1. Complex Number

■ Definition

Let the set of all real numbers be denoted by $\mathbb{R}$.
Geometrically, $\mathbb{R}$ can be represented by an open straight line.
Let $\mathbb{R}^2$ be the set of ordered pairs of real numbers, ie.,

$$\mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \}$$

Geometrically, $\mathbb{R}^2$ can be represented by an infinite plane.

The set $\mathbb{C}$ of complex numbers $z = (x, y)$ is just $\mathbb{R}^2$ equipped with the following properties.
Let

$$z_1 = (x_1, y_1) \qquad z_2 = (x_2, y_2)$$

Equality is defined by:

$$z_1 = z_2 \iff x_1 = x_2 \; \& \; y_1 = y_2$$

Addition is defined by:

$$z_1 + z_2 \equiv (x_1 + x_2, y_1 + y_2)$$

Multiplication is defined by:

$$z_1 \cdot z_2 \equiv (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Note that

$$(0, 1) \cdot (0, 1) = (0 - 1, 0 + 0) = (-1, 0)$$

It is straightforward to show that $\mathbb{C}$ is a field as well as a real vector space.

In more elementary treatments, z is defined as

$$z = x + i y$$
$$= \operatorname{Re} z + i \operatorname{Im} z$$

with

$$x, y \in \mathbb{R}, \qquad i^2 = -1$$

The foregoing definitions then becomes
Equality:

$$z_1 = z_2 \iff x_1 = x_2 \; \& \; y_1 = y_2$$

Addition:

$$z_1 + z_2 \equiv x_1 + i y_1 + x_2 + i y_2$$
$$= x_1 + x_2 + i (y_1 + y_2)$$

Multiplication:

$$z_1 \cdot z_2 \equiv (x_1 + i y_1)(x_2 + i y_2)$$
$$= x_1 x_2 - y_1 y_2 + i (x_1 y_2 + x_2 y_1)$$

These two approaches are made equivalent by setting $i = (0, 1)$.
The vector characteristic of z is apparent if we write

$$z = x 1 + y i$$

with

$$1 = (1, 0), \qquad i = (0, 1)$$

being the unit vectors along the real (x) & imaginary (y) axis, respectively.

The 1st approach is more appealing to mathematicians since it doesn't require the extra definition of i by $i^2 = -1$.

Properties

Modulus

Since $\mathbb{C}$ is a $2-D$ vector space, we can introduce the polar coordinates (r, ϕ) & write

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$z = r\cos\phi + i\, r\sin\phi$$

where ϕ is the angle between the vector z & the real axis and

$$r = \sqrt{x^2 + y^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$$

is its Euclidean length. In complex analysis, r is often called the absolute value or modulus of z & denoted by $|z|$. Note that

$$|z| \geq 0$$

with

$$|z| = 0 \qquad \Leftrightarrow \qquad z = 0$$

Complex Conjugate

The reflection of $z = (x, y)$ about the real axis gives a new vector $\bar{z} = z^* = (x, -y)$ called the complex conjugate of z. Thus

$$z\bar{z} = (x, y)(x, -y) = \left(x^2 + y^2, \ 0\right) = |z|^2$$

$$z + \bar{z} = (2x, 0) = 2\operatorname{Re} z$$

$$z - \bar{z} = (0, 2y) = 2\operatorname{Im} z$$

Triangle Inequality

The triangle inequality in geometry states that the sum of any 2 sides of a triangle is larger or equal to the 3rd side.

Since the vectors z_1, z_2 & $z_3 = z_2 - z_1$ form a triangle, we have

$$|z_1| + |z_2| \geq |z_2 - z_1|$$

$$|z_1| + |z_3| \geq |z_2| = |z_3 + z_1|$$

Thus, for arbitrary complex numbers z_1 & z_2

$$|z_1| + |z_2| \geq |z_2 \pm z_1|$$

Similarly, the rule that the difference of any 2 sides of a triangle is smaller or equal to the 3rd side leads to

$$\left| |z_1| - |z_2| \right| \leq |z_2 \pm z_1|$$

Combining the 2 results, we have

$$|z_1| + |z_2| \geq |z_2 \pm z_1| \geq \left| |z_1| - |z_2| \right|$$

By induction, it can be shown that

$$\sum_{j=1}^{n} |z_j| \geq \left| \sum_{j=1}^{n} z_j \right|$$

Cauchy-Schwarz Inequality

$$\sum_{j=1}^{n} |z_j|^2 \cdot \sum_{k=1}^{n} |w_k|^2 \geq \left| \sum_{j=1}^{n} z_j w_j \right|^2$$

- **Proof**

Let $\lambda \in \mathbb{C}$.

$$\longrightarrow \qquad \sum_{j=1}^{n} \left| z_j - \lambda \overline{w}_j \right|^2 \geq 0$$

Now:

$$\sum_{j=1}^{n} \left| z_j - \lambda \overline{w}_j \right|^2 = \sum_{j=1}^{n} \left(\left| z_j \right|^2 + |\lambda|^2 \left| w_j \right|^2 - z_j \overline{\lambda} w_j - \overline{z}_j \lambda \overline{w}_j \right)$$

$$= \sum_{j=1}^{n} \left\{ \left| z_j \right|^2 + |\lambda|^2 \left| w_j \right|^2 - 2 \operatorname{Re}\left(z_j \overline{\lambda} w_j \right) \right\}$$

$$= \sum_{j=1}^{n} \left| z_j \right|^2 + |\lambda|^2 \sum_{j=1}^{n} \left| w_j \right|^2 - 2 \operatorname{Re}\left(\overline{\lambda} \sum_{j=1}^{n} z_j \; w_j \right)$$

Setting

$$\lambda = \frac{\sum_{j=1}^{n} z_j \; w_j}{\sum_{j=1}^{n} \left| w_j \right|^2}$$

we have

$$\sum_{j=1}^{n} \left| z_j \right|^2 + |\lambda|^2 \sum_{j=1}^{n} \left| w_j \right|^2 - 2 \operatorname{Re}\left(\overline{\lambda} \sum_{j=1}^{n} z_j \; w_j \right)$$

$$= \sum_{j=1}^{n} \left| z_j \right|^2 + \frac{\left| \sum_{j=1}^{n} z_j \; w_j \right|^2}{\sum_{j=1}^{n} \left| w_j \right|^2} - 2 \frac{\left| \sum_{j=1}^{n} z_j \; w_j \right|^2}{\sum_{j=1}^{n} \left| w_j \right|^2}$$

$$= \sum_{j=1}^{n} \left| z_j \right|^2 - \frac{\left| \sum_{j=1}^{n} z_j \; w_j \right|^2}{\sum_{j=1}^{n} \left| w_j \right|^2}$$

$$\geq 0$$

$$\Longrightarrow$$

$$\sum_{j=1}^{n} \left| z_j \right|^2 \cdot \sum_{j=1}^{n} \left| w_j \right|^2 \geq \left| \sum_{j=1}^{n} z_j \; w_j \right|^2$$

- **Euler's Formula**

The exponential function of a complex number $z = x + i\,y$ is defined as

$$e^z \equiv e^x \left(\cos y + i \sin y \right)$$

Setting $x = 0$, we have

$$e^{i\,y} = \cos y + i \sin y \qquad\qquad y \in \mathbb{R}$$

which is known as the Euler formula.

The rationale of this definition is as follows.
It is well known in calculus that for $x \in \mathbb{R}$, we can define

$$e^x \equiv \sum_{n=0} \frac{x^n}{n!}$$

$$\cos x \equiv \sum_{n=0} \frac{x^{2n}}{(2n)!}$$

$$\sin x \equiv \sum_{n=0} \frac{x^{2n+1}}{(2n+1)!}$$

Using the technique of analytic continuation (to be described much later), we can define

$$e^{iy} \equiv \sum_{n=0} \frac{(iy)^n}{n!}$$

Splitting the sum into even & odd parts then gives the Euler formula.

Since the concept of neither series nor analytic continuation is as yet defined, we need logically to take the Euler formula as a definition.

Now, for any $z_1, z_2 \in \mathbb{C}$, we have

$$\begin{aligned} e^{z_1+z_2} &= e^{x_1+x_2} (\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2} [\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i (\sin y_1 \cos y_2 + \cos y_1 \sin y_2)] \\ &= e^{x_1+x_2} [\cos(y_1 + y_2) + \sin(y_1 + y_2)] \\ &= e^{x_1+x_2} e^{i(y_1+y_2)} \\ &= e^{z_1} e^{z_2} \end{aligned}$$

which is the same as the real case.

Using the polar coordinates, we have

$$z = r(\cos\phi + i \sin\phi)$$

The Euler formula then gives

$$z = r e^{i\phi} = |z| e^{i\phi}$$

where ϕ here is often called the phase or argument of z.
The corresponding notation is

$$\phi = \arg z$$

In the usual usage of polar coordinates, ϕ is restricted to the values $[0, 2\pi)$.
In complex analysis, the value of ϕ is in general not restricted for reasons to be discussed later.
On the other hand, in the relation

$$z = r(\cos\phi + i \sin\phi) \qquad \text{with} \qquad r = \text{constant}$$

all the distinct values of z can be represented by ϕ within an interval of length 2π.
The principal value $\operatorname{Arg} z$ of $\arg z$ is defined as

$$\operatorname{Arg} z = \phi \in (-\pi, \pi] \qquad \text{or} \qquad \operatorname{Arg} z = \phi \in [0, 2\pi)$$

An arbitrary $\arg z$ can always be written as

$$\arg z = \operatorname{Arg} z + 2n\pi \qquad \text{with} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Moivre's Formula

Consider a complex number with unit modulus, ie

$$z = e^{i\phi} = \cos\phi + i\sin\phi$$

Thus

$$z^n = e^{in\phi} = (\cos\phi + i\sin\phi)^n$$

On the other hand, the Euler formula gives

$$e^{in\phi} = \cos n\phi + i\sin n\phi$$

Hence

$$\cos n\phi + i\sin n\phi = (\cos\phi + i\sin\phi)^n$$

which is called the Moivre's formula.

The right hand side can be expanded using the binomial formula. Equating the real or imaginary parts of both sides then give $\cos n\phi$ or $\sin n\phi$ as a polynomial of $\cos\phi$ & $\sin\phi$ of degree n.

Roots

Consider

$$z = r\, e^{i\phi} = r\, e^{i\phi_0 + i\,2n\pi} \qquad \text{where} \qquad n = 0, \pm 1, \pm 2, \ldots$$

where

$$\phi_0 = \operatorname{Arg} z \in [0, 2\pi)$$

The k th roots w of z are solutions of the equation

$$w^k = z \qquad\qquad k = 1, 2, \ldots$$

Thus,

$$w = z^{1/k} = r^{1/k}\, e^{i\phi_0/k + i\,2\pi n/k} \qquad \text{where} \qquad n = 0, \pm 1, \pm 2, \ldots$$

so that

$$|w| = r^{1/k}$$

$$\arg w = \frac{\phi_0}{k} + 2\pi\frac{n}{k} \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots$$

which means there are k distinct values of the principal argument,

$$\operatorname{Arg} w = \frac{\phi_0}{k} + 2\pi\frac{m}{k} \qquad \text{where } m = 0, 1, \ldots, k-1$$

In other words, there are k distinct k th roots of any non-zero complex number.
These roots are located on a circle of radius $r^{1/k}$ about the origin & have an angular separation of $2\pi/k$ between adjacent points.

Topology

Neighborhood $N_\epsilon(z_0)$

The neighborhood $N_\epsilon(z_0)$ of a point z_0 in the complex plane is defined as the set

$$N_\epsilon(z_0) = \{ z : |z - z_0| < \epsilon \}$$

which is just an open disk of radius ϵ centered at z_0.
The corresponding deleted neighborhood $N_\epsilon'(z_0)$ is

$$N_\epsilon'(z_0) = N_\epsilon(z_0) \setminus \{ z_0 \}$$

which is just the neighborhood disk with its center punched out.

- **Interior Point**

z_0 is an interior point of a set S
$\Longleftrightarrow$ $\exists\ \epsilon > 0 \ \ni\ N_\epsilon(z_0) \subset S$

- **Exterior Point**

z_0 is an exterior point of a set S
$\Longleftrightarrow$ $\exists\ \epsilon > 0 \ \ni\ N_\epsilon(z_0) \bigcap S = \Phi$

- **Boundary Point**

z_0 is a boundary point of a set S
$\Longleftrightarrow$ z_0 is neither an interior nor an exterior point of S.

- **Boundary**

The boundary B of a set S is the set of all boundary points of S.

- **Open Set**

A set S is open if it contains none of its boundary points.

- **Closed Set**

A set S is closed if it contains all of its boundary points.

Note that a set can be neither open nor closed. It can also be both open & closed.

- **Closure**

The closure $\overline{S}$ of a set S is the union of S & its boundary B, ie.
$\overline{S} = S \bigcup B$

- **Connected Set / Domain / Region**

An open set S is connected if any pair of points in it can be connected by a polygonal line.

A connected open set is called a domain.

A domain with some, none, or all of its boundary points is called a region.

- **Bounded Set**

A set S is bounded
$\Longleftrightarrow$ $\exists\ R > 0 \quad \ni \quad |z| < R \quad \forall\ z \in S$

- **Unbounded Set**

A set S is unbounded if it is not bounded.

- **Accumulation Point**

A point z_0 is an accumulation point of a set S

$\Longleftrightarrow$ $N_\epsilon{}'(z_0) \bigcap S \neq \Phi \qquad \forall\ \epsilon > 0$

S is closed if it contains all of its accumulation points.