4. Integrals

Functions with One Real Variable: *f* (t)

Definition

Let $f(t) = u(t) + i\,v(i)$ $\quad$ u, v real & piecewise continuous

$$I = \int_a^b dt\; f(t) \equiv \int_a^b dt\; u(t) + i \int_a^b dt\; v(t) = \operatorname{Re} I + i \operatorname{Im} I$$

Writing $f(t) = f(x(t), y(t))$, we see that $\int_a^b dt\; u(t)$ & $\int_a^b dt\; v(t)$ are real line integrals along a curve in the 2-D Euclidean plane parametrized by:

$x = x(t),\; y = y(t).$

All results for line integrals thus apply here.

Note however that f is periodic so that the **mean value theorem** must be applied with care.

Formulae

$$\int_a^b = \int_a^c + \int_c^b$$

$$\int dt\, f + \int dt\, g = \int dt\, (f + g)$$

AntiDerivatives

Let $F' = f$ $\quad$ (F is the antiderivative of f)

then $\quad \int_a^b dt\, f(t) = F(b) - F(a)$

$$\left| \int_a^b dt\, f(t) \right| \le \int_a^b dt\, |f(t)|$$

Coutours

Definitions:

An **arc** or **curve** is a mapping from $A \subset \mathbb{R}$ to $B \subset \mathbb{C}$

$c : A \longrightarrow B$

$\quad t \mapsto z(t) = x(t) + i\,y(t)$

A **simple (Jordan) curve** is a non-self-crossing curve, ie.

$t_1 \neq t_2 \longrightarrow z(t_1) \neq z(t_2)$

A **closed curve** is a curve whose end points coincide, ie.

$z(a) = z(b)$ $\quad$ where $\quad A = [a, b]$

If x', y' & hence z' exist, the curve is **differentiable**.
Note: x, y continuous $\longrightarrow$ x', y' exist.

If x', y' & hence z' exist & are continuous, the curve is **smooth**.

A **contour** is a piecewise smooth curve.

A **closed contour** is a piecewise smooth closed curve.

Let $z = c(t)$ be a differentiable curve.
The **arc length** $\|c\|$ of the curve c is defined as $|L|$ where

$$L = \int_c dz = \int_{z(a)}^{z(b)} dz = \int_a^b dt\,|z'|$$

$\|c\| = |L|$

The **arc length** of a piecewise continuous curve is the sum of the arc lengths of its continuous segments.

- **Jordan Curve Theorem:**

A simple closed curve divides the complex plane into a bounded **interior** and an unbounded **exterior** region.

- **Reparametrization:**

Let $z = c(t)$ with $t \in [a, b]$ be a curve.
Let $t = \phi(\tau)$ with $\tau \in [\alpha, \beta]$ & $a = \phi(\alpha)$, $b = \phi(\beta)$, $\tau = \phi$^$-1\,(t)$
then
$z = \tilde{c}(\tau) = c(\phi(\tau)) = c(t) = \tilde{c}\left(\phi^{-1}(\mathrm{t})\right)$ is a reparametrization of the curve $c(t)$.

$$\longrightarrow \quad I = \int_a^b dt\, f(t) = \int_\alpha^\beta d\tau \frac{\mathrm{d}\phi}{\mathrm{d}\tau} f(\phi(\tau))$$

Contour Integrals

Definition

Let C be a contour parametrized by $z = z(t)$ for $t \in [a, b]$
The contour integral of f over C is defined as:

$$\int_c dz\, f(z) = \int_a^b dt\, z'\, f(z(t))$$

Caution: $\boldsymbol{a \le} \mathrm{t} \boldsymbol{\le b}$ is always assumed in Churchill's text.

If C is a closed contour, the contour integral is usually written as

$$\oint_c dz\, f(z) \qquad \text{or simply} \quad \oint dz\, f(z)$$

with C assumed to be traversed in the counter-clockwise direction.

- **Formulae**

- $\displaystyle \int_{-c} dz\, f(z) = -\int_{c} dz\, f(z)$

Let C be parametrized by $z = z(t)$ for $t \in [a, b]$
Then – C denotes $z = z(t)$ for $t \in [b, a]$

$$\int_{-c} \mathbf{dz}\, f(z) = \int_b^b \mathrm{dt}\, z'\, f(z(t)) = -\int_a^b \mathrm{dt}\, z'\, f(z(t)) = -\int_c \mathbf{dz}\, f(z)$$

Note:
If we follow Churchill's convention, the above proof is more elaborate.
Let C be parametrized by $z = c(t)$ for $\boldsymbol{a \leq} \mathrm{t} \boldsymbol{\leq b.}$
This is reparametrized through $t = -\tau \longrightarrow \tau = -t$.
so that $z = c(t) = c(-\tau) = \tilde{c}(\tau)$
with $a \leq t \leq b \longrightarrow -b \leq \tau \leq -a$

Using $d\tau = -dt$, $\dfrac{d\,\tilde{c}(\tau)}{d\tau} = \dfrac{dt}{d\tau}\dfrac{d\,c(\mathrm{t})}{dt} = -\dfrac{dc}{dt}$, $f(\tilde{c}) = f(c)$, we have:

$$\int_{-C} dz\, f(z) = \int_{-b}^{-a} d\tau \frac{d\,\tilde{c}}{d\tau} f(\tilde{c}) = \int_b^a dt \frac{dc}{dt} f(c)$$

$$= -\int_a^b dt \frac{dc}{dt} f(c) = -\int_C dz\, f(z)$$

- $\displaystyle \int_{C} dz\, f(z) = \int_{C_1} dz\, f(z) + \int_{C_2} dz\, f(z)$

Let C_1 be $z = z_1(t)$ for $t \in [a, b]$
C_2 be $z = z_2(t)$ for $t \in [b, c]$

$$\boldsymbol{C} = C_1 + C_2 \equiv \begin{cases} C_1 & \text{for } t \in [a, b] \\ C_2 & \text{for } t \in [b, c] \end{cases}$$

then

$$\int_C dz\, f(z) = \int_{C_1} dz\, f(z) + \int_{C_2} dz\, f(z)$$

- $\displaystyle \left| \int_C dz\, f(z) \right| \leq \| c \| \cdot \max |f|$

By definition:

$$\left| \int_C dz\, f(z) \right| \leq \int_a^b dt\, |z'|\, |f(z(t))| \leq \max |f| \cdot \int_a^b dt\, |z'| = \max |f| \cdot \| c \| \quad *$$

AntiDerivatives

Definition:

F is the antiderivative of f if $\frac{dF}{dz} = F' = f$.

Theorem:

Let f be continuous on D, $c \subset D$, then

$$\exists\, F' = f$$
$$\Longleftrightarrow \quad \int_c dz\, f \quad \text{depends only on the end points of } c.$$
$$\Longleftrightarrow \quad \oint_c dz\, f = 0$$

Cauchy - Goursat Theorem

Let f be analytic on & interior to a closed path c which encloses a simply connected region,

$$\Longrightarrow \quad \oint_c dz\, f = 0$$

Cauchy's proof (*f ' continuous*):

Let $z = x + i y$, $f = u + i v$

$$\longrightarrow \quad dz = dx + i\, dy$$
$$dz\, f = dx\, u - dy\, v + i\,(dx\, v + dy\, u)$$
$$\oint dz\, f(z) = \oint_c \{dx\, u - dy\, v + i\,(dx\, v + dy\, u)\}$$

Using the Green's theorem (*f ' continuous* ⟶ partials of u & v continuous) :

$$\oint dz\, f(z) = \int_S dx\, dy \left\{ \left(-v_x - u_y\right) + i\left(u_x - v_y\right)\right\} = 0 \qquad \text{(C-RR)}$$

Goursat's proof:

Let the closed region on & inside c be denoted by R.
Let us cover R with a grid of squares each with length s. Parts of the squares on the boundary of R will be outside R. For these partial squares, only the portions inside R are included.

Lemma:
$\forall\, \epsilon > 0$, $\exists$ s $\ni$ for each square, say the jth one, $\exists$ a point z_j $\ni$

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon \;\forall \text{ z in the square.}$$

Proof:
f analytic ⟶ f' exists $\forall$ z ϵ R. Hence, $\forall\, \epsilon > 0$ & z_j, $\exists\, \delta_j > 0 \ni$

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon \forall \; |z - z_j| < \delta_j$$

The said grid can be constructed by choosing $\sqrt{2}$ s $\leq \max \delta_j$

Now, define the function:

$$\Delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)$$

$\longrightarrow$ $|\Delta_j(z)| < \epsilon$ $\forall$ z inside square j.

$$\Delta_j(z_j) = 0$$

$$f(z) = f(z_j) + [(\Delta_j(z) + f'(z_j)](z - z_j)$$

Let c_j be the closed contour around the perimeter of square j.

$\longrightarrow$ $\oint_{c_j} dz = 0$ and $\oint_{c_j} dz\, z = 0$

Hence: $\oint_{c_j} \text{dz}\, f(z) = \oint_{c_j} \text{dz}\, \Delta_j(z)(z - z_j)$

$\longrightarrow$ $\left|\oint_{c_j} \text{dz}\, f(z)\right| < \oint_{c_j} dz\, \epsilon |z - z_j| < \epsilon \cdot (4s + L_j) \cdot \sqrt{2}\, s$

where L_j is the length of the contour inside a partial square.

$\longrightarrow$ $\sum_j \oint_{c_j} dz\, f(z) = \oint_c dz\, f = \sum_j \oint_{c_j} dz\, \Delta_j(z)(z - z_j)$

$\therefore$ $\left|\oint_c \text{dz}\, f\right| < \sum_j \epsilon \cdot (4s + L_j) \cdot \sqrt{2}\, s < \sqrt{2}\, \epsilon (4S^2 + LS)$

where S is the length of some square enclosing R & L the length of c.
Since $\epsilon > 0$ can be arbitrarily small

$\therefore$ $\left|\oint_c dz\, f\right| = 0 \longrightarrow \oint_c dz\, f = 0$

- ## Connectedness:

Simply Connected region:

$\Longleftrightarrow$ every simple closed curve encloses only point in the region.

$\Longleftrightarrow$ every simple closed curve can be continuously deformed into a point.

Multiply Connected region:

$\Longleftrightarrow$ not simply connected.

■ Stokes theorem:

$$\int_S d\,\boldsymbol{S}\cdot\nabla\times\boldsymbol{V} = \oint_C d\,\boldsymbol{l}\cdot\boldsymbol{V}$$ where C encloses S & all partials of V are continuous.

$$\nabla\times V = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix} \longrightarrow (\nabla\times V)_z = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}$$

Caution: subscripts in this section denotes components of vectors, NOT their partials.
For S & hence C lying entirely in the x-y plane, we have

$$\int_S d\,x\,d\,y\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) = \oint_C \left(d\,x\,V_x + d\,y\,V_y\right)$$

Putting $V_x = P,\ \ V_y = Q$, we obtain the **Green's theorem**:

$$\int_S d\,x\,d\,y\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \oint_C (d\,x\,P + d\,y\,Q)$$

■ Cauchy Integral Formula

Let f be analytic on & interior to a closed path c which encloses a simply connected region,

$$\Longrightarrow \quad \oint_c d\,z\,\frac{f(z)}{z-z_0} = 2\pi i\,f(z_0)$$

where z_0 is interior to c.

■ Proof:

Let c_0 be a circle centered at z_0 with radius $r < \min|z - z_0|\ \forall\ z \epsilon c$.

f analytic inside c $\longrightarrow \frac{f(z)}{z-z_0}$ analytic between c & c_0.

C-GT $\longrightarrow \oint_c d\,z\,\frac{f(z)}{z-z_0} = \oint_{c_0} d\,z\,\frac{f(z)}{z-z_0}$

f analytic $\longrightarrow f$ continuous $\longrightarrow$

$$\forall\,\epsilon > 0,\ \exists\,\delta > 0 \ni |f(z) - f(z_0)| < \epsilon \quad \forall \quad |z - z_0| < \delta$$

Choose $r < \delta \longrightarrow$

$$\left|\oint_{c_0} d\,z\,\frac{f(z)-f(z_0)}{z-z_0}\right| \le \oint_{c_0} d\,z\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \le \oint_{c_0} d\,z\,\frac{\epsilon}{r} = \frac{\epsilon}{r}\int_0^{2\pi} d\,\phi\,i\,r\,e^{i\phi} = 0$$

Hence:

$$\oint_{c_0} d\,z\,\frac{f(z)}{z-z_0} = \oint_{c_0} d\,z\,\frac{f(z_0)}{z-z_0} = f(z_0)\,\frac{1}{r}\int_0^{2\pi} d\,\phi\,i\,r = 2\pi i\,f(z_0)$$

Derivatives of Analytic Functions

Theorems:

$$f\,'(z_0) = \frac{1}{2\pi i}\oint_c dz\, \frac{f(z)}{(z-z_0)^2}$$

$f = u + i v$ analytic $\Longrightarrow$ $f^{(n)}$ analytic $\forall$ n
$\Longrightarrow$ u, v have continuous partials of all orders.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c dz\, \frac{f(z)}{(z-z_0)^{n+1}}$$

proof:

CIF $\longrightarrow$ $f(z_0) = \dfrac{1}{2\pi i}\oint_c dz\, \dfrac{f(z)}{z-z_0}$

$$\longrightarrow f\,'(z_0) = \frac{1}{2\pi i}\oint_c dz\, f(z)\, \frac{d}{d z_0}\left(\frac{1}{z-z_0}\right)$$
$$= \frac{1}{2\pi i}\oint_c dz\, \frac{f(z)}{(z-z_0)^2}$$

Since f is analytic & $z \neq z_0$ in the integral, $f\,'$ exists.
The above is valid $\forall z_0$ inside C, which $f\,'$ is analytic inside C.
Repeating the argument gives the theorems for $f^{(n)}$.

Theorem:

f analytic in simply connected region D $\Longleftrightarrow$ f continuous & $\oint_c dz\, f = 0 \quad \forall\ c \in D$.

Liouville's Theorem

Cauchy's Inequality

Let f be analytic on & inside circle C denoted by $|z - z_0| = R$.
Let $M_R = \max |f|$ on C.

$$\Longrightarrow \quad \left| f^{(n)}(z_0) \right| \le \frac{n!\, M_R}{R^n}$$

Proof:

$$\left| f^{(n)}(z_0) \right| \le \frac{n!}{2\pi}\oint_c dz\, \frac{|f(z)|}{|z-z_0|^{n+1}} \le \frac{n!}{2\pi}\oint_c dz\, \frac{M_R}{R^{n+1}} = \frac{n!\, M_R}{R^n}$$

Liouville's Theorem

f entire & bounded $\Longrightarrow$ $f = constant.$

Proof:

f entire ∴ Cauchy's Inequality ($n = 1$) ⟶ $\left| f'(z) \right| \leq \frac{M_R}{R} \quad \forall\, z, R.$

f bounded ⟶ $\exists\, M \ni |f| \leq M \quad \forall$ z

Hence: $\left| f'(z) \right| \leq \frac{M}{R} \quad \forall\, R$ and M is independent of z.

In particular, as $R \longrightarrow \infty$, $|f'|$ and hence $f' = 0 \implies f$ = constant

- **Foundamental Theorem of Algebra**

Every polynomial has at least one root in ℂ:

$$\exists\ z_0 \in \mathbb{C} \ni P_N(z) = \sum_{n=0}^{N} a_n z^n = 0$$

Proof:

P_N is entire ⟶ $f = \frac{1}{P_N}$ is analytic except for points at which $P_N = 0$.

Also, f is bounded if $P_N \neq 0\ \forall$ z.

Hence, if $P_N \neq 0\ \forall$ z, f is entire & bounded.

By Liouville's Theorem, f = constant, which is a contradiction.

Corollary:

- **There are exactly n roots for P_n(z)**

Proof:

Foundamental Theorem of Algebra ⟶ $\exists\, z_0 \ni P_n(\text{z}) = (z - z_0) P_{n-1}(\text{z})$

Applying the theorem to P_{n-1}(z) ⟶ $\exists\, z_1 \ni P_n(\text{z}) = (z - z_0)(z - z_1) P_{n-2}(\text{z})$

Repeating the above procedure until P_0(z) = C = constant is reached, we have

$$P_n(\text{z}) = (z - z_0)(z - z_1)..(z - z_{n-1})\ \text{C}$$

Maximum Moduli

- **Gauss' Mean Value Theorem**

Let f be analytic on & within the circle C: $|z - z_0| \leq r$

$$\implies \quad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} d\theta\, f\left(z_0 + r\, e^{i\theta}\right)$$

- **Proof:**

CIF ⟶ $f(z_0) = \frac{1}{2\pi i} \oint_c dz\, \frac{f(z)}{z - z_0}$

Using $z = z_0 + r\, e^{i\theta} \longrightarrow \frac{dz}{z - z_0} = i\, d\theta$ does the job.

- **Lemma:**

Let f be analytic in $N_\epsilon(z_0)$
$|f(z)| \leq |f(z_0)| \quad \forall\ z \in N_\epsilon(z_0)$
$\Longrightarrow \quad f(z) = \text{constant} \quad \forall\ z \in N_\epsilon(z_0)$

- **Proof**

Gauss' Mean Value Theorem $\longrightarrow f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, f\left(z_0 + r\, e^{i\theta}\right)$

$$\to \quad \left|f(z_0)\right| \leq \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \left|f\left(z_0 + r\, e^{i\theta}\right)\right|$$

$|f(z)| \leq |f(z_0)| \longrightarrow \quad \left|f\left(z_0 + r\, e^{i\theta}\right)\right| \leq |f(z_0)|$

$$\longrightarrow \quad \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \left|f\left(z_0 + r\, e^{i\theta}\right)\right| \leq \frac{1}{2\pi}\int_0^{2\pi} d\theta\, |f(z_0)| = |f(z_0)|$$

$$\Longrightarrow \quad |f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \left|f\left(z_0 + r\, e^{i\theta}\right)\right|$$

$$\longrightarrow \quad \int_0^{2\pi} d\theta\, \left\{ |f(z_0)| - \left|f\left(z_0 + r\, e^{i\theta}\right)\right| \right\} = 0$$

$\therefore \quad |f(z_0)| = \left|f\left(z_0 + r\, e^{i\theta}\right)\right| = |f(z)| \quad \forall\ z \in N_\epsilon(z_0)$
$\longrightarrow \quad f(z) = f(z_0) \quad \forall\ z \in N_\epsilon(z_0)$

Theorem of Maximum Moduli:

f analytic in a simply connected domain $D \Longrightarrow f$ has no max in D

- **Proof:**

D simply connected $\longrightarrow \quad \forall\ z \in D$, $\exists$ a contour c joining it to z_0
Let d be the shortest distant from c to boundary of D.
Let $\{z_j\}$, with $z_N = z$, be a set of points on c with all neighboring distances < r < d.
A domain R is formed by placing a disk of radius r at each z_j.
Note that disks at adjacent points overlap.
Applying the above lemma sequentially to these disks, we see that f = constant if f has a maximum. Hence, f has no max if it is not a constant.

- **Corollary:**

f analytic in $D \Longrightarrow$ max of f are on boundary of D.

- **Proof:**

Since f has no maximum inside D, any max must reside on the boundary.