3. Elementary Functions

Real elementary functions are continued into the complex plane:

$$f(x) \longrightarrow f(z)$$

▪ Exponential Functions

$$e^x \longrightarrow e^z$$

▪ Conditions

1. e^z is entire
2. $e^{z_1+z_2} = e^{z_1} e^{z_2}$

▪ Euler Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Proof:

Condition 2 $\longrightarrow$

$e^z = e^{x+iy} = e^x e^{iy} = e^x[u(y) + i\,v(y)] = e^x u + i\,e^x v$

Condition 1 $\longrightarrow$ C-RR $\longrightarrow$

$(e^x u)_x = (e^x v)_y \qquad \longrightarrow u = v'$

$(e^x u)_y = -(e^x v)_x \qquad \longrightarrow u' = -v$

$\therefore\ u'' = -v' = -u$

$\quad v'' = u' = -v$

$\longrightarrow\ u = a\cos y + b\sin y$

$\qquad v = c\cos y + d\sin y$

$u = v' \longrightarrow b = -c \quad a = d$ (same for $u' = -v$)

$e^{i0} = 1 \longrightarrow u(0) = 1 \longrightarrow a = 1 \longrightarrow d = 1$

$\qquad v(0) = 0 \longrightarrow c = 0 \longrightarrow b = 0$

$\therefore\ u = \cos y \quad v = \sin y$

$\longrightarrow$ $\boldsymbol{e^{iy} = \cos y + i\sin y}$

▪ Derivative: $\frac{d}{dz} e^z = e^z$

Proof:

Since f ' exists, we have:

$\frac{d}{dz} e^z = \frac{\partial}{\partial x} e^z = \frac{\partial}{\partial x}\left(e^x e^{iy}\right) = e^x e^{iy} = e^z$

▪ Periodicity: $e^{i2\pi n} = 1$

$\therefore\ \exp(z + 2\pi n i) = \exp(z)$

Trigonometric Functions

Procedure:

Euler formula $\longrightarrow$

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

$\longrightarrow$ $$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Continuation into z-plane:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Formulae:

All the formulae in the real case are directly applicable in the complex case.

Zeros:

$$\mathbf{\sin z = 0 \Longleftrightarrow z = n\pi}$$
$$\mathbf{\cos z = 0 \Longleftrightarrow z = \left(n + \tfrac{1}{2}\right)\pi}$$

Proof:

$\sin z = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy$

Now: $\cos iy = \dfrac{e^{-y} + e^{y}}{2} = \cosh y$

$\sin iy = \dfrac{e^{-y} - e^{y}}{2i} = i \sinh y$

$\therefore$ $\sin z = \sin x \cosh y + i \cos x \sinh y$

Now: $\cosh y \neq 0 \ \forall \ y$ real

$\therefore$ $\sin z = 0 \longleftrightarrow \sin x = 0$ and either $\cos x = 0$ or $\sinh y = 0$

Since $\sin x = 0 \longleftrightarrow x = n\pi$

$\therefore$ $\cos x = 0 \longleftrightarrow x = \left(n + \dfrac{1}{2}\right)\pi$ is incompatible with $\sin x = 0$

Thus we must have: $\sinh y = 0 \longleftrightarrow y = 0$

ie. $\sin z = 0 \longleftrightarrow z = (n\pi, 0) = n\pi \quad n = 0, \pm 1, \pm 2, \ldots$

Proof for the cos case is similar.

Hyperbolic Functions

Procedure:

$$\cosh x = \frac{e^x + e^{-x}}{2} \longrightarrow \cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2} \longrightarrow \sinh z = \frac{e^z - e^{-z}}{2}$$

Formulae:

All the formulae in the real case are directly applicable in the complex case.

Zeros:

$$\sinh z = 0 \Longleftrightarrow z = n\pi i$$
$$\cosh z = 0 \Longleftrightarrow z = \left(n + \frac{1}{2}\right)\pi i$$

Prove it yourself.

Logarithmic Functions

Procedure:

Real case:
Let $e^w = x$
We define $w = \ln x$, ie, ln is the inverse function of exp.
Note that
1. $e^w > 0$ ∴ ln x is defined only for x > 0.
2. Both exp & ln are 1-1 functions so their inverse is well-defined.
3. $e^{\ln x} = x = \ln e^x$

Complex case:
Let $e^w = z$
We define $w = \ln z$ with the understanding that $e^w = e^{\ln z} = z$.

Since $e^{w+2n\pi i} = e^w$, the complex exp function is not 1-1 & doesn't have an inverse. One consequence is that $\ln e^z = \ln e^{z+2n\pi i}\ z + 2n\pi i$. One can also say that ln is a **multi-valued function**.
However, if w is restricted to a rectangular strip of width 2π along the imaginary axis, exp is 1-1 & the corresponding ln is its true inverse. Each of the w strip is called a **branch** of the function ln z.

Writing $z = r e^{i\theta} = r e^{i\theta+2n\pi i}$, we have $\ln z = \ln r + i(\theta + 2n\pi)$
By restricting θ to some interval of length 2π, each value of n gives a branch of $\ln z$.
The **principal branch** corresponds to choosing $\theta \in (-\pi, \pi)$ & is denoted by Ln.

From the viewpoint of the z plane, an entire z plane is mapped to only 1 branch of a multi-valued function. Each z plane is called a **Riemann sheet**. The number of these sheets is equal to the number of branches, which is infinite for $\ln z$. These sheets are laid sequentially on top of one another according to some rule. To facilitate the transition between sheets, we introduce a **branch cut** in each sheet & join the resultant edges to the adjacent sheets.

For ln, a branch cut can be any straight line starting from the origin toward infinity. A point which all branch cuts must go through is called a **branch point,** For ln z, this is the point $z = 0$.

Caution:

Churchill used log to denote complex logarithmic & ln the real one.

- **Properties:**

$$e^{\ln z} = z$$

$$\ln e^{z} = z + 2n\pi i$$

$$\ln z = \ln r + i(\theta + 2n\pi)$$
$$\operatorname{Ln} z = \operatorname{Ln} r + i\theta \qquad \theta \in (-\pi, \pi)$$

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$
$$= \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 2n\pi)$$

$$\frac{d \ln z}{dz} = \frac{1}{z}$$

- **Complex Exponents**

- **Definition:**

$$z^c \equiv e^{c \ln z}$$

Principal value: $z^c \equiv e^{c \operatorname{Ln} z}$

- **Formula:**

$$\frac{d}{dz} c^z = c^z \ln c$$

Proof:

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \ln c} = e^{z \ln c} \ln c = c^z \ln c$$

- **Inverse Trigonometric & Hyperbolic Functions**

- **Definition:**

Let $\sin w = z$, we define $\sin^{-1} z \equiv w$

Note: $\sin^{-1}$ is multi-valued. The above definition refers to the principal branch.

All other inverse functions under the present heading are defined analogeously.

- **Formulae:**

$$\sin^{-1} z = -i \ln\left(i z + \sqrt{1 - z^2}\right)$$

Proof:

Let $w = \sin^{-1} z$

$$z = \sin w = \frac{e^{i w} - e^{-i w}}{2 i} = \frac{y - \frac{1}{y}}{2 i} \qquad \text{where } y = e^{i w}$$

$$\longrightarrow \; y^2 - 2 i z y - 1 = 0$$

$$\therefore \qquad y = i z + \sqrt{-z^2 + 1} = e^{i w}$$

$$\longrightarrow \qquad i w = \ln\left(i z + \sqrt{-z^2 + 1} \right) = i \sin^{-1} z$$