2. Analytic Functions

■ Introduction

Once the field properties of the complex numbers are established, (complex) functions of complex variables can be defined. As is the case with real analysis, the next step is calculus, which introduces the concepts of limits, continuity, differentiation and integration.
Since complex numbers are $2-D$ vectors, their calculus is the same as that of the vector space. For example, treating functions as vector fields, we expect integrations to be path-dependent. The analog of conserved vector fields, which are path-independent, is analytic functions. Something like the Gauss theorem, which relates integral on a closed surface with the source within, manifests itself as the residue theorem for contour integrals.

Note however that the above analogy should not be pushed too far since the complex number field is not identical to a $2-D$ real vector space. The crucial difference is that the "unit vector" along the imaginary axis has a negative magnitude $i \cdot i = -1$.

A more proper view of the situation is as follows.
Consider two complex numbers z_1 & z_2.

$$\begin{aligned} z_1^* z_2 &= (x_1 - i y_1)(x_2 + i y_2) \\ &= x_1 x_2 + y_1 y_2 + i(x_1 y_2 - x_2 y_1) \\ &= \vec{z}_1 \cdot \vec{z}_2 + i \, \vec{z}_1 \times \vec{z}_2 \end{aligned}$$

where $\vec{z}$ is a real $2-D$ vector with components x & y along the $x-$ & $y-$ axis, repectively.

■ Limits

■ Definition:

$$\lim_{z\to z_0} f(z) = w_0$$

$$\begin{aligned} &\Longleftrightarrow \forall \epsilon > 0, \quad \exists \delta > 0 \\ &\qquad \ni \quad | z - z_0 | < \delta \quad \Longrightarrow \quad | f(z) - w_0 | < \epsilon \\ &\Longleftrightarrow z \in N_\delta'(z_0) \Rightarrow f(z) \in N_\epsilon'(w_0) \\ &\Longleftrightarrow \lim_{\Delta z\to 0} f(z_0 + \Delta z) = w_0 \\ &\Longleftrightarrow f(z) \underset{z\to z_0}{\longrightarrow} w_0 \end{aligned}$$

■ Theorems:

$$f(z) = u + i v \underset{z\to z_0}{\longrightarrow} f_0 = u_0 + i v_0$$

$$\Longrightarrow u \to u_0, \quad v \to v_0$$

$$f(z) \underset{z\to z_0}{\longrightarrow} f_0, \quad g(z) \underset{z\to z_0}{\longrightarrow} g_0$$

$$\Longrightarrow f \begin{pmatrix} \pm \\ \cdot \\ \circ \\ / \end{pmatrix} g \underset{z\to z_0}{\longrightarrow} f_0 \begin{pmatrix} \pm \\ \cdot \\ \circ \\ / \end{pmatrix} g_0 \qquad (g_0 \neq 0 \text{ for } f/g)$$

- **Infinity**

$w \to \infty \iff \frac{1}{w} \to 0$

$\iff \forall\, \epsilon > 0\ \exists\, \delta > 0 \ni \left| z - z_0 \right| < \delta \ \Rightarrow\ \left| \frac{1}{w} \right| < \epsilon$

Continuity

- **Definition**

$f(z)$ is continuous at $z_0 \iff \lim_{z \to z_0} f(z) = f(z_0)$

Derivatives

- **Definition**

f is **differentiable** $\iff$

$$\frac{d f}{d z} = f' = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided the limit exists & is independent of how $\Delta z \to 0$

Analyticity

- **Definitions**

1. f is **analytic** at $z_0 \iff \exists\, \epsilon > 0 \ni f'$ exists $\forall\, z \in N_\epsilon(z_0)$

2. **analytic = holomorphic = regular**

3. f is **analytic** in $D \iff f$ is **analytic** $\forall\, z \in D$. (D is necessarily open)

4. z_0 is an **isolated singularity** $\iff \exists\, \epsilon > 0 \ni f'$ exists $\forall\, z \in N_\epsilon(z_0) \setminus \{z_0\}$ but not at z_0

5. f is **entire** $\iff f$ is **analytic** $\forall$ z finite.

- **Theorems:**

1. Let f, g be analytic in D

$\Longrightarrow f \begin{pmatrix} \pm \\ \cdot \\ \circ \\ / \end{pmatrix} g$ are analytic in D ($g \neq 0$ in D assumed for f/g)

2. $f' = 0$ in $D \Longrightarrow f$ = constant in D

- **Cauchy-Riemann Relations (C-RR):**

$\boldsymbol{u_x = v_y\ ,\ u_y = -v_x}$

$f = u + i\,v$

C-RR $\iff f$ is differentiable

C-RR + continuity of all partials $\iff f$ is analytic

Polar form:

$$u_r = \frac{v_\theta}{r}, \quad \frac{u_\theta}{r} = -v_r$$

- ***f* is analytic ⟹**

C-RR

$$\frac{\partial f}{\partial z^*} = 0$$

$$\frac{d f}{d z} = \frac{\partial f}{\partial x} = \frac{\partial f}{i \partial y}$$

u* , *v harmonic

$u_{xy} = u_{yx} \qquad v_{xy} = v_{yx}$

- **Reflection Principle**

Let D be a region containing a segment of & symmetric to the x - axis.
Let f be analytic in D.
$f(z)^* = f(z^*) \iff f(x) = f(x)^*$ in D

- **Harmonic Functions**

Definition (**harmonic functions**):
A real function f(x,y) is **harmonic** ⟺ $f_{xx} + f_{yy} = 0 = \nabla f$

Theorem:
$f = u + i v$ analytic ⟹ u, v harmonic.

Definition (**harmonic conjugates**):
Let u, v be harmonic & satisfy the C-RR, v is then the **harmonic conjugate** of u.

Theorem:
v is a harmonic conjugate of u ⟺ $-u$ is a harmonic conjugate of v

- **Analytic Continuation (AC)**

Definition:
Let f_1 & f_2 be analytic in D_1 & D_2, respectively.
If $f_1 = f_2$ in $D_1 \cap D_2$, f_2 is the **AC** of f_1 into D_2.

- $f(z) = \dfrac{(z^*)^2}{z}$

$$f(z + \Delta z) = \frac{(z^* + \Delta z^*)^2}{z + \Delta z}$$

$$f(0) \longrightarrow f(\Delta z) = \frac{(\Delta z^*)^2}{\Delta z} = \Delta r\, e^{-3\theta i} \longrightarrow 0$$

where $\Delta z = \Delta r\, e^{-\theta i}$

∴ **f** is continuous ∀ z

$$f(z) = \frac{(x - i\,y)^2}{x + i\,y} = \frac{x^3 - 3\,x\,y^2}{x^2 + y^2} + i\,\frac{y^3 - 3\,x^2\,y}{x^2 + y^2} = u + i\,v$$

$$u_x = \frac{3\,x^2 - 3\,y^2}{x^2 + y^2} - \frac{2\,x\left(x^3 - 3\,x\,y^2\right)}{\left(x^2 + y^2\right)^2}$$

$$u_y = \frac{6\,x\,y}{x^2 + y^2} - \frac{2\,y\left(x^3 - 3\,x\,y^2\right)}{\left(x^2 + y^2\right)^2}$$

$$v_x = -\frac{6\,x}{x^2 + y^2} - \frac{2\,x\left(y^3 - 3\,x^2\,y\right)}{\left(x^2 + y^2\right)^2}$$

$$v_y = \frac{3\,y^2 - 3\,x^2}{x^2 + y^2} - \frac{2\,y\left(y^3 - 3\,x^2\,y\right)}{\left(x^2 + y^2\right)^2}$$

The values of these partials at (x,y) = (0,0) depends on the order of taking the limits x ⟶ 0 & y ⟶ 0. This means they are **not continuous** there. Nonetheless, they do have well defined values which is found as follows.

Since $w_x = \left(\frac{\partial w}{\partial x}\right)_{y = \text{constant}}$, $w_x(0,\,0)$ is obtained by setting y = 0 first, then x ⟶ 0. Obviously, $w_y(0,\,0)$ means x = 0 then y ⟶ 0. Hence:

$$u_x(0,\,0) \longrightarrow \frac{3\,x^2}{x^2} - \frac{2\,x^4}{x^4} \longrightarrow 1$$

It's straightforward to show that the C-RR is satisfied. This means f is differentiable at z = 0. However, the partials are not continuous so f is not analytic there.

For z ≠ 0, the C-RR is obviously not satisfied . f is differentiable only at z = 0 even though it's continuous everywhere. That's why it's not analytic even at z = 0.

One trick to evaluate $w_x(0,\,0)$ is to use $w_x(0,\,0) = \frac{d}{d\,x}\,w(x,\,0)\Big|_{x=0}$. Hence:

$$u_x(0,\,0) = \frac{d}{d\,x}\,x = 1$$