{30}|^2 \right.$$
$$\left. - \frac{1}{\sqrt{6}}\left(\frac{a_E(30)}{a_E(10)}\right)[(\boldsymbol{L}\, Y_{10})\cdot(\boldsymbol{L}\, Y_{30})^* + (\boldsymbol{L}\, Y_{10})^*\cdot(\boldsymbol{L}\, Y_{30})]\right\}$$

As shown in § 16.4, by means of $L_\pm$ & L_z, one can show that

$$|\boldsymbol{L}\, Y_{l\,m}|^2 = \frac{1}{2}(l-m)(l+m+1)|Y_{l\,m+1}|^2 + \frac{1}{2}(l+m)(l-m+1)|Y_{l\,m-1}|^2 + m^2|Y_{l\,m}|^2$$

The same technique can be used to get

$$(\boldsymbol{L}\, Y_{l\,m})\cdot(\boldsymbol{L}\, Y_{l'\,m'})^* = \frac{1}{4}(L_+ + i L_-)Y_{l\,m}[(L_+ + i L_-)Y_{l'\,m'}]^*$$
$$+ \frac{1}{4}(L_+ - i L_-)Y_{l\,m}[(L_+ - i L_-)Y_{l'\,m'}]^* + L_z Y_{l\,m}(L_z Y_{l'\,m'})^*$$
$$= \frac{1}{2}(L_+ Y_{l\,m})(L_+ Y_{l'\,m'})^* + \frac{1}{2}(L_- Y_{l\,m})(L_- Y_{l'\,m'})^* + L_z Y_{l\,m}(L_z Y_{l'\,m'})^*$$
$$= \frac{1}{2}\sqrt{(l-m)(l+m+1)(l'-m')(l'+m'+1)}\; Y_{l\,m+1}\; Y_{l'\,m'+1}^*$$
$$+ \frac{1}{2}\sqrt{(l+m)(l-m+1)(l'+m')(l'-m'+1)}\; Y_{l\,m-1}\; Y_{l'\,m'-1}^*$$
$$+ m\,m'\, Y_{l\,m}\, Y_{l'\,m'}^*$$

This means (see § 3.5 for some values of $Y_{l\,m}$)

$$|\boldsymbol{L}\, Y_{10}|^2 = |Y_{11}|^2 + |Y_{1-1}|^2$$
$$= 2|Y_{11}|^2$$
$$= 2\left| -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right|^2$$
$$= \frac{3}{4\pi}\sin^2\theta \qquad (16.122)$$

$$|\boldsymbol{L}\, Y_{30}|^2 = 2\times 6\,|Y_{31}|^2$$

$$= \quad 12\left| -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\ \sin\theta\left(5\cos^2\theta-1\right)e^{i\phi} \right|^2$$

$$= \quad \frac{63}{16\pi}\sin^2\theta\left(5\cos^2\theta-1\right)^2$$

$$(\boldsymbol{L}\,Y_{10})\cdot(\boldsymbol{L}\,Y_{30})^* = \sqrt{6}\ (Y_{11}\,Y_{31}{}^* + Y_{1-1}\,Y_{3-1}{}^*)$$

$$= \quad \sqrt{6}\ (Y_{11}\,Y_{31}{}^* + Y_{11}{}^*\,Y_{31}) \qquad\qquad Y_{l-m} = (-)^m\,Y_{l\,m}{}^*$$

$$= \quad 2\sqrt{6}\ \mathrm{Re}\,(Y_{11}\,Y_{31}{}^*)$$

$$= \quad 2\sqrt{6}\ \mathrm{Re}\left\{\left[-\sqrt{\frac{3}{8\pi}}\ \sin\theta\,e^{i\phi}\right]\left[-\frac{1}{4}\sqrt{\frac{21}{4\pi}}\ \sin\theta\left(5\cos^2\theta-1\right)e^{-i\phi}\right]\right\}$$

$$= \quad \frac{3\sqrt{21}}{8\pi}\sin^2\theta\ \left(5\cos^2\theta-1\right)$$

$$= \quad (\boldsymbol{L}\,Y_{10})^*\cdot(\boldsymbol{L}\,Y_{30})$$

Hence

$$\frac{dP}{d\Omega} = \frac{3}{2}\,\frac{I^2}{c}\left(\frac{k\,d}{2}\right)^2\left[j_1\!\left(\frac{k\,d}{2}\right)\right]^2$$

$$\times\left\{\left|\boldsymbol{L}\,Y_{10}\right|^2 + \frac{1}{6}\left(\frac{a_E(30)}{a_E(10)}\right)^2\left|\boldsymbol{L}\,Y_{30}\right|^2 - \frac{2}{\sqrt{6}}\left(\frac{a_E(30)}{a_E(10)}\right)(\boldsymbol{L}\,Y_{10})\cdot(\boldsymbol{L}\,Y_{30})^*\right\}$$

Now

$$\{\ldots\} = \frac{3}{4\pi}\sin^2\theta + \frac{1}{6}\left(\frac{a_E(30)}{a_E(10)}\right)^2\frac{63}{16\pi}\sin^2\theta\left(5\cos^2\theta-1\right)^2$$

$$-\ \frac{2}{\sqrt{6}}\left(\frac{a_E(30)}{a_E(10)}\right)\frac{3\sqrt{21}}{8\pi}\sin^2\theta\ \left(5\cos^2\theta-1\right)$$

$$= \quad \frac{3}{4\pi}\sin^2\theta\left[1-\left(\frac{a_E(30)}{a_E(10)}\right)\sqrt{\frac{7}{2}}\ \left(5\cos^2\theta-1\right)+\left(\frac{a_E(30)}{a_E(10)}\right)^2\frac{7}{8}\ \left(5\cos^2\theta-1\right)^2\right]$$

$$= \quad \frac{3}{4\pi}\sin^2\theta\left[1-\left(\frac{a_E(30)}{a_E(10)}\right)\sqrt{\frac{7}{8}}\ \left(5\cos^2\theta-1\right)\right]^2$$

so that

$$\frac{dP}{d\Omega} = \frac{3}{2}\,\frac{I^2}{c}\left(\frac{k\,d}{2}\right)^2\left[j_1\!\left(\frac{k\,d}{2}\right)\right]^2\left(\frac{3}{4\pi}\sin^2\theta\right)\left[1-\left(\frac{a_E(30)}{a_E(10)}\right)\sqrt{\frac{7}{8}}\ \left(5\cos^2\theta-1\right)\right]^2$$

$$= \lambda \cdot 12 \frac{I^2}{c\,\pi^2} \left(\frac{3}{8\,\pi} \sin^2\theta \right) \left[1 - \left(\frac{a_E(30)}{a_E(10)} \right) \sqrt{\frac{7}{8}} \left(5\cos^2\theta - 1 \right) \right]^2 \qquad (16.123)$$

where

$$\lambda = \frac{\pi^2}{4} \left(\frac{k\,d}{2} \right)^2 \left[j_1\left(\frac{k\,d}{2} \right) \right]^2$$

and

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_3(x) = \left(\frac{15}{x^4} - \frac{6}{x^2} \right) \sin x - \left(\frac{15}{x^3} - \frac{1}{x} \right) \cos x$$

$$\frac{a_E(30)}{a_E(10)} = \sqrt{\frac{7}{18}} \frac{j_3\left(\frac{k\,d}{2} \right)}{j_1\left(\frac{k\,d}{2} \right)}$$

For the half- wave antenna, $k\,d = \pi$,

$$\lambda = \frac{\pi^2}{4} \left(\frac{\pi}{2} \right)^2 \left(\frac{1}{\left(\frac{\pi}{2} \right)^2} \right)^2 = 1$$

$$j_1\left(\frac{k\,d}{2} \right) = \frac{4}{\pi^2}$$

$$j_3\left(\frac{k\,d}{2} \right) = \frac{15 \cdot 16}{\pi^4} - \frac{6 \cdot 4}{\pi^2}$$

$$\frac{a_E(30)}{a_E(10)} = \sqrt{\frac{7}{18}} \left[\frac{60}{\pi^2} - 6 \right] = \sqrt{14} \left(\frac{10}{\pi^2} - 1 \right) = 4.95 \times 10^{-2}$$

$$\frac{d\,P}{d\,\Omega} = 12 \frac{I^2}{c\,\pi^2} \left(\frac{3}{8\,\pi} \sin^2\theta \right) \left[1 - \frac{7}{2} \left(\frac{10}{\pi^2} - 1 \right) \left(5\cos^2\theta - 1 \right) \right]^2$$

At $\theta = \frac{\pi}{2}$,

$$\frac{d\,P}{d\,\Omega} = 12 \frac{I^2}{c\,\pi^2} \left(\frac{3}{8\,\pi} \right) \left[1 + \frac{7}{2} \left(\frac{10}{\pi^2} - 1 \right) (-1) \right]^2$$

$$= \frac{I^2}{2\,c\,\pi} \times \frac{9}{\pi^2} \left[1 + \frac{7}{2} \left(\frac{10}{\pi^2} - 1 \right) \right]^2$$

$$= \frac{I^2}{2\,c\,\pi} \times 0.998$$

Note: the exact solution is

$$\frac{d\,P}{d\,\Omega} = \frac{I^2}{2\,\pi\,c} \left[\frac{\cos\left(\frac{\pi}{2} \cos\theta \right)}{\sin\theta} \right]^2$$

which becomes, for $\theta = \frac{\pi}{2}$,

$$\frac{d\,P}{d\,\Omega} = \frac{I^2}{2\,c\,\pi}$$

This close match of the multipole expansion with the exact solution is shown in Fig,164(a).

For the full- wave antenna, $k\,d = 2\,\pi$,

$$\lambda = \frac{\pi^2}{4}\,(\pi)^2\left(\frac{1}{\pi}\right)^2 = \frac{\pi^2}{4}$$

The total power is

$$P = \frac{c}{8\,\pi\,k^2}\sum_{l\,m} |\,a_E(\,l\,m\,)\,|^2 \qquad (16.79)$$

$$= \frac{c}{8\,\pi\,k^2}\sum_{l\,=\,\text{odd}} |\,a_E(\,l\,0\,)\,|^2 \qquad (16.125)$$

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8. Vector Plane Wave

Expansion of scalar plane wave $e^{i\,k\,r}$.

(cf §3.6 for the expansion of $\frac{1}{|\,\boldsymbol{x}-\boldsymbol{x}'\,|}$)

■ Direct Expansion

Let

$$\boldsymbol{k} = (\,0,\;0,\;k\,)$$
$$\boldsymbol{r} = r\,(\,\sin\theta\cos\phi,\;\sin\theta\sin\phi,\;\cos\theta\,)$$

$\Longrightarrow$

$$\boldsymbol{k}\cdot\boldsymbol{r} = k\,r\cos\theta$$
$$e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}} = e^{i\,k\,r\cos\theta}$$
$$\equiv \sum_{l=0}^{\infty} c_l(\,k\,r\,)\,P_l(\,\cos\theta\,)$$

Using

$$\int_{-1}^{1} d\,x\,P_l(x)\,P_{l'}(x) = \frac{2}{2\,l\;+\;1}\,\delta_{l\,l'} \qquad (3.21)$$

we have

$$\frac{2}{2\,l\;+\;1}\,c_l(\,k\,r\,) = \int_{-1}^{1} d\,x\,P_l\,(x)\;e^{i\,k\,r\,x} \qquad [\;x = \cos\theta\;]$$

$$= \;2\,i^l\,j_l(\,k\,r\,) \qquad [\text{ see Arfken p.807 }]$$

Reminder: j_l is the Fourier transform of P_l.

Thus

$$c_l(\,k\,r\,) = (\,2\,l+1\,)\,i^l\,j_l(\,k\,r\,)$$

$$e^{i\,k\,r\cos\theta} = \sum_{l=0}^{\infty}(\,2\,l+1\,)\,i^l\,j_l(\,k\,r\,)\,P_l(\,\cos\theta\,)$$

which can be generalized to an arbitrary coordinate system as

$$e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{l=0}^{\infty}(\,2\,l+1\,)\,i^l\,j_l(\,k\,r\,)\,P_l(\,\cos\gamma\,) \qquad (16.128)$$

where γ is the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$.

Using the addition theorem of spherical harmonics

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi')$$

we have

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{lm} i^l\, j_l(kr)\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi') \qquad (16.127)$$

with

$$\boldsymbol{r} = r(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) = (r,\ \theta,\ \phi)_s$$
$$\boldsymbol{k} = k(\sin\theta'\cos\phi',\ \sin\theta'\sin\phi',\ \cos\theta') = (k,\ \theta',\ \phi')_s$$

where the subscript s is used to denote spherical coordinates.

■ Green Function Approach

The green function for the Helmholtz eq. was already found to be

$$\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} = 4\pi i \sum_{lm} j_l(kr_<)\, h_l^{(1)}(kr_>)\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi') \qquad (16.22)$$

For

$$\boldsymbol{k} \,//\, \boldsymbol{r}' \qquad \text{and} \qquad r' \to \infty$$

we have

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\gamma}$$
$$\simeq r' - r\cos\gamma$$

where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$, as well as that between $\boldsymbol{r}$ and $\boldsymbol{k}$.

Hence

$$k|\boldsymbol{r}-\boldsymbol{r}'| \simeq k(r' - r\cos\gamma)$$
$$= kr' - \boldsymbol{k}\cdot\boldsymbol{r}$$
$$r_> = r' \qquad r_< = r$$
$$j_l(kr_<) = j_l(kr)$$
$$h_l^{(1)}(kr_>) = h_l^{(1)}(kr')$$
$$\simeq (-i)^{l+1}\frac{e^{ikr'}}{kr'}$$

so that (16.22) becomes

$$\frac{e^{ikr' - i\boldsymbol{k}\cdot\boldsymbol{r}}}{r'} \simeq 4\pi i \sum_{lm} (-i)^{l+1} j_l(kr)\,\frac{e^{ikr'}}{kr'}\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi') \qquad (16.126)$$

or

$$e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{lm} (-i)^l\, j_l(kr)\, Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi')$$

Note: the last equation is exact since it's independent of r'.

Taking the complex conjugate gives

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{lm} i^l\, j_l(kr)\, Y^*_{lm}(\theta,\phi)\, Y_{lm}(\theta',\phi') \qquad (16.127)$$

Note:

$$\sum_m Y^*_{l\,m}(\theta,\phi)\,Y_{l\,m}(\theta',\phi') = \sum_m Y_{l\,m}(\theta,\phi)\,Y^*_{l\,m}(\theta',\phi')$$

Proof:

Using

$$Y^*_{l\,m}(\theta,\phi) = (-)^m\,Y_{l\,-m}(\theta,\phi)$$

we have

$$\begin{aligned}\sum_m Y^*_{l\,m}(\theta,\phi)\,Y_{l\,m}(\theta',\phi') &= \sum_m (-)^m\,Y_{l\,-m}(\theta,\phi)\,Y_{l\,m}(\theta',\phi') \\ &= \sum_m (-)^{-m}\,Y_{l\,m}(\theta,\phi)\,Y_{l\,-m}(\theta',\phi') \qquad [\,m\to -m\,] \\ &= \sum_m Y_{l\,m}(\theta,\phi)\,Y^*_{l\,m}(\theta',\phi')\end{aligned}$$

■ $e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}}$

To summarize,

$$e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi\sum_{l\,m} i^l\,j_l(k\,r)\,Y_{l\,m}(\theta,\phi)\,Y^*_{l\,m}(\theta',\phi') \qquad (16.127)$$

$$= \sum_{l=0}^{\infty}(2\,l+1)\,i^l\,j_l(k\,r)\,P_l(\cos\gamma) \qquad (16.128)$$

$$= \sum_{l=0}^{\infty}\sqrt{4\pi\,(2\,l+1)}\;i^l\,j_l(k\,r)\,Y_{l\,0}(\gamma) \qquad (16.129)$$

■ Orthogonality Relations

We give here proof to the following equations

$$\int[\,f_{l'}(r)\,\boldsymbol{X}_{l'\,m'}]^*\cdot[\,g_l(r)\,\boldsymbol{X}_{l\,m}\,]\,d\,\Omega = f_l^*\,g_l\,\delta_{l\,l'}\,\delta_{m\,m'}$$

$$\int[\,f_{l'}(r)\,\boldsymbol{X}_{l'\,m'}]^*\cdot[\,\nabla\times g_l(r)\,\boldsymbol{X}_{l\,m}\,]\,d\,\Omega = 0 \qquad (16.132)$$

$$\int[\,\nabla\times f_{l'}(r)\,\boldsymbol{X}_{l'\,m'}]^*\cdot[\,\nabla\times g_l(r)\,\boldsymbol{X}_{l\,m}\,]\,d\,\Omega$$

$$= \delta_{l\,l'}\,\delta_{m\,m'}\left\{f_l^*\,g_l + \frac{1}{k^2\,r^2}\frac{\partial}{\partial r}\left[r\,f_l^*\,\frac{\partial}{\partial r}(r\,g_l)\right]\right\}$$

Using

$$\int d\,\Omega\;\boldsymbol{X}_{l\,m}{}^*\cdot\boldsymbol{X}_{l'\,m'} = \delta_{l\,l'}\,\delta_{m\,m'} \qquad (16.44)$$

we have

$$\begin{aligned}\int[\,f_{l'}(r)\,\boldsymbol{X}_{l'\,m'}]^*\cdot[\,g_l(r)\,\boldsymbol{X}_{l\,m}\,]\,d\,\Omega &= f_{l'}{}^*\,g_l\int d\,\Omega\;\boldsymbol{X}_{l\,m}{}^*\cdot\boldsymbol{X}_{l'\,m'} \\ &= f_l^*\,g_l\,\delta_{l\,l'}\,\delta_{m\,m'}\end{aligned}$$

Using

$$\nabla = \hat{\boldsymbol{r}} \frac{\partial}{\partial r} - i \frac{1}{r} \hat{\boldsymbol{r}} \times \boldsymbol{L}$$

$$i \nabla \times \boldsymbol{L} = \boldsymbol{r} \nabla^2 - \nabla \left(1 + r \frac{\partial}{\partial r} \right)$$

we have

$$\nabla \times \boldsymbol{X}_{l m} = \alpha \nabla \times \boldsymbol{L} Y_{l m} \qquad \alpha = \frac{1}{\sqrt{l(l+1)}}$$

$$= \frac{\alpha}{i} \left(\boldsymbol{r} \nabla^2 - \nabla \right) Y_{l m}$$

$$= \frac{\alpha}{i} \left(\boldsymbol{r} \nabla^2 - \hat{\boldsymbol{r}} \frac{\partial}{\partial r} + i \frac{1}{r} \hat{\boldsymbol{r}} \times \boldsymbol{L} \right) Y_{l m}$$

$$= \frac{\alpha}{i} \boldsymbol{r} \nabla^2 Y_{l m} + \frac{1}{r} \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l m}$$

$$= -\frac{\alpha}{i} \boldsymbol{r} \frac{l(l+1)}{r^2} Y_{l m} + \frac{1}{r} \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l m}$$

$$\nabla \times [g_l(r) \boldsymbol{X}_{l m}] = \nabla g_l \times \boldsymbol{X}_{l m} + g_l \nabla \times \boldsymbol{X}_{l m}$$

$$= \frac{d g_l}{d r} \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l m} + g_l \nabla \times \boldsymbol{X}_{l m}$$

$$= \frac{d g_l}{d r} \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l m} + g_l \left[\frac{\alpha}{i} \boldsymbol{r} \nabla^2 Y_{l m} + \frac{1}{r} \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l m} \right]$$

$$= \left[\frac{d g_l}{d r} + \frac{1}{r} g_l \right] \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l m} - g_l \frac{\alpha}{i} \boldsymbol{r} \frac{l(l+1)}{r^2} Y_{l m}$$

$$= \left[\frac{1}{r} \frac{d r g_l}{d r} \right] \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l m} - g_l \frac{\alpha}{i} \hat{\boldsymbol{r}} \frac{l(l+1)}{r} Y_{l m}$$

where we've used

$$\nabla^2 Y_{l m} = \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\boldsymbol{L}^2}{r^2} \right) Y_{l m}$$

$$= -\frac{l(l+1)}{r^2} Y_{l m}$$

Hence, with

$$\boldsymbol{r} \cdot \boldsymbol{X}_{l m} = \boldsymbol{r} \cdot \boldsymbol{L} Y_{l m} = 0$$

and

$$\int d\Omega \; \boldsymbol{X}_{l m}^* \cdot \left(\hat{\boldsymbol{r}} \times \boldsymbol{X}_{l' m'} \right) = 0 \qquad (16.44)$$

we have

$$\int [f_{l'}(r) \boldsymbol{X}_{l' m'}]^* \cdot [\nabla \times g_l(r) \boldsymbol{X}_{l m}] \, d\Omega$$

$$= f_{l'}^* \int \boldsymbol{X}_{l' m'}^* \cdot \left[\frac{1}{r} \frac{d r g_l}{d r} \right] \hat{\boldsymbol{r}} \times \boldsymbol{X}_{l m} \, d\Omega$$

$$= 0 \qquad (16.132)$$

Finally

$$\int [\nabla\times f_{l'}(r)\,\boldsymbol{X}_{l'm'}]^*\cdot[\nabla\times g_l(r)\,\boldsymbol{X}_{lm}]\,d\Omega$$

$$= \int d\Omega\left[\frac{1}{r}\frac{d\,f_{l'}}{d\,r}\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'm'} - f_{l'}\frac{\alpha'}{i}\hat{\boldsymbol{r}}\frac{l'(l'+1)}{r}Y_{l'm'}\right]^*$$

$$\cdot\left[\frac{1}{r}\frac{d\,r\,g_l}{d\,r}\hat{\boldsymbol{r}}\times\boldsymbol{X}_{lm} - g_l\frac{\alpha}{i}\hat{\boldsymbol{r}}\frac{l(l+1)}{r}Y_{lm}\right] \qquad \alpha' = \frac{1}{\sqrt{l'(l'+1)}}$$

$$= \int d\Omega\left\{\left[\frac{1}{r}\frac{d\,r\,f_{l'}}{d\,r}\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'm'}\right]^*\cdot\left[\frac{1}{r}\frac{d\,r\,g_l}{d\,r}\hat{\boldsymbol{r}}\times\boldsymbol{X}_{lm}\right]\right.$$

$$\left.+\left[f_{l'}\frac{\alpha'}{i}\frac{l'(l'+1)}{r}Y_{l'm'}\right]^*\left[g_l\frac{\alpha}{i}\frac{l(l+1)}{r}Y_{lm}\right]\right\}$$

$$= \delta_{ll'}\,\delta_{mm'}\frac{1}{r^2}\left\{\frac{d\,r\,f_l^*}{d\,r}\cdot\frac{d\,r\,g_l}{d\,r}+l(l+1)\,f_l^*\,g_l\right\}$$

Now, f_l and g_l are solutions to the radial Helmholtz eq, ie.

$$\left[\frac{d^2}{d\,r^2}+\frac{2}{r}\frac{d}{d\,r}+k^2-\frac{l(l+1)}{r^2}\right]f_l = 0$$

$$= \left[\frac{1}{r^2}\frac{d}{d\,r}r^2\frac{d}{d\,r}+k^2-\frac{l(l+1)}{r^2}\right]f_l$$

$$= \left[\frac{1}{r}\frac{d^2}{d\,r^2}r+k^2-\frac{l(l+1)}{r^2}\right]f_l$$

so that

$$\frac{d}{d\,r}\left(\frac{d\,r\,g_l}{d\,r}\right) = \left[-k^2+\frac{l(l+1)}{r^2}\right]r\,g_l$$

Thus

$$\frac{d}{d\,r}\left[r\,f_l^*\frac{d\,r\,g_l}{d\,r}\right] = \frac{d\,r\,f_l^*}{d\,r}\cdot\frac{d\,r\,g_l}{d\,r}+r\,f_l^*\frac{d}{d\,r}\left(\frac{d\,r\,g_l}{d\,r}\right)$$

$$= \frac{d\,r\,f_l^*}{d\,r}\cdot\frac{d\,r\,g_l}{d\,r}+\left[-k^2r^2+l(l+1)\right]f_l^*\,g_l$$

and

$$\int [\nabla\times f_{l'}(r)\,\boldsymbol{X}_{l'm'}]^*\cdot[\nabla\times g_l(r)\,\boldsymbol{X}_{lm}]\,d\Omega$$

$$= \delta_{ll'}\,\delta_{mm'}\frac{1}{r^2}\left\{\frac{d}{d\,r}\left[r\,f_l^*\frac{d\,r\,g_l}{d\,r}\right]+k^2r^2\,f_l^*\,g_l\right\}$$

$$= \delta_{ll'}\,\delta_{mm'}\,k^2\left\{\frac{1}{k^2r^2}\frac{d}{d\,r}\left[r\,f_l^*\frac{d\,r\,g_l}{d\,r}\right]+f_l^*\,g_l\right\} \qquad (16.132)$$

■ Circularly Polarized Plane Wave

Consider a circularly polarized plane wave running along $\boldsymbol{\epsilon}_3 = \hat{\boldsymbol{z}}$

$$\boldsymbol{E}_\pm(\boldsymbol{x}) = (\boldsymbol{\epsilon}_1 \pm i\,\boldsymbol{\epsilon}_2)\,e^{ikz} \qquad (16.130)$$

$$\boldsymbol{B}_\pm(\boldsymbol{x}) = \boldsymbol{\epsilon}_3\times\boldsymbol{E}_\pm = (\boldsymbol{\epsilon}_2 \mp i\,\boldsymbol{\epsilon}_1)\,e^{ikz} = \mp i\,\boldsymbol{E}_\pm$$

Consider the multipole expansion

$$\boldsymbol{E} = \sum_{l m} \left[a_M(l m)\, g_l\, \boldsymbol{X}_{l m} - \frac{1}{i k} a_E(l m)\, \nabla \times (f_l\, \boldsymbol{X}_{l m}) \right]$$

$$\boldsymbol{B} = \sum_{l m} \left[\frac{1}{i k} a_M(l m)\, \nabla \times (g_l\, \boldsymbol{X}_{l m}) + a_E(l m)\, f_l\, \boldsymbol{X}_{l m} \right] \tag{16.46}$$

Since a plane wave is everywhere finite, only j_l is retained in f_l and g_l. Setting $a = a_M$, $b = a_E$, we have

$$\boldsymbol{E}_\pm(\boldsymbol{x}) = \sum_{l m} \left[a_\pm(l m)\, j_l\, \boldsymbol{X}_{l m} - \frac{1}{i k} b_\pm(l m)\, \nabla \times (j_l\, \boldsymbol{X}_{l m}) \right]$$

$$\boldsymbol{B}_\pm(\boldsymbol{x}) = \sum_{l m} \left[\frac{1}{i k} a_\pm(l m)\, \nabla \times (j_l\, \boldsymbol{X}_{l m}) + b_\pm(l m)\, j_l\, \boldsymbol{X}_{l m} \right] \tag{16.131}$$

Using the orthogonality relations listed in the last section, we have

$$\int d\Omega\; \boldsymbol{X}_{l m}^* \cdot \boldsymbol{E}_\pm(\boldsymbol{x}) = a_\pm(l m)\, j_l \tag{16.133}$$

and

$$\int d\Omega\; \boldsymbol{X}_{l m}^* \cdot \boldsymbol{B}_\pm(\boldsymbol{x}) = b_\pm(l m)\, j_l \tag{16.134}$$

With

$$\boldsymbol{E}_\pm(\boldsymbol{x}) = (\boldsymbol{\epsilon}_1 \pm i\, \boldsymbol{\epsilon}_2)\, e^{i k z}$$

we have

$$a_\pm(l m)\, j_l = \int d\Omega\; \boldsymbol{X}_{l m}^* \cdot (\boldsymbol{\epsilon}_1 \pm i\, \boldsymbol{\epsilon}_2)\, e^{i k z}$$

$$= \int d\Omega\; e^{i k z} \frac{1}{\sqrt{l(l+1)}} \left[L_x \pm i L_y \right] Y_{l m}^*$$

$$= \frac{1}{\sqrt{l(l+1)}} \int d\Omega\; e^{i k z}\, L_\pm Y_{l m}^*$$

$$= \frac{1}{\sqrt{l(l+1)}} \int d\Omega\; e^{i k z}\, (L_\mp Y_{l m})^* \tag{16.135}$$

$$= \sqrt{\frac{(l \pm m)(l \mp m + 1)}{l(l+1)}} \int d\Omega\; e^{i k z}\, Y_{l\, m \mp 1}^* \tag{16.136}$$

Using

$$e^{i k z} = \sum_{l=0}^{\infty} \sqrt{4 \pi (2 l + 1)}\; i^l\, j_l(k r)\, Y_{l 0}(\theta)$$

and

$$\int d\,\Omega\; Y_{l\,m\mp 1}{}^{*}\, Y_{l'0} = \;\delta_{m,\pm 1}\,\delta_{l\,l'}$$

we have

$$a_{\pm}(\,l\,m\,)\; j_l = \delta_{m,\pm 1}\sqrt{\frac{(\,l\,+1\,)\,l}{l\,(\,l\,+\,1\,)}}\;\sqrt{4\,\pi\,(\,2\,l+1\,)}\;\,i^l\; j_l$$

$$=\;\delta_{m,\pm 1}\sqrt{4\,\pi\,(\,2\,l+1\,)}\;\,i^l\; j_l$$

ie.,

$$a_{\pm}(\,l\,m\,) = \delta_{m,\pm 1}\sqrt{4\,\pi\,(\,2\,l+1\,)}\;\,i^l \qquad (16.137)$$

Since

$$\boldsymbol{B}_{\pm}(\boldsymbol{x}) = \mp\, i\,\boldsymbol{E}_{\pm}$$

we have

$$b_{\pm}(\,l\,m\,)\; j_l = \mp\, i\int d\,\Omega\;\; \boldsymbol{X}_{l\,m}{}^{*}\cdot\boldsymbol{E}_{\pm}(\boldsymbol{x})$$

$$=\;\mp\, i\, a_{\pm}(\,l\,m\,)\; j_l$$

ie.

$$b_{\pm}(\,l\,m\,) = \mp\, i\, a_{\pm}(\,l\,m\,) \qquad (16.138)$$

Putting everything together, (16.131) becomes

$$\boldsymbol{E}_{\pm}(\boldsymbol{x}) = \sum_{l=1}^{\infty}\sqrt{4\,\pi\,(\,2\,l+1\,)}\;\,i^l\left[\;j_l\,\boldsymbol{X}_{l\,,\pm 1}\pm\frac{1}{k}\,\nabla\times\left(j_l\,\boldsymbol{X}_{l\,,\pm 1}\right)\right]$$

$$\boldsymbol{B}_{\pm}(\boldsymbol{x}) = \sum_{l=1}^{\infty}\sqrt{4\,\pi\,(\,2\,l+1\,)}\;\,i^l\left[\;-\frac{i}{k}\,\nabla\times\left(j_l\,\boldsymbol{X}_{l\,,\pm 1}\right)\mp\, i\; j_l\,\boldsymbol{X}_{l\,,\pm 1}\right] \qquad (16.139)$$

Note that l begins at $l = 1$ since $m = \pm 1$ and we must have $l \geq |\,m\,|$.

9. Scattering by a Sphere

■ General Considerations

We consider an incident plane wave scattered, with possible absorption, by a spherical object of radius a.

The fields outside the sphere are

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{E}_{\text{inc}} + \boldsymbol{E}_{\text{sc}}$$

$$\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{B}_{\text{inc}} + \boldsymbol{B}_{\text{sc}} \qquad (16.140)$$

Now, as shown in the last section, the incident plane wave is equivalent to a superposition of multipole fields (spherical waves). The action of scattering can then be thought of as a shuffling of the magnitude (as represented by the superposition coefficients) of these multipole fields.

We can use eqs.16.131 or 16.139 as our scattered fields. By definition, scattered fields must go like $\frac{e^{ikr}}{kr}$ as $kr \to \infty$ [see § 9.14]. Therefore, we must replace $j_l(kr)$, which goes like $\frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr}$, with $h_l^{(1)}(kr)$, which goes like $(-i)^{l+1}\frac{e^{ikr}}{kr}$, as $kr \to \infty$. Thus, we write

$$\boldsymbol{E}_{\pm\,\mathrm{sc}} = \frac{1}{2}\sum_{l=1}^{\infty}\sqrt{4\pi(2l+1)}\; i^l\left[\alpha_\pm(l)\; h_l^{(1)}\,\boldsymbol{X}_{l,\pm1} \pm \frac{1}{k}\,\beta_\pm(l)\,\nabla\times\left(h_l^{(1)}\,\boldsymbol{X}_{l,\pm1}\right)\right]$$

$$\boldsymbol{B}_{\pm\,\mathrm{sc}} = \frac{1}{2}\sum_{l=1}^{\infty}\sqrt{4\pi(2l+1)}\; i^l\left[-\frac{i}{k}\,\alpha_\pm(l)\;\nabla\times\left(h_l^{(1)}\,\boldsymbol{X}_{l,\pm1}\right) \mp i\;\beta_\pm(l)\;h_l^{(1)}\,\boldsymbol{X}_{l,\pm1}\right] \qquad (16.141)$$

where the coefficients $\alpha_\pm(l)$ and $\beta_\pm(l)$ represent the effects of the scattering. The factor $\frac{1}{2}$ is introduced for later convenience. We also expect α & β go to zero in the absence of scattering.

The restriction to $m = \pm 1$ in (16.141) is inherited from the plane wave expansion (16.139). We expect this to be valid if the scattering potential does not disturb the angular dependence of the incident wave. Eg., if the potential is central (radial), or if the scattering object is spherical.

As usual, the time- averaged scattered power P_{sc} is given by

$$\begin{aligned}
P_{\mathrm{sc}} &= \oint d\sigma\, \boldsymbol{n}\cdot\boldsymbol{S}_{\mathrm{sc}} \\
&= \frac{1}{2}\frac{c}{4\pi}\,\mathrm{Re}\; a^2\int d\Omega\; \boldsymbol{n}\cdot(\boldsymbol{E}_{\mathrm{sc}}\times\boldsymbol{B}_{\mathrm{sc}}^{*}) \\
&= \frac{c\,a^2}{8\pi}\,\mathrm{Re}\int d\Omega\;(\boldsymbol{n}\times\boldsymbol{E}_{\mathrm{sc}})\cdot\boldsymbol{B}_{\mathrm{sc}}^{*} \\
&= -\frac{c\,a^2}{8\pi}\,\mathrm{Re}\int d\Omega\;\boldsymbol{E}_{\mathrm{sc}}\cdot(\boldsymbol{n}\times\boldsymbol{B}_{\mathrm{sc}}^{*}) \qquad (16.142)
\end{aligned}$$

where a is the radius of any sphere surrounding the scatterer and $\boldsymbol{n} = \hat{\boldsymbol{r}}$.

Similarly, the time- averaged absorbed power P_{abs} is given by [see § 9.14]

$$\begin{aligned}
P_{\mathrm{abs}} &= -\oint d\sigma\, \boldsymbol{n}\cdot\boldsymbol{S} \\
&= \frac{c\,a^2}{8\pi}\,\mathrm{Re}\int d\Omega\;\boldsymbol{E}\cdot(\boldsymbol{n}\times\boldsymbol{B}^{*}) \qquad (16.143)
\end{aligned}$$

where the (total) fields are given by (16.140).

Since the fields enter through their cross products with $\boldsymbol{n}$, only their transverse parts (perpendicular to $\boldsymbol{n}$) give finite contribution to these integrals.

With reference to (16.141), $\boldsymbol{X}_{lm}$ is transverse. From § 16.8, we have

$$\nabla\times[g_l(r)\,\boldsymbol{X}_{lm}] = \left[\frac{1}{r}\frac{d\,r\,g_l}{dr}\right]\hat{\boldsymbol{r}}\times\boldsymbol{X}_{lm} - g_l\,\frac{1}{i}\,\hat{\boldsymbol{r}}\,\frac{\sqrt{l(l+1)}}{r}\,Y_{lm} \qquad (16.144)$$

of which only the 1st term is transverse.

Hence

$$\boldsymbol{E}_{\pm\,\mathrm{sc}}^{\perp} = \frac{1}{2}\sum_{l}\sqrt{4\pi(2l+1)}\; i^l\left[\alpha_\pm(l)\; h_l^{(1)}\,\boldsymbol{X}_{l,\pm1} \pm \frac{1}{k}\,\beta_\pm(l)\left(\frac{1}{r}\frac{d\,r\,h_l^{(1)}}{dr}\right)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right]$$

$$\boldsymbol{B}_{\pm\,\mathrm{sc}}^{\perp} = \frac{1}{2}\sum_{l}\sqrt{4\pi(2l+1)}\; i^l\left[-\frac{i}{k}\,\alpha_\pm(l)\left(\frac{1}{r}\frac{d\,r\,h_l^{(1)}}{dr}\right)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1} \mp i\;\beta_\pm(l)\;h_l^{(1)}\,\boldsymbol{X}_{l,\pm1}\right]$$

$$\hat{\boldsymbol{r}}\times\boldsymbol{B}_{\pm\,\mathrm{sc}}^{\pm\,*}=\frac{1}{2}\sum_{l}\sqrt{4\pi(2l+1)}\,(-i)^{l}\left[\frac{i}{k}\,\alpha_{\pm}(l)\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\boldsymbol{X}_{l,\pm1}\mp i\,\beta_{\pm}(l)\,h_l^{(1)}\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right]^{*}$$

$$=\frac{1}{2}\sum_{l}\sqrt{4\pi(2l+1)}\,(-i)^{l}\left[-\frac{i}{k}\,\alpha_{\pm}(l)^{*}\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(2)}}{d\,r}\right)\boldsymbol{X}_{l,\pm1}{}^{*}\pm i\,\beta_{\pm}(l)^{*}\,h_l^{(2)}\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}{}^{*}\right]$$

where we've used

$$h_l^{(1)}(x)^{*}=h_l^{(2)}(x)\qquad\text{for real } x.$$

Thus

$$\int d\,\Omega\;\boldsymbol{E}_{\pm\mathrm{sc}}\cdot(\boldsymbol{n}\times\boldsymbol{B}_{\pm\mathrm{sc}}{}^{*})$$

$$=\pi\sum_{l\,l'}\sqrt{(2l+1)(2l'+1)}\;i^{l}\,(-i)^{l'}\int d\,\Omega$$

$$\times\left[\alpha_{\pm}(l)\,h_l^{(1)}\,\boldsymbol{X}_{l,\pm1}\pm\frac{1}{k}\,\beta_{\pm}(l)\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right]$$

$$\cdot\left[-\frac{i}{k}\,\alpha_{\pm}(l')^{*}\left(\frac{1}{r}\,\frac{d\,r\,h_{l'}^{(2)}}{d\,r}\right)\boldsymbol{X}_{l',\pm1}{}^{*}\pm i\,\beta_{\pm}(l')^{*}\,h_{l'}^{(2)}\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l',\pm1}{}^{*}\right]$$

Using the orthogonality of the vector harmonics

$$\int d\,\Omega\;\boldsymbol{X}_{l\,m}{}^{*}\cdot\boldsymbol{X}_{l'\,m'}=\delta_{l\,l'}\,\delta_{m\,m'}$$

$$\int d\,\Omega\,\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\right)\cdot\boldsymbol{X}_{l\,m}{}^{*}=0\qquad(16.44)$$

$$\int d\,\Omega\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l\,m}\right)^{*}\cdot\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l'\,m'}\right)=\delta_{l\,l'}\,\delta_{m\,m'}$$

we have

$$\int d\,\Omega\;\boldsymbol{E}_{\pm\mathrm{sc}}\cdot(\boldsymbol{n}\times\boldsymbol{B}_{\pm\mathrm{sc}}{}^{*})$$

$$=\pi\sum_{l}(2l+1)\left\{-\frac{i}{k}\,|\,\alpha_{\pm}(l)\,|^{2}\,h_l^{(1)}\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(2)}}{d\,r}\right)+\frac{i}{k}\,|\,\beta_{\pm}(l)\,|^{2}\,h_l^{(2)}\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right\}$$

The Wronskian $W\left(h_l^{(1)},\,h_l^{(2)}\right)$ is

$$W\left(h_l^{(1)}(x),\,h_l^{(2)}(x)\right)\equiv\begin{vmatrix}h_l^{(1)} & h_l^{(2)}\\ h_l^{(1)\,\prime} & h_l^{(2)\,\prime}\end{vmatrix}=h_l^{(1)}\,h_l^{(2)\,\prime}-h_l^{(2)}\,h_l^{(1)\,\prime}=\frac{2}{i\,x^{2}}$$

where $f\,'=\dfrac{d\,f}{d\,x}$.

For x real, we have

$$h_l^{(1)}\,h_l^{(1)*\,\prime}-h_l^{(1)*}\,h_l^{(1)\,\prime}=\frac{2}{i\,x^{2}}$$

$$=2\,i\,\mathrm{Im}\,h_l^{(1)}\,h_l^{(1)*\,\prime}$$

or

$$\mathrm{Im}\,h_l^{(1)}\,h_l^{(1)*\,\prime}=-\frac{1}{x^{2}}$$

and

$$\mathrm{Im}\left\{h_l^{(1)}(k\,r)\left(\frac{d\,h_l^{(1)}(k\,r)}{d\,r}\right)^{*}\right\}=-\frac{1}{k\,r^{2}}$$

Now

$$\text{Re}\int d\Omega\;\; \boldsymbol{E}_{\pm\text{sc}}\cdot(\boldsymbol{n}\times\boldsymbol{B}_{\pm\text{sc}}{}^{*})$$

$$= \frac{\pi}{k}\sum_l (2l+1)\,\text{Re}\left\{ i\left[-|\alpha_\pm(l)|^2 h_l^{(1)}\left(\frac{1}{r}\frac{d\,r\,h_l^{(2)}}{d\,r}\right) + |\beta_\pm(l)|^2 h_l^{(2)}\left(\frac{1}{r}\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right]\right\}$$

$$= \frac{\pi}{k}\sum_l (2l+1)\;\text{Im}\left\{ |\alpha_\pm(l)|^2 h_l^{(1)}\left(\frac{1}{r}\frac{d\,r\,h_l^{(2)}}{d\,r}\right) - |\beta_\pm(l)|^2 h_l^{(2)}\left(\frac{1}{r}\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right\}$$

$$= \frac{\pi}{k}\sum_l (2l+1)\left\{|\alpha_\pm(l)|^2 + |\beta_\pm(l)|^2\right\}\text{Im}\left\{ h_l^{(1)}\left(\frac{1}{r}\frac{d\,r\,h_l^{(2)}}{d\,r}\right)\right\}$$

Now

$$h_l^{(1)}\left(\frac{1}{r}\frac{d\,r\,h_l^{(2)}}{d\,r}\right) = h_l^{(1)}\left(\frac{1}{r}h_l^{(2)} + \frac{d\,h_l^{(2)}}{d\,r}\right)$$

$$= \frac{1}{r}\left|h_l^{(1)}\right|^2 + h_l^{(1)}\left(\frac{d\,h_l^{(1)*}}{d\,r}\right)$$

so that

$$\text{Im}\left\{h_l^{(1)}\left(\frac{1}{r}\frac{d\,r\,h_l^{(2)}}{d\,r}\right)\right\} = \text{Im}\left\{h_l^{(1)}\left(\frac{d\,h_l^{(1)*}}{d\,r}\right)\right\}$$

$$= -\frac{1}{k\,r^2}$$

and

$$\text{Re}\int d\Omega\;\; \boldsymbol{E}_{\text{sc}}\cdot(\boldsymbol{n}\times\boldsymbol{B}_{\text{sc}}{}^{*}) = -\frac{\pi}{k^2 r^2}\sum_l (2l+1)\left\{|\alpha_\pm(l)|^2 + |\beta_\pm(l)|^2\right\}$$

Hence, with $r = a$,

$$P_{\pm\,\text{sc}} = \frac{c}{8\pi}\frac{\pi}{k^2}\sum_l (2l+1)\left\{|\alpha_\pm(l)|^2 + |\beta_\pm(l)|^2\right\}$$

Since the flux of the incoming plane wave is $\frac{4\pi}{c}$, the total scattering cross section is

$$\sigma_{\pm\,\text{sc}} = \frac{P_{\pm\,\text{sc}}}{c/4\pi}$$

$$= \frac{\pi}{2k^2}\sum_l (2l+1)\left\{|\alpha_\pm(l)|^2 + |\beta_\pm(l)|^2\right\} \qquad (16.144)$$

To evaluate P_{abs}, we decompose the incident transverse plane wave according to (16.139) so that

$$\boldsymbol{E}_\pm^{\pm}(\boldsymbol{x}) = \boldsymbol{E}_{\pm\,\text{inc}} + \boldsymbol{E}_{\pm\,\text{sc}}^{\pm}$$

$$= \frac{1}{2}\sum_l \sqrt{4\pi(2l+1)}\; i^l\left\{\left[2\,j_l + \alpha_\pm(l)\;h_l^{(1)}\right]\boldsymbol{X}_{l,\pm1}\right.$$

$$\left.\pm\frac{1}{k}\left[2\,\frac{1}{r}\frac{d\,r\,j_l}{d\,r} + \beta_\pm(l)\left(\frac{1}{r}\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right]\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right\}$$

$$\boldsymbol{B}_\pm^{\pm}(\boldsymbol{x}) = \boldsymbol{B}_{\pm\,\text{inc}} + \boldsymbol{B}_{\pm\,\text{sc}}^{\pm}$$

$$= \frac{1}{2}\sum_l \sqrt{4\pi(2l+1)}\; i^l\left\{-\frac{i}{k}\left[2\,\frac{1}{r}\frac{d\,r\,j_l}{d\,r} + \alpha_\pm(l)\left(\frac{1}{r}\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right]\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right.$$

$$\left.\mp i\left[2\,j_l + \beta_\pm(l)\;h_l^{(1)}\right]\boldsymbol{X}_{l,\pm1}\right\}$$

[Note: the decomposition of the plane wave can be obtained from that of the transverse fields by setting α, $\beta \to 2$ and $h_l^{(1)} \to j_l$.]

$$\hat{\boldsymbol{r}}\times\boldsymbol{B}_{\pm}^{\pm *} = \frac{1}{2}\sum_l \sqrt{4\pi(2l+1)}\,(-i)^l \left\{-\frac{i}{k}\left[2\,\frac{1}{r}\,\frac{d\,r\,j_l}{d\,r} + \alpha_{\pm}(l)^*\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(2)}}{d\,r}\right)\right]\boldsymbol{X}_{l,\pm 1}{}^* \right.$$
$$\left. \pm i\left[2\,j_l + \beta_{\pm}(l)^*\,h_l^{(2)}\right]\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm 1}{}^*\right\}$$

where $j_l(x)$ is real for real x.

Thus

$$\int d\,\Omega\;\; \boldsymbol{E}_{\pm}\cdot(\boldsymbol{n}\times\boldsymbol{B}_{\pm}{}^*)$$
$$= \pi\sum_l (2l+1)\left\{-\frac{i}{k}\left[2\,j_l + \alpha_{\pm}(l)\,h_l^{(1)}\right]\left[2\,\frac{1}{r}\,\frac{d\,r\,j_l}{d\,r} + \alpha_{\pm}(l)^*\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(2)}}{d\,r}\right)\right]\right.$$
$$\left. + \frac{i}{k}\left[2\,\frac{1}{r}\,\frac{d\,r\,j_l}{d\,r} + \beta_{\pm}(l)\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right]\left[2\,j_l + \beta_{\pm}(l)^*\,h_l^{(2)}\right]\right\}$$
$$= -\frac{i}{k}\,\pi\sum_l (2l+1)\left\{\,|\alpha_{\pm}(l)|^2\,h_l^{(1)}\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(2)}}{d\,r}\right) - |\,\beta_{\pm}(l)\,|^2\,h_l^{(2)}\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right.$$
$$\left. + \frac{2}{r}\left[\alpha_{\pm}(l)\,h_l^{(1)}\,\frac{d\,r\,j_l}{d\,r} + \alpha_{\pm}(l)^*\,j_l\,\frac{d\,r\,h_l^{(2)}}{d\,r} - \beta_{\pm}(l)\,\frac{d\,r\,h_l^{(1)}}{d\,r}\,j_l - \beta_{\pm}(l)^*\,h_l^{(2)}\,\frac{d\,r\,j_l}{d\,r}\right]\right\}$$

$$\mathrm{Re}\int d\,\Omega\;\; \boldsymbol{E}_{\pm}\cdot(\boldsymbol{n}\times\boldsymbol{B}_{\pm}{}^*)$$
$$= \frac{\pi}{k}\sum_l (2l+1)\,\mathrm{Im}\left\{\,|\,\alpha_{\pm}(l)\,|^2\,h_l^{(1)}\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(2)}}{d\,r}\right) - |\,\beta_{\pm}(l)\,|^2\,h_l^{(2)}\left(\frac{1}{r}\,\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right.$$
$$\left. + \frac{2}{r}\left[\alpha_{\pm}(l)\,h_l^{(1)}\,\frac{d\,r\,j_l}{d\,r} + \alpha_{\pm}(l)^*\,j_l\,\frac{d\,r\,h_l^{(2)}}{d\,r} - \beta_{\pm}(l)\,\frac{d\,r\,h_l^{(1)}}{d\,r}\,j_l - \beta_{\pm}(l)^*\,h_l^{(2)}\,d\,r\,\frac{j_l}{d\,r}\right]\right\}$$
$$= \frac{\pi}{k}\sum_l (2l+1)\left\{-\left[\,|\,\alpha_{\pm}(l)\,|^2 + |\,\beta_{\pm}(l)\,|^2\,\right]\frac{1}{k\,r^2}\right.$$
$$\left. + \frac{2}{r}\,\mathrm{Im}\left[\alpha_{\pm}(l)\,h_l^{(1)}\,\frac{d\,r\,j_l}{d\,r} + \alpha_{\pm}(l)^*\,j_l\,\frac{d\,r\,h_l^{(2)}}{d\,r} - \beta_{\pm}(l)\,\frac{d\,r\,h_l^{(1)}}{d\,r}\,j_l - \beta_{\pm}(l)^*\,h_l^{(2)}\,\frac{d\,r\,j_l}{d\,r}\right]\right\}$$

where results from the P_{sc} were directly quoted here.

Now

$$\alpha_{\pm}(l)\;h_l^{(1)}\,\frac{d\,r\,j_l}{d\,r} + \alpha_{\pm}(l)^*\,j_l\,\frac{d\,r\,h_l^{(2)}}{d\,r}$$
$$= \alpha_{\pm}(l)\;h_l^{(1)}\left[\,j_l + r\,\frac{d\,j_l}{d\,r}\right] + \alpha_{\pm}(l)^*\,j_l\left[\,h_l^{(2)} + r\,\frac{d\,h_l^{(2)}}{d\,r}\,\right]$$
$$= 2\,\mathrm{Re}\left[\alpha_{\pm}(l)\;h_l^{(1)}\,j_l\,\right] + r\left[\alpha_{\pm}(l)\;h_l^{(1)}\,\frac{d\,j_l}{d\,r} + \alpha_{\pm}(l)^*\,j_l\,\frac{d\,h_l^{(2)}}{d\,r}\right]$$

so that

$$\mathrm{Im}\Big[\ \alpha_\pm\ (l)\ h_l^{(1)}\ \frac{d\ r\ j_l}{d\ r} + \alpha_\pm(l)^*\ j_l\ \frac{d\ r\ h_l^{(2)}}{d\ r}\Big]$$

$$= \ r\ \mathrm{Im}\Big[\ \alpha_\pm(l)\ h_l^{(1)}\ \frac{d\ j_l}{d\ r} + \alpha_\pm(l)^*\ j_l\ \frac{d\ h_l^{(2)}}{d\ r}\Big]$$

$$= \ \frac{r}{2\,i}\Big[\ \alpha_\pm(l)\ h_l^{(1)}\ \frac{d\ j_l}{d\ r} - \alpha_\pm(l)^*\ h_l^{(2)}\ \frac{d\ j_l}{d\ r} + \alpha_\pm(l)^*\ j_l\ \frac{d\ h_l^{(2)}}{d\ r} - \alpha_\pm(l)\ j_l\ \frac{d\ h_l^{(1)}}{d\ r}\Big]$$

$$= \ \frac{r}{2\,i}\Big\{\ \alpha_\pm(l)\ k\ W\big[\ h_l^{(1)},\ j_l\ \big] - \alpha_\pm(l)^*\ k\ W\big[\ h_l^{(2)},\ j_l\ \big]\Big\}$$

$$= \ \frac{r}{2\,i}\Big\{\ \alpha_\pm(l)\Big(-\ \frac{i}{k\,r^2}\Big) - \alpha_\pm(l)^*\ \Big(\ \frac{i}{k\,r^2}\Big)\Big\} \qquad \text{[see (16.15)]}$$

$$= \ -\frac{1}{2\,k\,r}\big[\ \alpha_\pm(l) + \alpha_\pm(l)^*\ \big]$$

$$= \ -\frac{1}{k\,r}\ \mathrm{Re}\,\alpha_\pm(l)$$

Similarly, writing

$$\beta_\pm(l)\ \frac{d\ r\ h_l^{(1)}}{d\ r}\ j_l + \beta_\pm(l)^*\ h_l^{(2)}\ \frac{d\ r\ j_l}{d\ r}$$

$$= \ \Big[\ \beta_\pm(l)\ h_l^{(1)}\ \frac{d\ r\ j_l}{d\ r} + \beta_\pm(l)^*\ \frac{d\ r\ h_l^{(2)}}{d\ r}\ j_l\ \Big]^*$$

we see that

$$\mathrm{Im}\Big[\ \beta_\pm\ (l)\ \frac{d\ r\ h_l^{(1)}}{d\ r}\ j_l + \beta_\pm(l)^*\ h_l^{(2)}\ \frac{d\ r\ j_l}{d\ r}\ \Big]$$

$$= \ -\mathrm{Im}\Big\{\Big[\ \beta_\pm(l)\ h_l^{(1)}\ \frac{d\ r\ j_l}{d\ r} + \beta_\pm(l)^*\ \frac{d\ r\ h_l^{(2)}}{d\ r}\ j_l\ \Big]^*\ \Big\}$$

$$= \ \frac{1}{k\,r}\ \mathrm{Re}\,\beta_\pm(l)$$

Hence

$$\mathrm{Re}\int d\,\Omega\ \ \boldsymbol{E}_\pm\cdot(\,\boldsymbol{n}\times\boldsymbol{B}_\pm{}^*\,)$$

$$= \ \frac{\pi}{k}\sum_l\ (\,2\,l+1\,)\Big\{-\big[\ |\,\alpha_\pm(l)\,|^2 + |\ \beta_\pm(l)\,|^2\ \big]\ \frac{1}{k\,r^2}\ -\ \frac{2}{k\,r^2}\ \mathrm{Re}\,[\alpha_\pm(l) + \beta_\pm(l)\,]\Big\}$$

$$= \ -\frac{\pi}{k^2\,r^2}\sum_l\ (\,2\,l+1\,)\big\{\,|\,\alpha_\pm(l)\,|^2 + |\,\beta_\pm(l)\,|^2 + \mathrm{Re}\,[\alpha_\pm(l) + \beta_\pm(l)\,]\,\big\}$$

Thus, with $r = a$,

$$P_{\pm\ \mathrm{abs}} = -\ \frac{c}{8\,\pi}\ \frac{\pi}{k^2}\sum_l\ (\,2\,l+1\,)\ \big\{\ |\,\alpha_\pm(l)\,|^2 + |\,\beta_\pm(l)\,|^2\ \ + 2\ \mathrm{Re}\,[\alpha_\pm(l) + \beta_\pm(l)\,]\,\big\}$$

and

$$\sigma_{\pm\ \mathrm{abs}} = -\ \frac{\pi}{2\,k^2}\sum_l\ (\,2\,l+1\,)\ \big\{\,|\,\alpha_\pm(l)\,|^2 + |\,\beta_\pm(l)\,|^2\ \ + 2\ \mathrm{Re}\,[\alpha_\pm(l) + \beta_\pm(l)\,]\,\big\}$$

so that

$$\sigma_{\pm\,t} = \sigma_{\pm\ \mathrm{sc}} + \sigma_{\pm\ \mathrm{abs}}$$

$$= \ -\ \frac{\pi}{k^2}\sum_l\ (\,2\,l+1\,)\ \mathrm{Re}\,[\alpha_\pm(l) + \beta_\pm(l)\,] \qquad (16.146)$$

Now

$$|\alpha|^2 + 2\,\mathrm{Re}\,\alpha = \alpha\,\alpha^* + \alpha + \alpha^*$$
$$= (\alpha+1)(\alpha^*+1) - 1$$
$$= |\alpha+1|^2 - 1$$

Hence

$$\sigma_{\pm\,\mathrm{abs}} = -\frac{\pi}{2k^2}\sum_l (2l+1)\left\{|\alpha_\pm(l)+1|^2 + |\beta_\pm(l)+1|^2 - 2\right\} \qquad (16.145)$$

After you work out Prob 16.11(a), you'll find

$$\frac{d\sigma_{\pm\mathrm{sc}}}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_l \sqrt{2l+1}\left[\alpha_\pm(l)\,\boldsymbol{X}_{l,\pm1} \pm i\,\beta_\pm(l)\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right]\right|^2 \qquad (16.147)$$

Actually, (16.147) can also be obtained by following the derivation of $\sigma_{\pm\mathrm{sc}}$ without doing the Ω integral. This will be left as an exercise for those interested.

■ Sphere with Z_s

We now apply the foregoing results to a sphere specified by a surface impedance Z_s defined by [see § 8.1]

$$\boldsymbol{E}_{\mathrm{tan}} = Z_s\,\boldsymbol{K}_{\mathrm{eff}}$$
$$= Z_s\,\frac{c}{4\pi}\,\boldsymbol{n}\times\boldsymbol{H}_{\mathrm{tan}} \qquad (16.148)$$

where $\boldsymbol{E}_{\mathrm{tan}}$ and $\boldsymbol{H}_{\mathrm{tan}}$ are the tangential electric and magnetic fields, $\boldsymbol{K}_{\mathrm{eff}}$ the effective surface current, and $\boldsymbol{n}$ the normal to the surface.

For a good conductor [see footnote on p.339, Jackson]

$$Z_s = \frac{1-i}{\sigma\,\delta}$$

where σ is the conductivity, and δ the skin depth given by

$$\delta = \frac{c}{\sqrt{4\pi}}\sqrt{\frac{2}{\mu_c\,\omega\,\sigma}} \qquad (8.8)$$

In the last section, $\boldsymbol{E}_{\pm\,\mathrm{tan}}$ was denoted as $\boldsymbol{E}_\pm^+$.

$$\boldsymbol{E}_{\pm\,\mathrm{tan}} = \boldsymbol{E}_\pm^+$$
$$= \frac{1}{2}\sum_{l=1}^{\infty}\sqrt{4\pi(2l+1)}\;i^l\left\{\left[2\,j_l + \alpha_\pm(l)\,h_l^{(1)}\right]\boldsymbol{X}_{l,\pm1} \pm \frac{1}{k}\left[2\,\frac{1}{r}\frac{d\,r\,j_l}{dr} + \beta_\pm(l)\left(\frac{1}{r}\frac{d\,r\,h_l^{(1)}}{dr}\right)\right]\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right\}$$
$$= \sum_l \sqrt{4\pi(2l+1)}\;i^l\left\{\left[j_l + \frac{1}{2}\alpha_\pm(l)\,h_l^{(1)}\right]\boldsymbol{X}_{l,\pm1} \pm \frac{1}{kr}\frac{d}{dkr}\left[kr\left(j_l + \frac{1}{2}\beta_\pm(l)\,h_l^{(1)}\right)\right]\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right\}$$

For a nonmagnetic sphere, $\mu = 1$ so that

$$\boldsymbol{n}\times\boldsymbol{H}_{\tan} = \boldsymbol{n}\times\boldsymbol{B}$$

$$= \hat{\boldsymbol{r}}\times\boldsymbol{B}_{\pm}^{+}$$

$$= \frac{1}{2}\sum_{l}\sqrt{4\pi(2l+1)}\; i^{l}\left\{\frac{i}{k}\left[2\frac{1}{r}\frac{d\,r\,j_l}{d\,r} + \alpha_{\pm}(l)\left(\frac{1}{r}\frac{d\,r\,h_l^{(1)}}{d\,r}\right)\right]\boldsymbol{X}_{l,\pm1}\right.$$

$$\left.\mp i\left[2\,j_l + \beta_{\pm}(l)\,h_l^{(1)}\right]\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right\}$$

$$= \sum_{l}\sqrt{4\pi(2l+1)}\; i^{l}\left\{\frac{i}{k\,r}\frac{d}{d\,k\,r}\left[k\,r\left(j_l + \frac{1}{2}\alpha_{\pm}(l)\;h_l^{(1)}\right)\right]\boldsymbol{X}_{l,\pm1}\right.$$

$$\left.\mp i\left[j_l + \frac{1}{2}\beta_{\pm}(l)\,h_l^{(1)}\right]\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right\}$$

Since $\boldsymbol{X}_{l,\pm1}$ and $\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}$ are independent, Eq.(16.148) then implies

$$j_l + \frac{1}{2}\alpha_{\pm}(l)\;h_l^{(1)} = Z_s\,\frac{c}{4\pi}\,\frac{i}{k\,r}\,\frac{d}{d\,k\,r}\left[k\,r\left(j_l + \frac{1}{2}\alpha_{\pm}(l)\;h_l^{(1)}\right)\right]$$

$$\frac{1}{k\,r}\,\frac{d}{d\,k\,r}\left[k\,r\left(j_l + \frac{1}{2}\beta_{\pm}(l)\,h_l^{(1)}\right)\right] = -Z_s\,\frac{c}{4\pi}\,i\left[j_l + \frac{1}{2}\beta_{\pm}(l)\,h_l^{(1)}\right] \tag{16.149}$$

The 2nd eq. can be rewritten as

$$j_l + \frac{1}{2}\beta_{\pm}(l)\,h_l^{(1)} = \frac{4\pi}{c\,Z_s}\,\frac{i}{k\,r}\,\frac{d}{d\,k\,r}\left[k\,r\left(j_l + \frac{1}{2}\beta_{\pm}(l)\,h_l^{(1)}\right)\right]$$

which has the same form as the eq for $\alpha_{\pm}(l)$ except that $\frac{c\,Z_s}{4\pi}$ is replaced by its reciprocal $\frac{4\pi}{c\,Z_s}$.

Setting $x = k\,r$, we have

$$j_l + \frac{1}{2}\alpha_{\pm}(l)\;h_l^{(1)} = \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left[x\left(j_l + \frac{1}{2}\alpha_{\pm}(l)\;h_l^{(1)}\right)\right]$$

so that

$$\frac{1}{2}\alpha_{\pm}(l)\left[h_l^{(1)} - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right)\right] = \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}(x\,j_l) - j_l$$

and

$$\alpha_{\pm}(l) = -2\,\frac{j_l - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}(x\,j_l)}{h_l^{(1)} - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right)}$$

Using

$$j_l = \frac{1}{2}\left(h_l^{(1)} + h_l^{(2)}\right)$$

we have

$$\alpha_{\pm}(l)\left[h_l^{(1)} - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right)\right] = \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left[x\left(h_l^{(1)} + h_l^{(2)}\right)\right] - h_l^{(1)} - h_l^{(2)}$$

or

$$\alpha_{\pm}(l) = -1 + \frac{-h_l^{(2)} + \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(2)}\right)}{h_l^{(1)} - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right)}$$

$$\alpha_{\pm}(l) + 1 = -\frac{h_l^{(2)} - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(2)}\right)}{h_l^{(1)} - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right)} \tag{16.150}$$

With the substitution $\frac{c Z_s}{4\pi} \to \frac{4\pi}{c Z_s}$, we have

$$\beta_\pm(l) + 1 = -\frac{h_l^{(2)} - \frac{4\pi}{c Z_s} \frac{i}{x} \frac{d}{dx}\left(x h_l^{(2)}\right)}{h_l^{(1)} - \frac{4\pi}{c Z_s} \frac{i}{x} \frac{d}{dx}\left(x h_l^{(1)}\right)}$$

Note that $\alpha_+(l) = \alpha_-(l)$ and similarly for β.

▪ $Z_s = 0, \infty$, or purely imaginary

For $Z_s = 0$,

$$\alpha_\pm(l) + 1 = -\frac{h_l^{(2)}}{h_l^{(1)}} \qquad \beta_\pm(l) + 1 = -\frac{\frac{d}{dx}\left(x h_l^{(2)}\right)}{\frac{d}{dx}\left(x h_l^{(1)}\right)}$$

For $Z_s \to \infty$,

$$\alpha_\pm(l) + 1 = -\frac{\frac{d}{dx}\left(x h_l^{(2)}\right)}{\frac{d}{dx}\left(x h_l^{(1)}\right)} \qquad \beta_\pm(l) + 1 = -\frac{h_l^{(2)}}{h_l^{(1)}}$$

For Z_s purely imaginary, let

$$Z_s = i\zeta \qquad \zeta = \text{real}$$

$$\alpha_\pm(l) + 1 = -\frac{h_l^{(2)} - \frac{c\zeta}{4\pi} \frac{1}{x} \frac{d}{dx}\left(x h_l^{(2)}\right)}{h_l^{(1)} - \frac{c\zeta}{4\pi} \frac{1}{x} \frac{d}{dx}\left(x h_l^{(1)}\right)}$$

Since $h_l^{(1)} = h_l^{(2)*}$ for real argument, all these cases have the form

$$\alpha_\pm(l) + 1 = \frac{A_l}{A_l^*} = e^{2 i \delta_l} \qquad \text{where } A_l = |A_l|\, e^{i\delta_l}$$

$$\beta_\pm(l) + 1 = \frac{B_l}{B_l^*} = e^{2 i \delta_l'} \qquad \text{where } B_l = |B_l|\, e^{i\delta_l'}$$

For example, for $Z_s = 0$, we set $A_l = i\, h_l^{(2)}$, $A_l^* = -i\, h_l^{(1)}$; $B_l = i \frac{d}{dx}\left(x h_l^{(2)}\right)$, $B_l^* = -i \frac{d}{dx}\left(x h_l^{(1)}\right)$.

Results for $Z_s \to \infty$, can be obtained from those for $Z_s = 0$ by the substitution $A_l \leftrightarrow B_l$.

In a word, for $Z_s = 0, \infty$, or purely imaginary, we can write

$$\alpha_\pm(l) = e^{2 i \delta_l} - 1 \qquad \text{and} \qquad \beta_\pm(l) = e^{2 i \delta_l'} - 1 \qquad (16.151)$$

where δ_l and δ_l' are real and called the scattering phase shifts.

Now

$$e^{2 i \delta} = \frac{A}{A^*} = \frac{a + i b}{a - i b} = \frac{(a + i b)^2}{a^2 + b^2}$$

$$e^{i\delta} = \frac{a + i b}{\sqrt{a^2 + b^2}}$$

$$\tan\delta = \frac{b}{a} = \frac{\operatorname{Im} A}{\operatorname{Re} A}$$

Using

$$h_l^{(1)} = j_l + i\,n_l \qquad\qquad h_l^{(2)} = j_l - i\,n_l$$

we have, for $Z_s = 0$,

$$\tan\delta_l = \frac{\text{Im}\left[i\,h_l^{(2)}\right]}{\text{Re}\left[i\,h_l^{(2)}\right]} = \frac{j_l}{n_l}$$

$$\tan\delta_l\,' = \frac{\text{Im}\left[i\,\frac{d}{d\,x}\left(x\,h_l^{(2)}\right)\right]}{\text{Re}\left[i\,\frac{d}{d\,x}\left(x\,h_l^{(2)}\right)\right]} = \frac{\frac{d}{d\,x}(x\,j_l)}{\frac{d}{d\,x}(x\,n_l)} \qquad (16.152)$$

where all terms are understood to be evaluated at $x = k\,a$.

■ Long Wavelength Limit ($k\,a \ll 1$)

Consider

$$\alpha_\pm(l) = -2\,\frac{j_l - \frac{c\,Z_s}{4\,\pi}\,\frac{i}{x}\,\frac{d}{d\,x}(x\,j_l)}{h_l^{(1)} - \frac{c\,Z_s}{4\,\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right)}$$

In the limit $x = k\,a \ll 1$,

$$j_l(x) \;\to\; \frac{x^l}{(2\,l+1)!!}\left(1 - \frac{x^2}{2\,(2\,l+3)} + \ldots\right)$$

$$n_l(x) \;\to\; -\frac{(2\,l-1)!!}{x^{l+1}}\left(1 - \frac{x^2}{2\,(1-2\,l)} + \ldots\right) \qquad (16.12)$$

so that

$$\frac{i}{x}\,\frac{d}{d\,x}(x\,j_l) \;\to\; \frac{i}{x}\,\frac{d}{d\,x}\,\frac{x^{l+1}}{(2\,l+1)!!} = i(l+1)\,\frac{x^{l-1}}{(2\,l+1)!!}$$

and

$$h_l^{(1)}(x) \to i\,n_l \to -i\,\frac{(2\,l-1)!!}{x^{l+1}}$$

$$\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right) \;\to\; \frac{1}{x}\,\frac{d}{d\,x}\,\frac{(2\,l-1)!!}{x^l} = -\frac{l\,(2\,l-1)!!}{x^{l+2}}$$

Hence

$$\alpha_\pm(l) \to -2\,\frac{\frac{x^l}{(2\,l+1)!!} - \frac{c\,Z_s}{4\,\pi}\,i(l+1)\,\frac{x^{l-1}}{(2\,l+1)!!}}{-i\,\frac{(2\,l-1)!!}{x^{l+1}} + \frac{c\,Z_s}{4\,\pi}\,\frac{l\,(2\,l-1)!!}{x^{l+2}}}$$

$$= -2\,\frac{\frac{x^{l-1}}{(2\,l+1)!!}\left[x - \frac{c\,Z_s}{4\,\pi}\,i\,(l+1)\right]}{-i\,\frac{(2\,l-1)!!}{x^{l+2}}\left[x + \frac{c\,Z_s}{4\,\pi}\,i\,l\right]}$$

$$= -2\,i\,\frac{x^{2\,l+1}}{(2\,l+1)\,[(2\,l-1)!!]^2}\left(\frac{x - \frac{c\,Z_s}{4\,\pi}\,i\,(l+1)}{x + \frac{c\,Z_s}{4\,\pi}\,i\,l}\right) \qquad (16.153)$$

$\beta_\pm(l)$ is obtained by setting $\frac{c\,Z_s}{4\,\pi} \;\to\; \frac{4\,\pi}{c\,Z_s}$,

$$\beta_\pm(l) \to -2\,i\,\frac{x^{2\,l+1}}{(2\,l+1)\,[(2\,l-1)!!]^2}\left(\frac{x - \frac{4\,\pi}{c\,Z_s}\,i\,(l+1)}{x + \frac{4\,\pi}{c\,Z_s}\,i\,l}\right)$$

Short Wavelength Limit ($k\,a \gg 1$)

Consider

$$\alpha_{\pm}(l) + 1 = -\frac{h_l^{(2)} - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(2)}\right)}{h_l^{(1)} - \frac{c\,Z_s}{4\pi}\,\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right)}$$

In the limit $x = k\,a \gg 1$,

$$h_l{}^{(1)}(x) \;\to\; (-i)^{l+1}\,\frac{e^{i\,x}}{x} \qquad (16.13)$$

$$h_l{}^{(2)}(x) \;\to\; i^{l+1}\,\frac{e^{-i\,x}}{x}$$

so that

$$\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(1)}\right) \;\to\; \frac{1}{x}\,(-i)^l\,\frac{d}{d\,x}\,e^{i\,x}$$

$$= \frac{1}{x}\,(-i)^{l-1}\,e^{i\,x}$$

$$\frac{i}{x}\,\frac{d}{d\,x}\left(x\,h_l^{(2)}\right) \;\to\; \frac{1}{x}\,i^{l+2}\,\frac{d}{d\,x}\,e^{-i\,x}$$

$$= \frac{1}{x}\,i^{l+1}\,e^{-i\,x}$$

Hence

$$\alpha_{\pm}(l) \to -1 - \frac{i^{l+1}\,\frac{e^{-i\,x}}{x} - \frac{c\,Z_s}{4\pi}\,\frac{1}{x}\,i^{l+1}\,e^{-i\,x}}{(-i)^{l+1}\,\frac{e^{i\,x}}{x} - \frac{c\,Z_s}{4\pi}\,\frac{1}{x}\,(-i)^{l-1}\,e^{i\,x}}$$

$$= -1 - \frac{i^{l+1}\,\frac{e^{-i\,x}}{x}}{(-i)^{l+1}\,\frac{e^{i\,x}}{x}}\left(\frac{1 - \frac{c\,Z_s}{4\pi}}{1 + \frac{c\,Z_s}{4\pi}}\right)$$

$$= -1 - i^{2\,(l+1)}\,e^{-2\,i\,x}\left(\frac{1 - \frac{c\,Z_s}{4\pi}}{1 + \frac{c\,Z_s}{4\pi}}\right)$$

$$= -1 - e^{-2\,i\,x + i\,(l+1)\,\pi}\left(\frac{1 - \frac{c\,Z_s}{4\pi}}{1 + \frac{c\,Z_s}{4\pi}}\right) \qquad (\,i = e^{i\,\pi/2}\,)$$

$$= -1 + e^{2\,i\left[-x + (l+1)\,\frac{\pi}{2}\right]}\left(\frac{-1 + \frac{c\,Z_s}{4\pi}}{1 + \frac{c\,Z_s}{4\pi}}\right) \qquad (16.154)$$

- **Perfectly Conducting Sphere ($Z_s = 0$),**
  Long Wavelength Limit ($k\,a \ll 1$)

With $Z_s = \dfrac{1 - i}{\sigma\,\delta}$, we see that a perfect conductor ($\sigma \to \infty$) corresponds to $Z_s = 0$.

In the long wavlength limit, $x = k\,a \ll 1$, and

$$\alpha_\pm(l) \to -2\,i\,\frac{x^{2l+1}}{(2\,l+1)\,[(2\,l-1)!!]^2}\left(\frac{x - \frac{c\,Z_s}{4\pi}\,i\,(l+1)}{x + \frac{c\,Z_s}{4\pi}\,i\,l}\right) \qquad (16.153)$$

so that, with $Z_s = 0$,

$$\alpha_\pm(l) \to -2\,i\,\frac{x^{2l+1}}{(2\,l+1)\,[(2\,l-1)!!]^2}$$

Obviously, the most significant term is $l = 1$:

$$\alpha_\pm(1) \to -2\,i\,\frac{x^3}{3}$$

Similarly

$$\beta_\pm(l) \to -2\,i\,\frac{x^{2l+1}}{(2\,l+1)\,[(2\,l-1)!!]^2}\left(\frac{x - \frac{4\pi}{c\,Z_s}\,i\,(l+1)}{x + \frac{4\pi}{c\,Z_s}\,i\,l}\right)$$

$$\underset{Z_s=0}{=} -2\,i\,\frac{x^{2l+1}}{(2\,l+1)\,[(2\,l-1)!!]^2}\left(-\frac{l+1}{l}\right)$$

$$= -\frac{l+1}{l}\,\alpha_\pm(l)$$

so that

$$\alpha_\pm(1) = -\frac{1}{2}\,\beta_\pm(1)$$

Now

$$\frac{d\,\sigma_{\pm\mathrm{sc}}}{d\,\Omega} = \frac{\pi}{2\,k^2}\left|\sum_l \sqrt{2\,l+1}\,\left[\alpha_\pm(l)\,\boldsymbol{X}_{l,\pm1} \pm i\,\beta_\pm(l)\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{l,\pm1}\right]\right|^2 \qquad (16.147)$$

so that in the present limit

$$\frac{d\,\sigma_{\pm\mathrm{sc}}}{d\,\Omega} \to \frac{\pi}{2\,k^2}\left|\sqrt{3}\,\left[\alpha_\pm(1)\,\boldsymbol{X}_{1,\pm1} \pm i\,\beta_\pm(1)\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}\right]\right|^2$$

$$\simeq \frac{\pi}{2\,k^2}\left|\sqrt{3}\,\alpha_\pm(1)\right|^2\left|\boldsymbol{X}_{1,\pm1} \mp 2\,i\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}\right|^2$$

$$= \frac{\pi}{2\,k^2}\,\frac{4}{3}\,(k\,a)^6\left|\boldsymbol{X}_{1,\pm1} \mp 2\,i\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}\right|^2$$

$$= \frac{2\,\pi}{3}\,a^2(k\,a)^4\left|\boldsymbol{X}_{1,\pm1} \mp 2\,i\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}\right|^2 \qquad (16.255)$$

Now

$$\left|\boldsymbol{X}_{1,\pm1} \mp 2\,i\,\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}\right|^2$$

$$= \left|X_{1,\pm1}\right|^2 + 4\left|\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}\right|^2 \pm 2\,i\left[\boldsymbol{X}_{1,\pm1}\cdot\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}{}^*\right) - \boldsymbol{X}_{1,\pm1}{}^*\cdot\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}\right)\right]$$

$$= \left|X_{1,\pm1}\right|^2 + 4\left|\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}\right|^2 \mp 4\,\mathrm{Im}\left[\boldsymbol{X}_{1,\pm1}\cdot\left(\hat{\boldsymbol{r}}\times\boldsymbol{X}_{1,\pm1}{}^*\right)\right]$$

We've already shown in § 16.4 that

$$\left|X_{1,\pm1}\right|^2 = \frac{3}{16\,\pi}\left(1+\cos^2\theta\right) \qquad (16.156)$$

Using

$$\left| \hat{\boldsymbol{r}} \times \boldsymbol{X} \right|^2 = (\hat{\boldsymbol{r}} \times \boldsymbol{X}) \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{X}^*)$$
$$= \hat{\boldsymbol{r}} \cdot [\boldsymbol{X} \times (\hat{\boldsymbol{r}} \times \boldsymbol{X}^*)]$$
$$= \hat{\boldsymbol{r}} \cdot [\hat{\boldsymbol{r}} (\boldsymbol{X} \cdot \boldsymbol{X}^*) - (\hat{\boldsymbol{r}} \cdot \boldsymbol{X}) \boldsymbol{X}^*]$$
$$= |\boldsymbol{X}|^2 \qquad \text{for} \qquad \hat{\boldsymbol{r}} \cdot \boldsymbol{X} = 0$$

we have

$$\left| \hat{\boldsymbol{r}} \times \boldsymbol{X}_{1,\pm1} \right|^2 = \left| X_{1,\pm1} \right|^2 \tag{16.156}$$

Also, from § 16.2,

$$(\hat{\boldsymbol{r}} \times \boldsymbol{L} f) \cdot \boldsymbol{L}^* g = \frac{\partial f}{\partial \theta} \frac{1}{\sin\theta} \frac{\partial g}{\partial \phi} - \frac{1}{\sin\theta} \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \theta}$$

so that

$$\boldsymbol{X}_{1,\pm1} \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{X}_{1,\pm1}{}^*) = \frac{1}{2} \boldsymbol{L} Y_{1,\pm1} \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{L}^* Y_{1,\pm1}{}^*)$$
$$= \frac{1}{2} \left[\frac{\partial Y_{1,\pm1}}{\partial \theta} \frac{1}{\sin\theta} \frac{\partial Y_{1,\pm1}{}^*}{\partial \phi} - \frac{1}{\sin\theta} \frac{\partial Y_{1,\pm1}}{\partial \phi} \frac{\partial Y_{1,\pm1}{}^*}{\partial \theta} \right]^*$$
$$= \frac{1}{2} \left[\frac{\partial Y_{1,\pm1}{}^*}{\partial \theta} \frac{1}{\sin\theta} \frac{\partial Y_{1,\pm1}}{\partial \phi} - \frac{1}{\sin\theta} \frac{\partial Y_{1,\pm1}{}^*}{\partial \phi} \frac{\partial Y_{1,\pm1}}{\partial \theta} \right]$$
$$= \frac{i}{\sin\theta} \operatorname{Im}\left[\frac{\partial Y_{1,\pm1}{}^*}{\partial \theta} \frac{\partial Y_{1,\pm1}}{\partial \phi} \right]$$

Using

$$Y_{1,\pm1} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$$

we have

$$\frac{\partial Y_{1,\pm1}{}^*}{\partial \theta} = -\sqrt{\frac{3}{8\pi}} \cos\theta \, e^{\mp i\phi}$$

$$\frac{\partial Y_{1,\pm1}}{\partial \phi} = \mp i \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}$$

$$\operatorname{Im}\left[\frac{\partial Y_{1,\pm1}{}^*}{\partial \theta} \frac{\partial Y_{1,\pm1}}{\partial \phi} \right] = \pm \frac{3}{8\pi} \sin\theta \cos\theta$$

so that

$$\boldsymbol{X}_{1,\pm1} \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{X}_{1,\pm1}{}^*) = \pm i \frac{3}{8\pi} \cos\theta \tag{16.157}$$

Hence

$$\left| \boldsymbol{X}_{1,\pm1} \mp 2 i \, \hat{\boldsymbol{r}} \times \boldsymbol{X}_{1,\pm1} \right|^2 = 5 \left| X_{1,\pm1} \right|^2 \mp 4 \operatorname{Im}\left[\boldsymbol{X}_{1,\pm1} \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{X}_{1,\pm1}{}^*) \right]$$
$$= \frac{15}{16\pi} (1 + \cos^2\theta) - \frac{3}{2\pi} \cos\theta$$

and

$$\frac{d\sigma_{\pm \text{sc}}}{d\Omega} \simeq a^2 (k a)^4 \left[\frac{5}{8} (1 + \cos^2\theta) - \cos\theta \right] \tag{16.158}$$

The total scattering cross section is

$$\begin{aligned}\sigma_{\pm\text{sc}} &= \int d\,\Omega\, d\,\frac{\sigma_{\pm\text{sc}}}{d\,\Omega}\\ &\simeq 2\pi \int_{-1}^{1} d\cos\theta\, a^2 (k\,a)^4 \left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta\right]\\ &= 2\pi\, a^2 (k\,a)^4 \left[\frac{5}{8}\left(2+\frac{2}{3}\right) - 0\right]\\ &= \frac{10}{3}\pi\, a^2 (k\,a)^4 \end{aligned} \tag{16.159}$$