17
Radiation Damping, Self–Fields of a Particle, Scattering and Absorption of Radiation by a Bound System

17.1 Introductory Considerations

Types of problems treated so far:

1. ρ, $\boldsymbol{J}$ given, calculate fields.
   eg., wave guides, cavities, multipoles.
2. Fields specified, calculate motion of charges & current.
   eg., motion of charges, energy losses.
3. Combined.
   [treatment is stepwise: 1st, trajectory is determined neglecting radiation; then radiation calculated from given trajectory.]
   eg., bremsstrahlung.

From the Larmor formula (14.22)

$$E_{\text{rad}} = P_{\text{rad}}\, T \sim \frac{2\, e^2\, a^2\, T}{3\, c^3} \qquad (17.1)$$

Let E_0 be the relevant energy of the system, radiative effects begin to be important if

$$E_{\text{rad}} \sim E_0 \qquad (17.2)$$

For a particle at rest initially,

$$E_0 \sim m\,(a\,T\,)^2$$

The criterion (17.2) is

$$\frac{2\, e^2\, a^2\, T}{3\, c^3} = m\,(a\,T\,)^2$$

which defines the characteristic time

$$\tau = \frac{2}{3}\,\frac{e^2}{m\,c^3} \qquad (17.3)$$

$$\simeq 6.26 \times 10^{-24} \text{ sec} \qquad \text{for electron}$$

$$\simeq c \times \left(10^{-13} \text{ cm}\right)$$

For quasi- periodic motion with amplitude d and characteristic frequency ω_0,

$$E_0 \sim m\,\omega_0^{\,2}\, d^2 \qquad a \sim \omega_0^{\,2}\, d \qquad T \sim \frac{1}{\omega_0}$$

The criterion (17.2) is

$$\frac{2\, e^2\, a^2}{3\, c^3\, \omega_0} = m\,\omega_0^{\,2}\, d^2 = m\,\frac{a^2}{\omega_0^{\,2}}$$

or

$$\omega_0 = \frac{3}{2}\,\frac{m\,c^3}{e^2} = \frac{1}{\tau}$$

where τ is given by (17.3)

17.2 Radiative Reaction Force

Radiative Reaction Force

Neglecting radiation, the Newton equation of motion is

$$m\,\dot{\boldsymbol{v}} = \boldsymbol{F}_{\text{ext}} \tag{17.5}$$

Radiated power is (Larmor formula)

$$P(t) = \frac{2}{3}\frac{e^2}{c^3}\dot{\boldsymbol{v}}^2 \tag{17.6}$$

To include the radiative loss, let

$$m\,\dot{\boldsymbol{v}} = \boldsymbol{F}_{\text{ext}} + \boldsymbol{F}_{\text{rad}} \tag{17.7}$$

where $\boldsymbol{F}_{\text{rad}}$ is called the *radiative reaction force*.

Now, we expect

1. $\boldsymbol{F}_{\text{rad}} = 0$ if $\dot{\boldsymbol{v}} = 0$.
2. $\boldsymbol{F}_{\text{rad}} \propto e^2$ since
   a) $P \propto e^2$.
   b) Sign of e is immaterial.
3. $\boldsymbol{F}_{\text{rad}}$ involves τ, which is the only significant parameter available.

Abraham- Lorentz Equation

To determine $\boldsymbol{F}_{\text{rad}}$, we set

$$\int_{t_1}^{t_2} d\,t\,\boldsymbol{F}_{\text{rad}}\cdot\boldsymbol{v} = -\int_{t_1}^{t_2} d\,t\,P(t)$$

$$= -\int_{t_1}^{t_2} d\,t\,\frac{2}{3}\frac{e^2}{c^3}\dot{\boldsymbol{v}}^2$$

which simply assigns the radiative power to the work done against $\boldsymbol{F}_{\text{rad}}$.

Now

$$-\int_{t_1}^{t_2} d\,t\,\dot{\boldsymbol{v}}^2 = -\int_{t_1}^{t_2} d\,t\,\dot{\boldsymbol{v}}\cdot\frac{d}{d\,t}\boldsymbol{v}$$

$$= -\dot{\boldsymbol{v}}\cdot\boldsymbol{v}\,\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} d\,t\,\ddot{\boldsymbol{v}}\cdot\boldsymbol{v}$$

so that

$$\int_{t_1}^{t_2} d\,t\,\boldsymbol{F}_{\text{rad}}\cdot\boldsymbol{v} = \frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2} d\,t\,\ddot{\boldsymbol{v}}\cdot\boldsymbol{v} - \left[\frac{2}{3}\frac{e^2}{c^3}\dot{\boldsymbol{v}}\cdot\boldsymbol{v}\right]_{t_1}^{t_2}$$

For circular, or periodic motion, or whenever $\dot{\boldsymbol{v}}\cdot\boldsymbol{v}\,\big|_{t_1}^{t_2} = 0$, we have

$$\int_{t_1}^{t_2} d\,t\left[\boldsymbol{F}_{\text{rad}} - \frac{2}{3}\frac{e^2}{c^3}\ddot{\boldsymbol{v}}\right]\cdot\boldsymbol{v} = 0$$

so that

$$\begin{aligned}\boldsymbol{F}_{\text{rad}} &= \frac{2}{3}\frac{e^2}{c^3}\ddot{\boldsymbol{v}} \\ &= m\,\tau\,\ddot{\boldsymbol{v}} \qquad (17.8)\end{aligned}$$

(17.7) thus becomes

$$m\left(\dot{\boldsymbol{v}} - \tau\,\ddot{\boldsymbol{v}}\right) = \boldsymbol{F}_{\text{ext}} \qquad (17.9)$$

which is called the *Abraham- Lorentz equation of motion*.

Since (17.9) involves 2nd time derivatives of $\boldsymbol{v}$, spurious solutions exist.

For example, when

$$\boldsymbol{F}_{\text{ext}} = 0$$

we have

$$\dot{\boldsymbol{v}} - \tau\,\ddot{\boldsymbol{v}} = 0$$

or

$$\frac{d}{d\,t}\dot{\boldsymbol{v}} = \frac{1}{\tau}\dot{\boldsymbol{v}}$$

with solutions

$$\dot{\boldsymbol{v}}(t) = 0 \qquad (17.10a)$$

or

$$\dot{\boldsymbol{v}}(t) = \dot{\boldsymbol{v}}(0)\,e^{t/\tau} \qquad (17.10b)$$

Eq.(17.10b) is clearly unphysical & is called the "runaway" solution. It can be rejected on the ground that it doesn't satisfy the requirement $\dot{\boldsymbol{v}}\cdot\boldsymbol{v}\,\big|_{t_1}^{t_2} = 0$ on which (17.9) is based. Another way to deal with this problem is to replace (17.9) with an integrodifferential equation (see §17.6).

This need to keep out the unphysical solutions means that (17.9) is useful only when the radiative correction is small & can be treated perturbatively from a known, physical, state.

■ Particle in Attractive, Conservative, Central Force Field

Let the potential be $V(r)$.
In the absence of radiative reaction, we have

$$\dot{\boldsymbol{v}} = -\frac{1}{m}\left(\frac{d\,V}{d\,r}\right)\hat{\boldsymbol{r}} \qquad (17.11)$$

and both energy E and angular momentum $\boldsymbol{L}$ are conserved.

Assigning the power loss to the Larmor (radiative) power, we have

$$\begin{aligned}\frac{d\,E}{d\,t} &= -\frac{2}{3}\frac{e^2}{c^3}\dot{\boldsymbol{v}}^2 \\ &= -\frac{2}{3}\frac{e^2}{c^3}\frac{1}{m^2}\left(\frac{d\,V}{d\,r}\right)^2 \\ &= -\frac{\tau}{m}\left(\frac{d\,V}{d\,r}\right)^2 \qquad (17.12)\end{aligned}$$

For small radiative losses,

$$\frac{dE}{dt} \simeq -\frac{\tau}{m}\left\langle \left(\frac{dV}{dr}\right)^2 \right\rangle \tag{17.13}$$

where $\langle \ldots \rangle$ denotes time average over a period of the motion.

From

$$\boldsymbol{L} = m\,\boldsymbol{r}\times\boldsymbol{v}$$

we have

$$\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= m\,\boldsymbol{r}\times\dot{\boldsymbol{v}} \\
&= \boldsymbol{r}\times(\boldsymbol{F}_{\text{ext}} + \boldsymbol{F}_{\text{rad}}) \\
&= \boldsymbol{r}\times\left[-\frac{1}{m}\left(\frac{dV}{dr}\right)\hat{\boldsymbol{r}} + m\,\tau\,\ddot{\boldsymbol{v}}\right] \qquad (17.14) \\
&= m\,\tau\,\boldsymbol{r}\times\ddot{\boldsymbol{v}}
\end{aligned}$$

On the other hand

$$\frac{d^2\boldsymbol{L}}{dt^2} = m\left[\boldsymbol{r}\times\ddot{\boldsymbol{v}} + \boldsymbol{v}\times\dot{\boldsymbol{v}}\right]$$

so that

$$m\,\boldsymbol{r}\times\ddot{\boldsymbol{v}} = \frac{d^2\boldsymbol{L}}{dt^2} - m\,\boldsymbol{v}\times\dot{\boldsymbol{v}} \tag{17.15}$$

Hence

$$\frac{d\boldsymbol{L}}{dt} = \tau\left(\frac{d^2\boldsymbol{L}}{dt^2} - m\,\boldsymbol{v}\times\dot{\boldsymbol{v}}\right)$$

For small radiative correction,

$$\frac{d^2\boldsymbol{L}}{dt^2} \simeq 0$$

$$\begin{aligned}
\boldsymbol{v}\times\dot{\boldsymbol{v}} &\simeq \boldsymbol{v}\times\left[-\frac{1}{m}\left(\frac{dV}{dr}\right)\hat{\boldsymbol{r}}\right] \qquad [\text{ see } (17.11)\ ] \\
&= \frac{1}{m^2}\left(\frac{dV}{dr}\right)\frac{1}{r}\boldsymbol{L} \\
&\simeq \frac{1}{m^2}\left\langle \frac{1}{r}\frac{dV}{dr}\right\rangle \boldsymbol{L}
\end{aligned}$$

so that

$$\frac{d\boldsymbol{L}}{dt} \simeq -\frac{\tau}{m}\left\langle \frac{1}{r}\frac{dV}{dr}\right\rangle \boldsymbol{L} \tag{17.16}$$

If the motion can be specified by a characteristic frequency ω_0,

$$\begin{aligned}
\frac{\tau}{m}\left\langle \frac{1}{r}\frac{dV}{dr}\right\rangle &\sim \frac{\tau}{m}\, m\,\omega_0^{\,2} \\
&= \omega_0^{\,2}\,\tau
\end{aligned}$$

17.3 Abraham- Lorentz Self- Force

From § 6.8, we have

$$\frac{d}{dt}\left(\boldsymbol{P}_{\text{mech}} + \boldsymbol{P}_{\text{field}}\right) = \oint d\boldsymbol{\sigma}\cdot T \qquad (6.122)\ (12.120,122)$$

where T is the Maxwell stress tensor

$$T_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)\delta_{ij}\right] \qquad (6.120)$$

and

$$\boldsymbol{P}_{\text{field}} = \frac{1}{4\pi c}\int d^3x\, \boldsymbol{E}\times\boldsymbol{B} \qquad (6.117)$$

$$= \int d^3x\, \boldsymbol{g}$$

$$\frac{d}{dt}\boldsymbol{P}_{\text{mech}} = \int d^3x\left(\rho\,\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}\right) \qquad (6.114)$$

Hence, the conservation of momentum is

$$\frac{d}{dt}\left(\boldsymbol{P}_{\text{mech}} + \boldsymbol{P}_{\text{field}}\right) = 0$$

provided

$$\oint d\boldsymbol{\sigma}\cdot T = 0$$

Abraham & Lorentz proposed that $\boldsymbol{P}_{\text{mech}}$ is also electromagnetic in origin so that the conservation of momentum is simply

$$\frac{d}{dt}\boldsymbol{G} = 0$$

or

$$\int d^3x\left(\rho\,\boldsymbol{E} + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}\right) = 0 \qquad (17.17)$$

where the fields are the "total" ones:

$$\boldsymbol{E} = \boldsymbol{E}_{\text{ext}} + \boldsymbol{E}_s$$

$$\boldsymbol{B} = \boldsymbol{B}_{\text{ext}} + \boldsymbol{B}_s \qquad (17.18)$$

where subscript s denotes a self- field.

Setting

$$\boldsymbol{F}_{\text{ext}} = \int d^3x\left(\rho\,\boldsymbol{E}_{\text{ext}} + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}_{\text{ext}}\right) \qquad (17.19)$$

we can write

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}_{\text{ext}}$$

$$= -\int d^3x\left(\rho\,\boldsymbol{E}_s + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}_s\right) \qquad (17.20)$$

Consider the (non-relativistic) case where

a) particle is instantaneously at rest, ie, $\boldsymbol{J} = 0$.

b) ρ is rigid & spherically symmetric.

Thus

$$\frac{d\boldsymbol{p}}{dt} = -\int d^3x\;\rho\;\boldsymbol{E}_s \qquad (17.21)$$

Let the self- potentials be $A^\alpha = (\Phi, \boldsymbol{A})$ so that

$$\boldsymbol{E}_s = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}$$

we have

$$\frac{d\boldsymbol{p}}{dt} = \int d^3x\,\rho\left(\nabla\Phi + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}\right) \qquad (17.22)$$

On the other hand, in the Lorentz gauge,

$$A^\alpha(\boldsymbol{x}, t) = \frac{1}{c}\int d^3x'\left[\frac{J^\alpha(\boldsymbol{x}', t')}{|\boldsymbol{x} - \boldsymbol{x}'|}\right]_{\text{ret}} \qquad (17.23)$$

where ret means $t' = t - R/c$, where $R = |\boldsymbol{x} - \boldsymbol{x}'|$.

For self- fields, R is of the order of the size of the particle, which is assumed to be small. Hence, we can write

$$[f(t')]_{\text{ret}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{R}{c}\right)^n\frac{\partial^n f(t)}{\partial t^n} \qquad (17.24)$$

Thus, (17.23) becomes

$$A^\alpha(\boldsymbol{x}, t) = \frac{1}{c}\sum_{n=0}^{\infty}\frac{(-)^n}{n!\,c^n}\int d^3x'\,R^{n-1}\frac{\partial^n}{\partial t^n}J^\alpha(\boldsymbol{x}', t)$$

$$\Phi(\boldsymbol{x}, t) = \sum_{n=0}^{\infty}\frac{(-)^n}{n!\,c^n}\int d^3x'\,R^{n-1}\frac{\partial^n}{\partial t^n}\rho(\boldsymbol{x}', t)$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{c}\sum_{n=0}^{\infty}\frac{(-)^n}{n!\,c^n}\int d^3x'\,R^{n-1}\frac{\partial^n}{\partial t^n}\boldsymbol{J}(\boldsymbol{x}', t)$$

so that (17.22) becomes

$$\begin{aligned}\frac{d\boldsymbol{p}}{dt} &= \int d^3x\,\rho(\boldsymbol{x}, t)\sum_{n=0}^{\infty}\frac{(-)^n}{n!\,c^n}\int d^3x'\left\{\nabla\left[R^{n-1}\frac{\partial^n}{\delta t^n}\rho(\boldsymbol{x}', t)\right]\right.\\ &\qquad\left.+\frac{1}{c^2}\frac{\partial}{\partial t}\left[R^{n-1}\frac{\partial^n}{\partial t^n}\boldsymbol{J}(\boldsymbol{x}', t)\right]\right\}\\ &= \sum_{n=0}^{\infty}\frac{(-)^n}{n!\,c^n}\int d^3x\int d^3x'\,\rho(\boldsymbol{x}, t)\,\frac{\partial^n}{\partial t^n}\left\{\rho(\boldsymbol{x}', t)\nabla R^{n-1} + \frac{1}{c^2}R^{n-1}\frac{\partial}{\partial t}\boldsymbol{J}(\boldsymbol{x}', t)\right\}\\ &= \text{Scalar Potential Part} + \text{Vector Potential Part}\end{aligned}$$

Consider the scalar potential part.

The $n = 0$ term is

$$\begin{aligned}&\int d^3x\int d^3x'\,\rho(\boldsymbol{x}, t)\;\rho(\boldsymbol{x}', t)\,\nabla\left(\frac{1}{R}\right)\\ &= \text{Electrostatic self- force}\\ &= -\int d^3x\int d^3x'\,\rho(\boldsymbol{x}, t)\;\rho(\boldsymbol{x}', t)\,\frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\\ &= 0 \qquad\qquad \text{since it changes sign under } \boldsymbol{x} \leftrightarrow \boldsymbol{x}'.\end{aligned}$$

Note that we've used

$$\begin{aligned}\nabla R^n &= n\,R^{n-1}\,\nabla R\\ &= n\,R^{n-2}\,(\boldsymbol{x} - \boldsymbol{x}')\\ &= n\,R^{n-2}\,\boldsymbol{R}\end{aligned}$$

The $n = 1$ term is

$$-\frac{1}{c}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,\frac{\partial}{\partial t}\{\rho(\boldsymbol{x}',t)\nabla R^0\} = 0$$

since

$$\nabla R^0 = \nabla\, 1 = 0$$

Therefore, the scalar potential part starts at $n = 2$, ie.

$$\sum_{n=2}^{\infty}\frac{(-)^n}{n!\,c^n}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,\frac{\partial^n}{\partial t^n}\{\rho(\boldsymbol{x}',t)\nabla R^{n-1}\}$$
$$= \sum_{m=0}^{\infty}\frac{(-)^{m+2}}{(m+2)!\,c^{m+2}}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,\frac{\partial^{m+2}}{\partial t^{m+2}}\{\rho(\boldsymbol{x}',t)\nabla R^{m+1}\}\qquad (m = n-2)$$

Hence,

$$\frac{d\boldsymbol{p}}{dt} = \sum_{m=0}^{\infty}\frac{(-)^m}{(m+2)!\,c^{m+2}}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,\frac{\partial^{m+2}}{\partial t^{m+2}}\{\rho(\boldsymbol{x}',t)\nabla R^{m+1}\}$$
$$+\sum_{n=0}^{\infty}\frac{(-)^n}{n!\,c^n}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,\frac{\partial^n}{\partial t^n}\left\{\frac{1}{c^2}R^{n-1}\frac{\partial}{\partial t}\boldsymbol{J}(\boldsymbol{x}',t)\right\}$$
$$= \sum_{m=0}^{\infty}\frac{(-)^m}{m!\,c^{m+2}}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,\frac{\partial^{m+1}}{\partial t^{m+1}}\left\{\frac{1}{(m+1)(m+2)}\nabla R^{m+1}\frac{\partial}{\partial t}\rho(\boldsymbol{x}',t)\right\}$$
$$+\sum_{n=0}^{\infty}\frac{(-)^n}{n!\,c^{n+2}}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,R^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\boldsymbol{J}(\boldsymbol{x}',t)$$
$$= \sum_{m=0}^{\infty}\frac{(-)^m}{m!\,c^{m+2}}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)R^{m-1}\frac{\partial^{m+1}}{\partial t^{m+1}}$$
$$\times\left\{\frac{1}{(m+1)(m+2)}\frac{\nabla R^{m+1}}{R^{m-1}}\frac{\partial}{\partial t}\rho(\boldsymbol{x}',t)+\boldsymbol{J}(\boldsymbol{x}',t)\right\}\qquad(17.25)$$

Now,

$$\frac{\nabla R^{m+1}}{R^{m-1}} = (m+1)\,\boldsymbol{R}$$

and

$$\frac{\partial}{\partial t}\rho(\boldsymbol{x}',t) = -\nabla'\cdot\boldsymbol{J}(\boldsymbol{x}',t)$$

so that

$$\frac{1}{(m+1)(m+2)}\frac{\nabla R^{m+1}}{R^{m-1}}\frac{\partial}{\partial t}\rho(\boldsymbol{x}',t)+\boldsymbol{J}(\boldsymbol{x}',t)$$
$$= -\frac{1}{m+2}\boldsymbol{R}\,\nabla'\cdot\boldsymbol{J}(\boldsymbol{x}',t)+\boldsymbol{J}(\boldsymbol{x}',t)$$

and

$$\frac{d\boldsymbol{p}}{dt} = \sum_{m=0}^{\infty}\frac{(-)^m}{m!\,c^{m+2}}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)R^{m-1}\frac{\partial^{m+1}}{\partial t^{m+1}}$$
$$\times\left\{-\frac{1}{m+2}\boldsymbol{R}\,\nabla'\cdot\boldsymbol{J}(\boldsymbol{x}',t)+\boldsymbol{J}(\boldsymbol{x}',t)\right\}$$

Consider now the integral

$$\boldsymbol{I} = -\frac{1}{m+2}\int d^3x'\, R^{m-1}\,\boldsymbol{R}\;\nabla'\cdot\boldsymbol{J}(\boldsymbol{x}',t)$$

The α th cartesian component can be integrated by part to give

$$\begin{aligned}
I_\alpha &= -\frac{1}{m+2}\int d^3x'\, R^{m-1}\,R_\alpha\,\frac{\partial}{\partial x_\beta'}\,J_\beta \\
&= \frac{1}{m+2}\int d^3x'\;\; J_\beta\,\frac{\partial}{\partial x_\beta'}\left(R^{m-1}R_\alpha\right) \qquad\qquad (\text{ surface part vanishes }) \\
&= \frac{1}{m+2}\int d^3x'\;\; \boldsymbol{J}\cdot\nabla'\left(R^{m-1}R_\alpha\right) \\
&= \frac{1}{m+2}\int d^3x'\;\; \boldsymbol{J}\cdot\left[R_\alpha\,\nabla'R^{m-1} + R^{m-1}\,\nabla'R_\alpha\right] \\
&= \frac{1}{m+2}\int d^3x'\;\; \left[-(m-1)\,R_\alpha\,R^{m-3}\,\boldsymbol{J}\cdot\boldsymbol{R} - R^{m-1}\,J_\alpha\right]
\end{aligned}$$

where we've used

$$\nabla'R^{m-1} = -(m-1)\,R^{m-3}\,\boldsymbol{R}$$

and

$$\begin{aligned}
\boldsymbol{J}\cdot\nabla'R_\alpha &= J_\beta\,\frac{\partial}{\partial x_\beta'}R_\alpha \\
&= J_\beta\,\frac{\partial}{\partial x_\beta'}(x_\alpha - x_\alpha') \\
&= -J_\beta\,\delta_{\beta\alpha} \\
&= -J_\alpha
\end{aligned}$$

Hence

$$\begin{aligned}
\boldsymbol{I} &= -\frac{1}{m+2}\int d^3x'\, R^{m-1}\,\boldsymbol{R}\;\nabla'\cdot\boldsymbol{J}(\boldsymbol{x}',t) \\
&= -\frac{1}{m+2}\int d^3x'\;\;\left[(m-1)\,R^{m-3}(\boldsymbol{J}\cdot\boldsymbol{R})\,\boldsymbol{R} + R^{m-1}\,\boldsymbol{J}\right] \\
&= -\frac{1}{m+2}\int d^3x'\;\; R^{m-1}\left[(m-1)\,\frac{1}{R^2}\,(\boldsymbol{J}\cdot\boldsymbol{R})\,\boldsymbol{R} + \boldsymbol{J}\right]
\end{aligned}$$

so that, with $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{x}',t)$,

$$\begin{aligned}
\frac{d\,\boldsymbol{p}}{d\,t} &= \sum_{m=0}^{\infty}\frac{(-)^m}{m!\,c^{m+2}}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,R^{m-1}\,\frac{\partial^{m+1}}{\partial t^{m+1}} \\
&\quad\times\left\{-\frac{1}{m+2}\left[(m-1)\,\frac{1}{R^2}\,(\boldsymbol{J}\cdot\boldsymbol{R})\,\boldsymbol{R} + \boldsymbol{J}\right] + \boldsymbol{J}\right\} \\
&= \sum_{m=0}^{\infty}\frac{(-)^m}{m!\,c^{m+2}}\int d^3x\int d^3x'\,\rho(\boldsymbol{x},t)\,R^{m-1}\,\frac{\partial^{m+1}}{\partial t^{m+1}} \\
&\quad\times\left\{-\left(\frac{m-1}{m+2}\right)\frac{1}{R^2}\,(\boldsymbol{J}\cdot\boldsymbol{R})\,\boldsymbol{R} + \left(\frac{m+1}{m+2}\right)\boldsymbol{J}\right\} \qquad (17.26)
\end{aligned}$$

For a rigid ρ, every part of the charge moves with the same velocity $\boldsymbol{v}(t)$, so that

$$\boldsymbol{J}(\boldsymbol{x}', t) = \rho(\boldsymbol{x}', t)\, \boldsymbol{v}(t)$$

$$\frac{\partial}{\partial t}\rho(\boldsymbol{x}', t) = -\nabla' \cdot [\rho(\boldsymbol{x}', t)\, \boldsymbol{v}(t)]$$

$$= -\boldsymbol{v}(t)\cdot\nabla'\rho(\boldsymbol{x}', t)$$

and

$$\frac{d\boldsymbol{p}}{dt} = \sum_{m=0}^{\infty} \frac{(-)^m}{m!\, c^{m+2}} \int d^3x \int d^3x'\, \rho(\boldsymbol{x}, t)\, R^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}}$$
$$\times\left\{ -\left(\frac{m-1}{m+2}\right)\frac{1}{R^2}\rho(\boldsymbol{x}', t)\,[\boldsymbol{v}(t)\cdot\boldsymbol{R}]\,\boldsymbol{R} + \left(\frac{m+1}{m+2}\right)\rho(\boldsymbol{x}', t)\,\boldsymbol{v}(t)\right\}$$
$$= \sum_{m=0}^{\infty} \frac{(-)^m}{m!\, c^{m+2}} \int d^3x \int d^3x'\, \rho(\boldsymbol{x}, t)\, R^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}}$$
$$\times\left\{ \rho(\boldsymbol{x}', t)\left[-\left(\frac{m-1}{m+2}\right)\frac{1}{R^2}[\boldsymbol{v}(t)\cdot\boldsymbol{R}]\,\boldsymbol{R} + \left(\frac{m+1}{m+2}\right)\boldsymbol{v}(t)\right]\right\}$$

Now, if ρ is spherically symmetric, the only relevant direction is $\boldsymbol{v}$.
In other word, the integrals have only non-vanishing component in the $\boldsymbol{v}$ direction.
This means $[\boldsymbol{v}(t)\cdot\boldsymbol{R}]\,\boldsymbol{R}$ may be replaced by

$$[\boldsymbol{v}\cdot\boldsymbol{R}]\left[\hat{\boldsymbol{v}}\cdot\boldsymbol{R}\right]\hat{\boldsymbol{v}} = [\boldsymbol{v}\cdot\boldsymbol{R}]^2 \frac{1}{v^2}\boldsymbol{v}$$

so that

$$\frac{d\boldsymbol{p}}{dt} = \sum_{m=0}^{\infty} \frac{(-)^m}{m!\, c^{m+2}} \int d^3x \int d^3x'\, \rho(r, t)\, R^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}}$$
$$\times\left\{ \rho(r', t)\,\boldsymbol{v}(t)\left[-\left(\frac{m-1}{m+2}\right)\left[\frac{\boldsymbol{v}(t)\cdot\boldsymbol{R}}{vR}\right]^2 + \frac{m+1}{m+2}\right]\right\}$$
$$= \sum_{m=0}^{\infty} \frac{(-)^m}{m!\, c^{m+2}} \int d^3x \int d^3x'\, \rho(r, t)\, R^{m-1} \frac{\partial^{m+1}}{\partial t^{m+1}}$$
$$\times\left\{ \rho(r', t)\,\boldsymbol{v}(t)\left[-\left(\frac{m-1}{m+2}\right)\left[\hat{\boldsymbol{v}}(t)\cdot\hat{\boldsymbol{R}}\right]^2 + \frac{m+1}{m+2}\right]\right\}$$

Now, the angular part of the integrals involves only the term $\left[\hat{\boldsymbol{v}}(t)\cdot\hat{\boldsymbol{R}}\right]^2$.

For a given t and $\boldsymbol{x}'$, we can choose the coordinates such that

$$\hat{\boldsymbol{v}}(t) = \hat{\boldsymbol{z}}$$
$$\hat{\boldsymbol{v}}(t)\cdot\hat{\boldsymbol{R}} = \cos\theta$$

$$\int d\,\Omega\,\left[\hat{\mathbf{v}}(t)\cdot\hat{\boldsymbol{R}}\right]^2 = \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\,\phi\,\cos^2\theta$$

$$= \frac{4\pi}{3}$$

Furthermore, using

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\,\phi = 4\pi$$

we can write

$$\int d\,\Omega\,\left[\hat{\mathbf{v}}(t)\cdot\hat{\boldsymbol{R}}\right]^2 = \frac{1}{3}\int d\,\Omega$$

This manuver is obviously equivalent to replacing $\left[\hat{\mathbf{v}}(t)\cdot\hat{\boldsymbol{R}}\right]^2$ with its averaged value $\frac{1}{3}$, which makes

$$-\left(\frac{m-1}{m+2}\right)\left[\hat{\mathbf{v}}(t)\cdot\hat{\boldsymbol{R}}\right]^2 + \frac{m+1}{m+2} = -\frac{1}{3}\left(\frac{m-1}{m+2}\right) + \frac{m+1}{m+2}$$

$$= \frac{2m+4}{3(m+2)}$$

$$= \frac{2}{3}$$

Hence

$$\frac{d\,\boldsymbol{p}}{d\,t} = \frac{2}{3}\sum_{m=0}^{\infty}\frac{(-)^m}{m\,!\,c^{m+2}}\int d^3x\int d^3x'\,\rho\,(r,\,t)\,R^{m-1}\,\frac{\partial^{m+1}}{\partial\,t^{m+1}}\left[\rho(\,r\,',\,t)\,\boldsymbol{v}(t)\right] \qquad (17.27)$$

Now

$$\frac{\partial^m}{\partial\,t^m}\left[\rho(\,r\,',\,t)\,\boldsymbol{v}(t)\right] = \sum_{n=0}^{m} C_n^m\left[\frac{\partial^n}{\partial\,t^n}\rho(\,r\,',\,t)\right]\left[\frac{\partial^{m-n}}{\partial\,t^{m-n}}\boldsymbol{v}(t)\right]$$

where

$$C_n^m = \frac{m!}{n!\,(m-n)!}$$

Hence

$$\frac{d\,\boldsymbol{p}}{d\,t} = \frac{2}{3}\sum_{m=0}^{\infty}\frac{(-)^m}{m\,!\,c^{m+2}}\sum_{n=0}^{m+1} C_n^{m+1}\left[\frac{\partial^{m+1-n}}{\partial\,t^{m+1-n}}\boldsymbol{v}(t)\right]\int d^3x\,\rho\,(r,\,t)$$

$$\times\int d^3x'\,R^{m-1}\left[\frac{\partial^n}{\partial\,t^n}\rho(\,r\,',\,t)\right]$$

Keeping only the $n = 0$ term for each m, we have

$$\frac{d\,\boldsymbol{p}}{d\,t} = \frac{2}{3}\sum_{m=0}^{\infty}\frac{(-)^m}{m\,!\,c^{m+2}}\left[\frac{\partial^{m+1}}{\partial\,t^{m+1}}\boldsymbol{v}(t)\right]\int d^3x\,\rho\,(r,\,t)\int d^3x'\,R^{m-1}\,\rho(\,r\,',\,t) \qquad (17.28)$$

Consider now the first few terms in (17.28)

$$\left(\frac{d\,\boldsymbol{p}}{d\,t}\right)_0 = \frac{2}{3}\frac{1}{c^2}\,\dot{\boldsymbol{v}}(t)\int d^3x\,\rho(r,t)\int d^3x'\,R^{-1}\rho(r',t)$$

$$= \frac{2}{3}\frac{1}{c^2}\,\dot{\boldsymbol{v}}(t)\int d^3x\int d^3x'\,\frac{\rho(r,t)\rho(r',t)}{R}$$

$$\left(\frac{d\,\boldsymbol{p}}{d\,t}\right)_1 = -\frac{2}{3}\frac{1}{c^3}\,\ddot{\boldsymbol{v}}(t)\int d^3x\int d^3x'\,\rho(r,t)\rho(r',t)$$

$$= -\frac{2}{3}\frac{1}{c^3}\,\ddot{\boldsymbol{v}}(t) \qquad (17.29)$$

$$\vdots$$

$$\left(\frac{d\,\boldsymbol{p}}{d\,t}\right)_n = (-)^n\,\frac{2}{3}\frac{1}{c^{n+2}}\,\overset{(n+1)}{\boldsymbol{v}}(t)\int d^3x\int d^3x'\,R^{n-1}\rho(r,t)\rho(r',t)$$

$$= (-)^n\,\frac{2}{3}\frac{e^2}{c^{n+2}}\,\overset{(n+1)}{\boldsymbol{v}}(t)\,a^{n-1}$$

where

$$\overset{(n+1)}{\boldsymbol{v}}(t) \equiv \frac{\partial^{n+1}}{\partial\,t^{n+1}}\boldsymbol{v}(t)$$

$$a^{n-1} \equiv \frac{1}{e^2}\int d^3x\int d^3x'\,R^{n-1}\rho(r,t)\rho(r',t)$$

$\simeq$ characteristic length of particle.

Now, $a^{n-1} \to 0$ as $a \to 0$ for $n \geq 2$.
Hence, for point particles, we need consider only terms $n = 0$ and $n = 1$.

For the $n = 1$,

$$\left(\frac{d\,\boldsymbol{p}}{d\,t}\right)_1 = -\frac{2}{3}\frac{e^2}{c^3}\,\ddot{\boldsymbol{v}}(t)$$

is simply $\boldsymbol{F}_{\text{rad}}$ given by (17.8).

For the $n = 0$ term

$$a^{-1} \equiv \frac{1}{e^2}\int d^3x\int d^3x'\,R^{-1}\rho(r,t)\rho(r',t)$$

$$= 2\,\frac{1}{e^2}\,U$$

where U is the electrostatic self- energy. Hence

$$\left(\frac{d\,\boldsymbol{p}}{d\,t}\right)_0 = \frac{4}{3}\frac{U}{c^2}\,\dot{\boldsymbol{v}}(t) \qquad (17.31)$$

$$= \frac{4}{3}\,m_e\,\dot{\boldsymbol{v}}(t)$$

where

$$m_e = \frac{U}{c^2} \qquad (17.32)$$

is the electromagnetic mass of the particle.

The Abraham- Lorentz eq thus becomes

$$\frac{d\,\boldsymbol{p}}{d\,t} \simeq \left(\frac{d\,\boldsymbol{p}}{d\,t}\right)_0 + \left(\frac{d\,\boldsymbol{p}}{d\,t}\right)_1$$

$$= \frac{4}{3} m_e\, \dot{\boldsymbol{v}}(t) - \frac{2}{3}\frac{e^2}{c^3}\, \ddot{\boldsymbol{v}}(t)$$

$$= \boldsymbol{F}_{\text{ext}} \qquad (17.33)$$

which is just (17.9) except for the $\frac{4}{3}$ factor.

17.4 Difficulties of A-L Model

Deficiencies of the A-L model:

1. Non-relativistic.

2. The strange factor $\frac{4}{3}$ in (17.33) which is due to the incorrect Lorentz covariance.

3. In order to ignore terms beyond $n = 1$, we assumed $a \to 0$.
   Since $U \sim \frac{e^2}{a}$, we have $m_e \sim \frac{e^2}{a\,c^2}$, which goes to ∞ as $a \to 0$.
   To keep m_e finite & agree with observed values, we need
   $$a \sim r_0 = \frac{e^2}{m\,c^2}$$
   where $r_0 \simeq 2.82 \times 10^{-13}$ cm is the classical electron radius.
   Higher terms must be included for violent motion.

4. Nonelectromagnetic forces of quantum mechancial origin are needed to hold the charge distribution together. Classical models such as A-L may be rather irrelevant.

17.5 Covariant Definitions

The symmetric electromagnetic stress tensor is

$$\Theta = \left(\begin{array}{c|c} u & c\,\boldsymbol{g} \\ \hline c\,\boldsymbol{g} & -T \end{array}\right) \qquad (12.115)$$

where

$$u = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)$$

$$\boldsymbol{g} = \frac{1}{4\pi c}\,\boldsymbol{E}\times\boldsymbol{B}$$

$$T^{i\,j} = \frac{1}{4\pi}\left[E_i\,E_j + B_i\,B_j - \frac{1}{2}\,\delta_{i\,j}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)\right] = T_{i\,j}$$

Thus

$$p^\mu = (u,\ c\,\boldsymbol{g}) = \Theta^{0\,\mu}$$

is NOT a 4-vector.

Let

$$U = \int d^3 x\, u$$

$$\boldsymbol{G} = \int d^3 x\, \boldsymbol{g}$$

It was shown that (§ 12.10(a))

$$P^\mu = (U,\ c\, \boldsymbol{G})$$

is not a 4-vector unless

$$\partial_\alpha \Theta^{\alpha\beta} = 0 \qquad \text{(source- free)}$$

■ Poincare Solution

The Poincare solution side- steps the question of the transformation properties of the electromagnetic energy and momentum.

Let the total stree tensor be

$$S^{\alpha\beta} = \Theta^{\alpha\beta} + P^{\alpha\beta}$$

where the Poincare stress $P^{\alpha\beta}$ is from non- electromagnetic origin.

Define the total 4-momentum as

$$P^\alpha = \frac{1}{c} \int d^3 x\, S^{\alpha 0} \qquad (17.35)$$

The condition for P^α to be a 4-vector is that

$$\int d^3 x^{(0)}\, S^{(0)\, i j} = 0 \qquad i,\ j = 1,\ 2,\ 3 \qquad (17.36)$$

where the superscript (0) denotes quantities in the rest frame ($\boldsymbol{P} = 0$) of the particle.

Furthermore, condition (17.36) is equivalent to

$$\partial_\alpha S^{\alpha\beta} = 0$$

■ Covariant Definitions

Starting with an arbitrary initial frame K', one defines the electromagnetic energy and momentum as

$$E_e' = \frac{1}{8\pi} \int d^3 x' \left(\boldsymbol{E}'^2 + \boldsymbol{B}'^2 \right)$$

$$\boldsymbol{P}_e' = \frac{1}{4\pi c} \int d^3 x'\, \boldsymbol{E}' \times \boldsymbol{B}' \qquad (17.37)$$

For a frame K in which K' is moving with velocity $c\, \boldsymbol{\beta}$, let

$$P_e^\alpha = \frac{1}{c} \int d\sigma_\beta\, \Theta^{\alpha\beta}$$

$$= \frac{1}{c} \int d^3 \sigma\, n_\beta\, \Theta^{\alpha\beta} \qquad (17.40)$$

where $d\sigma_\beta$ is a time-like 4-vector defined by

$$d\sigma_\beta = d^3 \sigma\, n_\beta \qquad (17.38)$$

Here, $d^3 \sigma$ is an invariant 3-dimensional "area" element on a spacelike hyperplane with normal n_β. By definition, $d^3 \sigma = d^3 x'$.

Using

$$n_\beta n^\beta = 1$$

and

$$n^{\beta'} = (1, \mathbf{0})$$

in the particle rest frame K', (17.38) becomes

$$d^3\sigma = n^\beta d\sigma_\beta$$

Also

$$n^0 = \gamma\left(n^{0'} + \boldsymbol{\beta}\cdot\boldsymbol{n}'\right) = \gamma$$

$$\boldsymbol{n}_{||} = \gamma\left(\boldsymbol{n}_{||}' + \boldsymbol{\beta}\, n_0'\right) = \gamma\,\boldsymbol{\beta}$$

$$\boldsymbol{n}_\perp = \boldsymbol{n}_\perp' = 0$$

$$\boldsymbol{n} = \boldsymbol{n}_{||} + \boldsymbol{n}_\perp$$

or

$$n^\beta = \gamma\left(1,\ \beta^i\right) = \gamma\,(1,\ \boldsymbol{\beta}) \qquad (17.39)$$

$$n_\beta = \gamma\,(1,\ \beta_i) = \gamma\,(1,\ -\boldsymbol{\beta})$$

Thus

$$P_e^0 = \frac{1}{c}\int d^3\sigma\,\gamma\left(\Theta^{00} + \beta_i\,\Theta^{0i}\right)$$

$$= \frac{1}{c}\int d^3\sigma\,\gamma\,(u - c\,\boldsymbol{\beta}\cdot\boldsymbol{g}) \qquad (17.41)$$

$$P_e^i = \frac{1}{c}\int d^3\sigma\,\gamma\left(\Theta^{i0} - \beta_j\,T^{ij}\right)$$

$$\boldsymbol{P}_e = \frac{1}{c}\int d^3\sigma\,\gamma\,(c\,\boldsymbol{g} + \boldsymbol{\beta}\cdot T) \qquad (\boldsymbol{\beta}\cdot T = \beta^j\,T^{ji} = \beta^j\,T^{ij})$$

$$= \frac{1}{c}\int d^3x\,\gamma^2\,(c\,\boldsymbol{g} + \boldsymbol{\beta}\cdot T)$$

where

$$d^3\sigma\,\gamma = d^3x'\,\gamma = d^3x\,\gamma^2$$

Obviously, different choice of K' leads to different P_e^μ. This is acceptable since only changes in P_e^μ is physically significant.

The natural choice of K' is the rest frame $K^{(0)}$ in which

$$\boldsymbol{P}_e^{(0)} = \frac{1}{4\pi c}\int d^3x\,\boldsymbol{E}^{(0)}\times\boldsymbol{B}^{(0)} = 0 \qquad (17.42)$$

The corresponding electromagnetic rest energy is, according to (17.37),

$$E_e^{(0)} = \frac{1}{8\pi}\int d^3x^{(0)}\left(\boldsymbol{E}^{(0)2} + \boldsymbol{B}^{(0)2}\right) \qquad (17.43)$$

$$\equiv m_e\,c^2$$

The $\boldsymbol{\beta}$ in (17.41) then denotes the velocity of $K^{(0)}$ in K.

■ All charges at rest in $K^{(0)}$

If all charges are at rest in some frame K', we have only electrostatic fields and

$$\boldsymbol{B}' = 0$$

Hence, (17.42) is satisfied trivially and K' can be chosen as $K^{(0)}$.

In an arbitrary frame K, we have (see 11.149)

$$\boldsymbol{E} = \gamma\, \boldsymbol{E}^{(0)} - \frac{\gamma^2}{\gamma+1}\, \boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\boldsymbol{E}^{(0)}\right)$$

$$\boldsymbol{B} = \gamma\, \boldsymbol{\beta}\times \boldsymbol{E}^{(0)}$$

so that

$$\boldsymbol{B} = \boldsymbol{\beta}\times\boldsymbol{E} \qquad (17.44)$$

The integrand in the 1st eq of (17.41) thus become

$$\begin{aligned} u - c\,\boldsymbol{\beta}\cdot\boldsymbol{g} &= \frac{1}{8\pi}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right) - \frac{1}{4\pi}\,\boldsymbol{\beta}\cdot(\boldsymbol{E}\times\boldsymbol{B}) \\ &= \frac{1}{8\pi}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right) - \frac{1}{4\pi}\,(\boldsymbol{\beta}\times\boldsymbol{E})\cdot\boldsymbol{B} \\ &= \frac{1}{8\pi}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right) - \frac{1}{4\pi}\,\boldsymbol{B}^2 \\ &= \frac{1}{8\pi}\left(\boldsymbol{E}^2-\boldsymbol{B}^2\right) \end{aligned}$$

which is Lorentz invariant.

$$\begin{aligned} P_e^0 &= \frac{1}{8\pi c}\int d^3\sigma\,\gamma\left(\boldsymbol{E}^2-\boldsymbol{B}^2\right) \\ &= \frac{1}{8\pi c}\,\gamma^2\int d^3x\left(\boldsymbol{E}^2-\boldsymbol{B}^2\right) \qquad (17.45) \end{aligned}$$

For the 2nd eq of (17.41),

$$T^{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)\right]$$

$$\begin{aligned} (\boldsymbol{\beta}\cdot T)^i = \beta^j T^{ij} &= \frac{1}{4\pi}\left[E_i(\boldsymbol{\beta}\cdot\boldsymbol{E}) + B_i(\boldsymbol{\beta}\cdot\boldsymbol{B}) - \frac{1}{2}\beta^i\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)\right] \\ &= \frac{1}{4\pi}\left[E_i(\boldsymbol{\beta}\cdot\boldsymbol{E}) - \frac{1}{2}\beta^i\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)\right] \end{aligned}$$

$$\begin{aligned} c\,\boldsymbol{g} &= \frac{1}{4\pi}\,\boldsymbol{E}\times\boldsymbol{B} \\ &= \frac{1}{4\pi}\,\boldsymbol{E}\times(\boldsymbol{\beta}\times\boldsymbol{E}) \\ &= \frac{1}{4\pi}\left[\boldsymbol{\beta}\,\boldsymbol{E}^2 - \boldsymbol{E}(\boldsymbol{\beta}\cdot\boldsymbol{E})\right] \end{aligned}$$

$$\begin{aligned} c\,\boldsymbol{g} + \boldsymbol{\beta}\cdot T &= \frac{1}{4\pi}\left[\boldsymbol{\beta}\,\boldsymbol{E}^2 - \boldsymbol{E}(\boldsymbol{\beta}\cdot\boldsymbol{E})\right] + \frac{1}{4\pi}\left[\boldsymbol{E}(\boldsymbol{\beta}\cdot\boldsymbol{E}) - \frac{1}{2}\boldsymbol{\beta}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right)\right] \\ &= \frac{1}{8\pi}\,\boldsymbol{\beta}\left(\boldsymbol{E}^2-\boldsymbol{B}^2\right) \end{aligned}$$

so that

$$\begin{aligned} \boldsymbol{P}_e &= \frac{1}{8\pi c}\,\gamma\,\boldsymbol{\beta}\int d^3\sigma\left(\boldsymbol{E}^2-\boldsymbol{B}^2\right) \\ &= \frac{1}{8\pi c}\,\gamma^2\,\boldsymbol{\beta}\int d^3x\left(\boldsymbol{E}^2-\boldsymbol{B}^2\right) \qquad (17.46) \\ &= \boldsymbol{\beta}\,P_e^0 \end{aligned}$$

Hence

$$P_e^\mu = P_e^0\,(1,\ \boldsymbol{\beta})$$
$$\equiv \gamma\, m_e\, c\,(1,\ \boldsymbol{\beta})$$

where

$$m_e = \frac{1}{\gamma\, c}\, P_e^0$$
$$= \frac{1}{8\,\pi\, c^2}\, \gamma \int d^3 x \left(\boldsymbol{E}^2 - \boldsymbol{B}^2 \right)$$
$$= \frac{1}{8\,\pi\, c^2} \int d^3 x^{(0)} \left(\boldsymbol{E}^{(0)\,2} - \boldsymbol{B}^{(0)\,2} \right)$$
$$= \frac{1}{8\,\pi\, c^2} \int d^3 x^{(0)}\, \boldsymbol{E}^{(0)\,2}$$

Writing

$$\boldsymbol{P}_e = \frac{1}{c} \int d^3 \sigma\, \gamma\, (c\, \boldsymbol{g} + \boldsymbol{\beta} \cdot T\ )$$

as

$$\boldsymbol{P}_e = \boldsymbol{P}_e^{(1)} + \boldsymbol{P}_e^{(2)}$$

where

$$\boldsymbol{P}_e^{(1)} = \gamma \int d^3 \sigma\, \boldsymbol{g} \qquad (17.47)$$
$$= \frac{1}{4\,\pi\, c}\, \gamma \int d^3 \sigma \left[\boldsymbol{\beta}\, \boldsymbol{E}^2 - \boldsymbol{E}\, (\boldsymbol{\beta} \cdot \boldsymbol{E}\,) \right] \qquad (17.48)$$

$$\boldsymbol{P}_e^{(2)} = \frac{1}{c}\, \gamma \int d^3 \sigma\, \boldsymbol{\beta} \cdot T$$
$$= \frac{1}{4\,\pi\, c}\, \gamma \int d^3 \sigma \left[\boldsymbol{E}\, (\boldsymbol{\beta} \cdot \boldsymbol{E}\,) - \frac{1}{2}\, \boldsymbol{\beta} \left(\boldsymbol{E}^2 + \boldsymbol{B}^2 \right) \right] \qquad (17.49)$$

In the relativistic limit

$$\boldsymbol{E} = \boldsymbol{E}^{(0)} + O\!\left(\beta^2\right)$$

Hence, to 1st order in β and assuming spherical symmetry for $\boldsymbol{E}^{(0)}$, we have

$$\boldsymbol{P}_e^{(1)} \simeq \frac{1}{4\,\pi\, c} \int d^3 x^{(0)} \left[\boldsymbol{\beta}\, \boldsymbol{E}^{(0)\,2} - \boldsymbol{E}^{(0)} \left(\boldsymbol{\beta} \cdot \boldsymbol{E}^{(0)} \right) \right]$$
$$= \frac{1}{4\,\pi\, c} \int d^3 x^{(0)} \left[\boldsymbol{\beta}\, \boldsymbol{E}^{(0)\,2} - \left(\boldsymbol{\beta} \cdot \boldsymbol{E}^{(0)} \right) \left(\hat{\boldsymbol{\beta}} \cdot \boldsymbol{E}^{(0)} \right) \hat{\boldsymbol{\beta}} \right]$$
$$= \frac{1}{4\,\pi\, c} \int d^3 x^{(0)} \left[\boldsymbol{\beta}\, \boldsymbol{E}^{(0)\,2} - \frac{1}{\beta^2} \left(\boldsymbol{\beta} \cdot \boldsymbol{E}^{(0)} \right)^2 \boldsymbol{\beta} \right]$$
$$= \frac{1}{4\,\pi\, c}\, \boldsymbol{\beta} \int d^3 x^{(0)}\ \boldsymbol{E}^{(0)\,2} \left[1 - \cos^2\theta \right] \qquad [\ \hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{E}}^{(0)} = \cos\theta\ ]$$
$$= \frac{1}{4\,\pi\, c}\, \boldsymbol{\beta} \int d^3 x^{(0)}\ \boldsymbol{E}^{(0)\,2} \left[1 - \frac{1}{3} \right] \qquad [\ \boldsymbol{E}^{(0)} = \boldsymbol{E}^{(0)}(r)\ ]$$
$$= \frac{4}{3}\, \boldsymbol{\beta}\, c\, m_e$$

Similarly,

$$\boldsymbol{P}_e^{(2)} \simeq \frac{1}{4\pi c}\int d^3x^{(0)}\left[\boldsymbol{E}^{(0)}\left(\boldsymbol{\beta}\cdot\boldsymbol{E}^{(0)}\right) - \frac{1}{2}\boldsymbol{\beta}\,\boldsymbol{E}^{(0)\,2}\right]$$

$$= \frac{1}{4\pi c}\boldsymbol{\beta}\int d^3x^{(0)}\,\boldsymbol{E}^{(0)\,2}\left[\cos^2\theta - \frac{1}{2}\right]$$

$$= \frac{1}{4\pi c}\boldsymbol{\beta}\int d^3x^{(0)}\,\boldsymbol{E}^{(0)\,2}\left[\frac{1}{3} - \frac{1}{2}\right]$$

$$= -\frac{1}{3}\boldsymbol{\beta}\, c\, m_e$$

Hence

$$\boldsymbol{P}_e = \boldsymbol{\beta}\, c\, m_e$$

The strange factor $\frac{4}{3}$ of the A-L model is thus removed.

17.6 Integrodifferential Equation of Motion

We now deal with the problem of unphysical solutions in the A-L model. Only the non- relativistic case will be considered explicitly since the generalization to relativistic motion involves no new principles.

Consider the A-L equation

$$m\left(\dot{\boldsymbol{v}} - \tau\,\ddot{\boldsymbol{v}}\right) = \boldsymbol{F} \qquad (17.9)$$

We wish to convert it into an equivalent eq of involving only 1st order time derivatives of $\boldsymbol{v}$.

Treating $\boldsymbol{F}$ as a function of t, (17.9) can be considered as a 1st order eq for $\boldsymbol{w} = \dot{\boldsymbol{v}}$.

$$\dot{\boldsymbol{w}} - \frac{1}{\tau}\boldsymbol{w} = -\frac{1}{m\tau}\boldsymbol{F} \qquad (a)$$

The integrating factor is [see Arfken § 8.2]

$$\alpha(t) = e^{-t/\tau}$$

so that (a) becomes

$$\alpha\left[\dot{\boldsymbol{w}} - \frac{1}{\tau}\boldsymbol{w}\right] = -\frac{1}{m\tau}\alpha\,\boldsymbol{F}$$

$$= \frac{d}{dt}(\alpha\,\boldsymbol{w})$$

and

$$\alpha\,\boldsymbol{w} = -\frac{1}{m\tau}\int dt\,\alpha\,\boldsymbol{F}_{\text{ext}}$$

or

$$m\,\dot{\boldsymbol{v}} = -\,e^{t/\tau}\,\frac{1}{\tau}\int_{t_0}^{t} d\,t'\,e^{-t'/\tau}\,\boldsymbol{F}(t') \qquad [\,\dot{\boldsymbol{v}}(t_0) = 0\,]$$

$$= \;\; e^{t/\tau}\,\frac{1}{\tau}\int_{t}^{t_0} d\,t'\,e^{-t'/\tau}\,\boldsymbol{F}(t') \qquad (17.50)$$

$$= \;\; \frac{1}{\tau}\int_{t}^{t_0} d\,t'\,e^{-(t'-t)/\tau}\,\boldsymbol{F}(t')$$

where the factor $e^{-(t'-t)/\tau}$ ensures only t' not too much further than τ from t contributes significantly to the integral. (17.50) also means that the effective force is an average of the force the particle is going to experience, $\boldsymbol{F}_{\text{ext}}(t' > t)$, with a weighting factor $e^{-(t'-t)/\tau}$. [see Jackson p.797-8 for discussion of this acausal behavior]

Consider

$$f(t'-t) = \lim_{\tau\to 0}\,\frac{1}{\tau}\,e^{-(t'-t)/\tau}$$

$$= \begin{cases} 0 & \text{for } t' \neq t \\ \infty & \text{for } t' = t \end{cases}$$

Furthermore

$$\lim_{\tau\to 0}\int_{t}^{\infty} d\,t'\,f(t'-t) = \; -\lim_{\tau\to 0}\; e^{-(t'-t)/\tau}\Big]_{t}^{\infty} = 1$$

Hence, one may set

$$\lim_{\tau\to 0}\,\frac{1}{\tau}\,e^{-(t'-t)/\tau} = \delta(t'-t) \qquad \text{for } t' \in (\,t,\,\infty)$$

Now, $\tau = \frac{2}{3}\frac{e^2}{c^3}$ is equal to 0 for neutral ($e = 0$) particles.

Thus, there is no radiation dampiing for $\tau = 0$ and (17.50) must reduce to the ordinary Newton's equation

$$m\,\dot{\boldsymbol{v}} = \boldsymbol{F}_{\text{ext}}$$

From the foregoing consideration, we see that this can be accomplished by setting $t_0 = \infty$.

Setting

$$s = \;\frac{1}{\tau}\,(\,t'-t\,)$$

we have

$$m\,\dot{\boldsymbol{v}}(t) = \frac{1}{\tau}\int_{t}^{\infty} d\,t'\,e^{-(t'-t)/\tau}\,\boldsymbol{F}(t')$$

$$= \int_0^\infty d\,s\, e^{-s}\, \boldsymbol{F}(s\,\tau + t) \tag{17.51}$$

$$= \sum_{n=0}^\infty \frac{\tau^n}{n!} \frac{d^n \boldsymbol{F}(t)}{d\,t^n} \int_0^\infty d\,s\, e^{-s}\, s^n \tag{17.52}$$

$$= \sum_{n=0}^\infty \tau^n \frac{d^n \boldsymbol{F}(t)}{d\,t^n} \tag{17.53}$$

where

$$\int_0^\infty d\,s\, e^{-s}\, s^n = \Gamma(n+1) = n!$$

17.7 Line Breadth & Level Shift of Oscillator

Consider an 1-dim harmonic oscillator.
In the absence of radiation damping, the eq of motion is

$$\ddot{x} + \omega_0^2\, x = 0$$

Using

$$f(t) = -\omega_0^2\, x(t)$$

(17.51) becomes

$$\ddot{x} = -\omega_0^2 \int_0^\infty d\,s\, e^{-s}\, x(t + s\,\tau) \tag{17.54}$$

Consider the ansatz

$$x(t) = x_0\, e^{-\alpha\, t} \tag{17.55}$$

For $\tau = 0$, we know that

$$x(t) = x_0\, e^{-i\,\omega_0\, t}$$

so that

$$\alpha = i\,\omega_0 \qquad \text{for} \qquad \tau = 0$$

Next

$$\ddot{x} = \alpha^2\, x$$

$$\int_0^\infty d\,s\, e^{-s}\, x(t + s\,\tau) = x_0\, e^{-\alpha\, t} \int_0^\infty d\,s\; e^{-s(1+\alpha\,\tau)}$$

$$= -x(t)\, \frac{1}{1 + \alpha\,\tau}\, e^{-s(1+\alpha\,\tau)}\Big]_0^\infty$$

$$= \frac{x(t)}{1 + \alpha\,\tau}$$

where $\mathrm{Re}(1 + \alpha\,\tau) > 0$ is assumed.

(17.54) thus becomes

$$\alpha^2 = -\omega_0^2\, \frac{1}{1 + \alpha\,\tau}$$

or

$$\alpha^3\,\tau + \alpha^2 + \omega_0^2 = 0 \tag{17.56}$$

Note that (17.56) can be obtained directly from the A-L eq. (17.9).
The purpose of going through (17.51) is to introduce the restriction Re$(1 + \alpha \tau) > 0$ which serves to eliminate the runaway solution.

Another way to ensure the runaway solution is kept out is to re-write (17.56) as

$$\alpha^2 = -\frac{\omega_0^2}{1 + \alpha\tau}$$

so that

$$\alpha = \pm i\,\omega_0 \,(1 + \alpha\tau)^{-1/2} \qquad \text{(i)}$$

For $\tau = 0$, we have $\alpha = \pm i\,\omega_0$ as expected.

For small τ, (i) can be solved by iteration starting with $\alpha = \pm i\,\omega_0$. The runaway solution will then never be reached.

Equivalently, we can expand $(1 + \alpha\tau)^{-1/2}$ in a Taylor series & obtain solutions to any desired power in $\omega_0 \tau$.

Thus, to order $O\left(\tau^2\right)$, we have

$$\alpha = \pm i\,\omega_0 \left[1 - \frac{1}{2}\alpha\tau + \frac{3}{8}\alpha^2\tau^2 \right] \qquad \text{(ii)}$$

Solving (ii) by iteration, a solution good to $O(\tau)$ is obtained by putting $\alpha = \pm i\,\omega_0$ into (ii) and keeping terms to $O(\tau)$:

$$\alpha = \pm i\,\omega_0 \left[1 \mp i\frac{1}{2}\omega_0\tau \right] \qquad \text{(iii)}$$

One good to $O\left(\tau^2\right)$ is obtained by putting (iii) into (ii) and keeping terms to $O\left(\tau^2\right)$:

$$\begin{aligned}
\alpha &= \pm i\,\omega_0 \left[1 \mp \frac{1}{2} i\,\omega_0 \left(1 \mp i\frac{1}{2}\omega_0\tau \right)\tau - \frac{3}{8}\omega_0^2\tau^2 \right] \\
&= \pm i\,\omega_0 \left[1 \mp \frac{1}{2} i\,\omega_0\tau - \frac{5}{8}\omega_0^2\tau^2 \right] \\
&= \pm i \left[\omega_0 - \frac{5}{8}\omega_0^3\tau^2 \right] + \frac{1}{2}\omega_0^2\tau \\
&= \pm i\,[\,\omega_0 + \Delta\omega\,] + \frac{1}{2}\Gamma \qquad (17.57)
\end{aligned}$$

where

$$\Delta\omega = -\frac{5}{8}\omega_0^3\tau^2 = \text{ level shift}$$

$$\Gamma = \omega_0^2\tau = \text{decay constant}$$

Note that

$$\begin{aligned}
x(t) &= x_0\, e^{-\alpha t} \\
&= x_0\, e^{-i(\omega_0 + \Delta\omega)t}\, e^{-\frac{1}{2}\Gamma t}
\end{aligned}$$

so that the energy of the particle is

$$E(t) \propto |\,x(t)\,|^2 \propto e^{-\Gamma t}$$

Now, the acceleration of the particle is

$$a(t) = \alpha^2 x(t)$$

or, in terms of their Fourier transform,

$$\begin{aligned} a(\omega) &= \alpha^2 x(\omega) \\ &= \alpha^2 \int_0^\infty d\,t\; e^{i\,\omega\,t}\, x(t) \\ &= \alpha^2 x_0 \int_0^\infty d\,t\; e^{i\,\omega\,t}\, e^{-\alpha\,t} \\ &= \alpha^2 x_0 \frac{1}{\alpha - i\,\omega} \end{aligned}$$

As shown in § 14.5, the radiated power is proportional to $a(t)^2$, and the radiated spectrum to $|\,a(\omega)|^2$. Hence

$$\begin{aligned} \frac{d\,I}{d\,\omega} &= A \left| \frac{1}{i\,(\omega_0 + \Delta\omega - \omega) - \frac{\Gamma}{2}} \right|^2 \\ &= \mathrm{A}\, \frac{1}{(\omega_0 + \Delta\omega - \omega)^2 + \left(\frac{\Gamma}{2}\right)^2} \end{aligned}$$

where A is an overall constant.

Using

$$\begin{aligned} \int_{-\infty}^{\infty} d\,\omega\, \frac{1}{(\omega_0 + \Delta\omega - \omega)^2 + \left(\frac{\Gamma}{2}\right)^2} &= 2\,\pi\,i\,\mathrm{Res}\left[\frac{1}{(\omega_0 + \Delta\omega - \omega)^2 + \left(\frac{\Gamma}{2}\right)^2} \right] \\ &= 2\,\pi\,i\, \frac{1}{2 \cdot \frac{\Gamma}{2}\, i} \qquad \text{(contour closed in upper plane)} \\ &= \frac{2\,\pi}{\Gamma} \end{aligned}$$

we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty} d\,\omega\, \frac{d\,I}{d\,\omega} \\ &= A\, \frac{2\,\pi}{\Gamma} \end{aligned}$$

so that

$$\frac{d\,I}{d\,\omega} = I\, \frac{\Gamma}{2\,\pi}\, \frac{1}{(\omega_0 + \Delta\omega - \omega)^2 + \left(\frac{\Gamma}{2}\right)^2} \qquad (17.58)$$

The spectrum thus peaks at $\omega_0 + \Delta\omega$ with half- width Γ. It is sometimes called the resonant line- shape.

Since

$$\begin{aligned} \Gamma &= {\omega_0}^2\, \tau \\ &= {\omega_0}^2\, \frac{2}{3}\, \frac{e^2}{m\,c^3} \qquad \text{(} m = 1 \text{ for our oscillator)} \end{aligned}$$

$\frac{\Gamma}{{\omega_0}^2}$ is a universal constant for electronic oscillators.

See Jackson p.801 for a discussion of $\Delta\,\omega$.

17.8 Scattering & Absorption of Oscillator

■ Scattering

Consider the scattering of radiation of frequency ω by a single nonrelativistic particle of mass m charge e and restoring force $-m\,\omega_0^2\,\boldsymbol{x}$.

Since we're concerned only with steady oscillations, the A-L eq (17.9) is sufficient:

$$m\left(\ddot{\boldsymbol{x}} - \tau\,\dddot{\boldsymbol{x}} + \omega_0^2\,\boldsymbol{x}\right) = \boldsymbol{F} \qquad \text{(i)}$$

In the dipole approximation,

$$\boldsymbol{F} = e\,\boldsymbol{\epsilon}\,E_0\,e^{-i\,\omega\,t}$$

We shall alos allow for other kinds of dissipative forces

$$-m\,\Gamma'\,\dot{\boldsymbol{x}}$$

(i) then becomes

$$m\left(\ddot{\boldsymbol{x}} + \Gamma'\,\dot{\boldsymbol{x}} - \tau\,\dddot{\boldsymbol{x}} + \omega_0^2\,\boldsymbol{x}\right) = e\,\boldsymbol{\epsilon}\,E_0\,e^{-i\,\omega\,t} \qquad (17.59)$$

The steady state solution is of the form

$$\boldsymbol{x}(t) = \boldsymbol{x}_0\,e^{-i\,\omega\,t}$$

so that (17.59) becomes

$$m\left(-\omega^2 - i\,\omega\,\Gamma' - i\,\omega^3\,\tau + \omega_0^2\right)\boldsymbol{x}_0 = e\,\boldsymbol{\epsilon}\,E_0$$

or

$$\boldsymbol{x}_0 = \boldsymbol{\epsilon}\,\frac{e\,E_0}{m\left(-\omega^2 - i\,\omega\,\Gamma' - i\,\omega^3\,\tau + \omega_0^2\right)}$$

$$= \boldsymbol{\epsilon}\,\frac{e\,E_0}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i\,\omega\,\Gamma_t}$$

where

$$\Gamma_t = \Gamma' + \omega^2\,\tau = \text{Total decay constant}$$

$$= \Gamma' + \left(\frac{\omega}{\omega_0}\right)^2\Gamma \qquad (17.61)$$

and

$$\Gamma = \omega_0^2\,\tau = \text{radiative decay constant.}$$

Hence

$$\boldsymbol{x}(t) = \boldsymbol{\epsilon}\,\frac{e\,E_0}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i\,\omega\,\Gamma_t}\,e^{-i\,\omega\,t} \qquad (17.60)$$

$$\ddot{\boldsymbol{x}}(t) = -\boldsymbol{\epsilon}\,\frac{e\,E_0}{m}\,\frac{\omega^2}{\omega_0^2 - \omega^2 - i\,\omega\,\Gamma_t}\,e^{-i\,\omega\,t}$$

Now

$$\boldsymbol{E}_{\text{rad}} = \frac{e}{c^2}\,\frac{1}{r}\left[\boldsymbol{n}\times\left(\boldsymbol{n}\times\ddot{\boldsymbol{x}}\right)\right]_{\text{ret}} \qquad (14.18)$$

$$= \frac{e}{c^2}\,\frac{1}{r}\left[\boldsymbol{n}\left(\boldsymbol{n}\cdot\ddot{\boldsymbol{x}}\right)-\ddot{\boldsymbol{x}}\right]_{\text{ret}}$$

$$= \frac{e}{c^2}\,\frac{1}{r}\left[-\boldsymbol{n}\,(\boldsymbol{n}\cdot\boldsymbol{\epsilon})+\boldsymbol{\epsilon}\right]\,\frac{e\,E_0}{m}\,\frac{\omega^2}{\omega_0{}^2 - \omega^2 - i\,\omega\,\Gamma_t}\,e^{-i\,\omega\,t+i\,k\,r}$$

where the retarded time is

$$t-\frac{1}{c}\,\boldsymbol{n}\cdot\boldsymbol{r} = t-\frac{1}{c}\,r$$

and $k = \dfrac{\omega}{c}$.

Thus, the radiation field with polarization ϵ' is

$$\epsilon'^{*}\cdot\boldsymbol{E}_{\text{rad}} = \frac{1}{r}[\,\epsilon'^{*}\cdot\boldsymbol{\epsilon}\,]\,\frac{e^2\,E_0}{m\,c^2}\,\frac{\omega^2}{\omega_0{}^2 - \omega^2 - i\,\omega\,\Gamma_t}\,e^{-i\,\omega\,t+i\,k\,r} \qquad (17.62)$$

where, by definition,

$$\epsilon'^{*}\cdot\boldsymbol{n} = 0$$

Using (14.101), we have

$$\frac{d\,\sigma(\omega,\ \boldsymbol{\epsilon}')}{d\,\Omega} = \left|\frac{r\,\boldsymbol{\epsilon}'^{*}\cdot\boldsymbol{E}_{\text{rad}}}{E_0}\right|^2$$

$$= |\,\epsilon'^{*}\cdot\boldsymbol{\epsilon}\,|^2\left(\frac{e^2}{m\,c^2}\right)^2\frac{\omega^4}{\left(\omega_0{}^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2} \qquad (17.63)$$

$$= \left(\frac{d\,\sigma}{d\,\Omega}\right)_T\frac{\omega^4}{\left(\omega_0{}^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2}$$

where $\left(\dfrac{d\,\sigma}{d\,\Omega}\right)_T$ is the Thomson cross section (for free electron scattering).

Using

$$\Gamma = \omega_0{}^2\,\tau = \omega_0{}^2\left(\frac{2}{3}\,\frac{e^2}{m\,c^3}\right)$$

we have

$$\frac{e^2}{m\,c^2} = \frac{3}{2}\,\tau\,c$$

$$= \frac{3}{2}\,\frac{c}{\omega_0{}^2}\,\Gamma$$

so that (17.63) can be written as

$$\frac{d\,\sigma(\omega,\ \boldsymbol{\epsilon}')}{d\,\Omega} = |\,\epsilon'^{*}\cdot\boldsymbol{\epsilon}\,|^2\,\frac{9}{4}\left(\frac{c}{\omega_0{}^2}\right)^2\frac{\Gamma^2\,\omega^4}{\left(\omega_0{}^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2} \qquad (17.63a)$$

- σ_{sc}

To find σ_{sc}, consider the following coordinates:

$$\epsilon = (0,\ 0,\ 1) = \hat{\boldsymbol{z}}$$
$$\boldsymbol{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$
$$\epsilon_1' = (-\cos\theta\cos\phi,\ -\cos\theta\sin\phi,\ \sin\theta)$$
$$\epsilon_2' = (\sin\phi,\ -\cos\phi,\ 0)$$
$$\epsilon_1' \times \epsilon_2' = \boldsymbol{n}$$

so that

$$\epsilon_1'^{*} \cdot \epsilon = \sin\theta$$
$$\epsilon_2'^{*} \cdot \epsilon = 0$$

Thus

$$\begin{aligned}
\sigma_{\mathrm{sc}}(\omega) &= \sum_{i=1,2} \int d\,\Omega\, \frac{d\,\sigma(\omega,\ \boldsymbol{\epsilon}_i')}{d\,\Omega} \\
&= \left(\frac{e^2}{m\,c^2}\right)^2 \frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2} \sum_{i=1,2} \int d\,\Omega\, \left|\epsilon_i'^{*} \cdot \epsilon\right|^2 \\
&= \left(\frac{e^2}{m\,c^2}\right)^2 \frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2} \int d\,\Omega \sin^2\theta \\
&= \left(\frac{e^2}{m\,c^2}\right)^2 \frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2} \frac{8\,\pi}{3} \\
&= 6\,\pi \left(\frac{c}{\omega_0^2}\right)^2 \frac{\Gamma^2\,\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2}
\end{aligned}$$

- **For low frequencies $\omega \ll \omega_0$**

$$\frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2} \simeq \frac{\omega^4}{\omega_0^4}$$

so that

$$\frac{d\,\sigma(\omega,\ \boldsymbol{\epsilon}')}{d\,\Omega} \simeq \left|\epsilon'^{*} \cdot \boldsymbol{\epsilon}\right|^2 \left(\frac{e^2}{m\,c^2}\right)^2 \left(\frac{\omega}{\omega_0}\right)^4 \qquad (17.64)$$

which agrees with the Rayleigh scattering law (see Chap 9).

- **For resonance scattering, $\omega \approx \omega_0$**

$$\frac{\omega^4}{\left(\omega_0{}^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2} = \frac{\omega^4}{(\omega_0 + \omega)^2 (\omega_0 - \omega)^2 + (\omega\,\Gamma_t)^2}$$

$$\simeq \frac{\omega^4}{4\,\omega^2 (\omega_0 - \omega)^2 + (\omega\,\Gamma_t)^2}$$

$$= \frac{1}{4} \frac{\omega^2}{(\omega_0 - \omega)^2 + (\Gamma_t / 2)^2}$$

$$\simeq \frac{1}{4} \frac{\omega_0{}^2}{(\omega_0 - \omega)^2 + (\Gamma_t / 2)^2}$$

so that

$$\frac{d\,\sigma(\omega, \boldsymbol{\epsilon}')}{d\,\Omega} \simeq |\epsilon'^* \cdot \boldsymbol{\epsilon}|^2 \left(\frac{e^2}{m\,c^2}\right)^2 \frac{1}{4} \frac{\omega_0{}^2}{(\omega_0 - \omega)^2 + (\Gamma_t / 2)^2}$$

$$= |\epsilon'^* \cdot \boldsymbol{\epsilon}|^2 \frac{9\,c^2}{16\,\omega_0{}^2} \frac{\Gamma^2}{(\omega_0 - \omega)^2 + (\Gamma_t / 2)^2} \qquad (17.65)$$

and

$$\sigma_{\text{sc}}(\omega) = \frac{3\,\pi\,c^2}{2\,\omega_0{}^2} \frac{\Gamma^2}{(\omega_0 - \omega)^2 + (\Gamma_t / 2)^2} \qquad (17.66)$$

The peak cross section is

$$\sigma_{\text{sc}}(\omega_0) = \frac{3\,\pi\,c^2}{2\,\omega_0{}^2} \frac{\Gamma^2}{(\Gamma_t / 2)^2}$$

$$= \frac{6\,\pi\,c^2}{\omega_0{}^2} \left(\frac{\Gamma}{\Gamma_t}\right)^2 \qquad (17.67)$$

- **At very high frequencies $\omega \gg \omega_0$**

$$\frac{\omega^4}{\left(\omega_0{}^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2} \simeq \frac{\omega^4}{\omega^4 + (\omega\,\Gamma_t)^2}$$

$$= \frac{1}{1 + (\Gamma_t / \omega)^2}$$

$$= \frac{1}{1 + \omega^2\,\tau^2}$$

where

$$\Gamma_t = \Gamma' + \left(\frac{\omega}{\omega_0}\right)^2 \Gamma$$

$$\simeq \left(\frac{\omega}{\omega_0}\right)^2 \Gamma$$

$$= \left(\frac{\omega}{\omega_0}\right)^2 \omega_0{}^2\,\tau$$

$$= \omega^2\,\tau$$

Hence

$$\frac{d\,\sigma(\omega,\,\boldsymbol{\epsilon}')}{d\,\Omega} \simeq \left(\frac{d\,\sigma}{d\,\Omega}\right)_T \frac{1}{1+\,\omega^2\,\tau^2}$$

See Jackson p.803-4 for further discussions.

Absorption

The absorption spectrum is given by (13.24) with Γ repaced by Γ_t,

$$\frac{d\,E}{d\,\omega} = \frac{e^2}{m}\,|\,E_0(\omega)|^2 \frac{2\,\omega^2\,\Gamma_t}{\left(\omega_0{}^2\,-\,\omega^2\right)^2\,+\,(\,\omega\,\Gamma_t\,)^2} \qquad (17.68)$$

where E is the total energy (scattering + dissipation) removed from the incident beam.

The total cross section is defined as

$$\begin{aligned}\sigma_t(\omega) &\equiv \frac{1}{F}\,\frac{d\,E}{d\,\omega} \\ &= \;4\,\pi\left(\frac{e^2}{m\,c}\right)\frac{\omega^2\,\Gamma_t}{\left(\omega_0{}^2\,-\,\omega^2\right)^2\,+\,(\,\omega\,\Gamma_t\,)^2} \qquad (17.69) \\ &= \;6\,\pi\left(\frac{c}{\omega_0}\right)^2\frac{\omega^2\;\Gamma\,\Gamma_t}{\left(\omega_0{}^2\,-\,\omega^2\right)^2\,+\,(\,\omega\,\Gamma_t\,)^2} \qquad (17.70)\end{aligned}$$

where

$$F = \frac{c}{2\,\pi}\,|\,E_0(\omega)|^2 = \text{ incident flux}$$

$$\frac{e^2}{m\,c} = \;\frac{3}{2}\,\frac{c^2}{\omega_0{}^2}\,\Gamma$$

Comparing with

$$\sigma_{\rm sc}(\omega) = \;6\,\pi\left(\frac{c}{\omega_0{}^2}\right)^2\,\frac{\Gamma^2\,\omega^4}{\left(\omega_0{}^2\,-\,\omega^2\right)^2\,+\,(\omega\,\Gamma_t)^2}$$

we have

$$\sigma_t(\omega) = \;\frac{\Gamma_t}{\Gamma}\left(\frac{\omega_0}{\omega}\right)^2\sigma_{\rm sc}(\omega)$$

As shown in the scattering problem,

$$\frac{\omega^4}{\left(\omega_0{}^2\,-\,\omega^2\right)^2\,+\,(\omega\,\Gamma_t)^2} \simeq \begin{cases} \left(\frac{\omega}{\omega_0}\right)^4 & \omega \ll \omega_0 \\ \frac{1}{4}\,\frac{\omega_0{}^2}{(\,\omega_0-\omega\,)^2\,+\,(\,\Gamma_t\,/\,2)^2} & \omega\;\sim\omega_0 \\ \frac{1}{1+\,\omega^2\,\tau^2} & \omega \gg \omega_0 \end{cases}$$

Hence

$$\sigma_t(\omega) \simeq \begin{cases} 6\,\pi\left(\frac{c}{\omega_0}\right)^2\;\frac{\omega^2\;\Gamma\,\Gamma_t}{\omega_0{}^4} & \omega \ll \omega_0 \\ \frac{3}{2}\,\pi\left(\frac{c}{\omega_0}\right)^2\;\frac{\Gamma\,\Gamma_t}{(\omega_0-\omega)^2\,+\,(\,\Gamma_t\,/\,2\,)^2} & \omega\;\sim\omega_0 \\ 6\,\pi\left(\frac{c}{\omega_0}\right)^2\;\frac{\Gamma\,\Gamma_t}{\omega^2} & \omega \gg \omega_0 \end{cases} \qquad (17.71)$$

where we've set $\frac{1}{1+\,\omega^2\,\tau^2} \simeq 1$.

Define the reaction cross section as

$$\sigma_r(\omega) = \sigma_t(\omega) - \sigma_{sc}(\omega)$$
$$= \left[\frac{\Gamma_t}{\Gamma}\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]\sigma_{sc}(\omega)$$

Using

$$\Gamma_t = \Gamma' + \left(\frac{\omega}{\omega_0}\right)^2 \Gamma \qquad (17.61)$$

we have

$$\frac{\Gamma_t}{\Gamma}\left(\frac{\omega_0}{\omega}\right)^2 - 1 = \frac{\Gamma'}{\Gamma}\left(\frac{\omega_0}{\omega}\right)^2$$

Thus

$$\sigma_{sc}(\omega) = A\left(\frac{\omega}{\omega_0}\right)^2 \Gamma$$
$$\sigma_t(\omega) = A\;\Gamma_t \qquad (17.72)$$
$$\sigma_r(\omega) = A\;\Gamma'$$

where

$$A = 6\pi\left(\frac{c}{\omega_0}\right)^2 \frac{\omega^2\,\Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2}$$

■ Dipole Sum Rule

Since the dipole sum rule is a consequence of causality, radiation damping effects must be neglected. This means Γ_t should be treated as a constant in doing the ω integral. Thus

$$\int_0^\infty \sigma_t(\omega)\,d\omega = 6\pi\left(\frac{c}{\omega_0}\right)^2 \int_0^\infty d\omega\,\frac{\omega^2\,\Gamma\,\Gamma_t}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\,\Gamma_t)^2}$$
$$= 3\pi\left(\frac{c}{\omega_0}\right)^2 \int_{-\infty}^\infty d\omega\,\frac{\Gamma\,\Gamma_t}{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + \Gamma_t^2}$$

Poles are at

$$\omega^2 \pm i\,\omega\,\Gamma_t - \omega_0^2 = 0$$

or

$$\omega = \frac{1}{2}\left[\pm i\,\Gamma_t \pm \sqrt{-\Gamma_t^2 + 4\,\omega_0^2}\right]$$
$$= \pm i\,\frac{\Gamma_t}{2} \pm \sqrt{-\left(\frac{\Gamma_t}{2}\right)^2 + \omega_0^2}$$

where the two ± signs are not correlated (there are 4 roots).

Closing the contour in the upper plane, the only contributing roots are

$$\omega = i\,\frac{\Gamma_t}{2} \pm \sqrt{-\left(\frac{\Gamma_t}{2}\right)^2 + \omega_0^2}$$
$$\simeq i\,\frac{\Gamma_t}{2} \pm \omega_0$$

The residues are

$$\sum \frac{1}{2\left(\frac{\omega_0{}^2}{\omega} - \omega\right)\left(-\frac{\omega_0{}^2}{\omega^2} - 1\right)} = \sum \frac{1}{2\,\omega} \frac{1}{1 - \left(\frac{\omega_0}{\omega}\right)^4}$$

Now

$$\begin{aligned}\frac{\omega_0}{\omega} &\simeq \frac{\omega_0}{i\,\frac{\Gamma_t}{2} \pm \omega_0} \\ &= \pm\left(1 \pm i\,\frac{\Gamma_t}{2\,\omega_0}\right)^{-1} \\ &\simeq \pm 1 - i\,\frac{\Gamma_t}{2\,\omega_0}\end{aligned}$$

so that

$$\begin{aligned}1 - \left(\frac{\omega_0}{\omega}\right)^4 &\simeq 1 - \left(1 \mp i\,\frac{\Gamma_t}{2\,\omega_0}\right)^4 \\ &\simeq \pm i\,\frac{2\,\Gamma_t}{\omega_0}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\,\omega}\frac{1}{1 - \left(\frac{\omega_0}{\omega}\right)^4} &\simeq \mp i\,\frac{\omega_0}{2\,\omega}\frac{1}{2\,\Gamma_t} \\ &\simeq \mp i\left(\pm 1 - i\,\frac{\Gamma_t}{2\,\omega_0}\right)\frac{1}{4\,\Gamma_t} \\ &= -\frac{i}{4\,\Gamma_t} \mp \frac{1}{8\,\omega_0}\end{aligned}$$

The sum of the residues is

$$-\frac{i}{2\,\Gamma_t}$$

so that

$$\int_{-\infty}^{\infty} d\,\omega\,\frac{\Gamma_t}{\left(\frac{\omega_0{}^2}{\omega} - \omega\right)^2 + \Gamma_t{}^2} = \pi$$

And

$$\begin{aligned}\int_0^{\infty} \sigma_t(\omega)\,d\,\omega &= 3\,\pi^2\left(\frac{c}{\omega_0}\right)^2\,\Gamma \\ &= 3\,\pi^2\,c^2\,\tau \\ &= 2\,\pi^2\,\frac{e^2}{m\,c}\end{aligned} \qquad (17.73)$$

See Jackson for further discussions.