# 5. Series

## Convergence

### Definition

A **sequence** $\{z_n\}$ has a limit z

$\Longleftrightarrow \quad \lim_{n \to \infty} z_n = z$

$\Longleftrightarrow \quad \forall\, \epsilon > 0, \exists\, N \ni \ | z_n - z | < \epsilon\ \forall\, n > N$

$\Longleftrightarrow \quad \lim_{n \to \infty} x_n = x \quad \lim_{n \to \infty} y_n = y$

where $z_n = x_n + i\, y_n, \quad z = x + i\, y$

A **series** $\sum_{n=1}^{\infty} z_n$ converges to sum $S$

$\Longleftrightarrow \quad \lim_{N \to \infty} \sum_{n=1}^{N} z_n = S$

$\Longleftrightarrow \quad \lim_{N \to \infty} S_N = S$ where $S_N = \sum_{n=1}^{N} z_n$ is a partial sum.

$\Longleftrightarrow \quad \forall\, \epsilon > 0, \exists\, M \ni \ | S - S_N | < \epsilon \quad \forall\, N > M$

$\Longleftrightarrow$ The sequence of partial sums $\{S_N\}$ has a limit S.

$\Longleftrightarrow \quad \lim_{N \to \infty} \rho_N = 0$ where $\rho_N$ is the remainder

$\Longleftrightarrow \quad \forall\, \epsilon > 0, \exists\, M \ni \ | \rho_N | < \epsilon \quad \forall\, N > M$

$\Longleftrightarrow \quad \lim_{N \to \infty} \sum_{n=1}^{N} x_n = \lim_{N \to \infty} \operatorname{Re} S_N = \operatorname{Re} S$

$\lim_{N \to \infty} \sum_{n=1}^{N} y_n = \lim_{N \to \infty} \operatorname{Im} S_N = \operatorname{Im} S$

A series $\sum_{n=1}^{\infty} z_n$ **converges absolutely** to $S$

$\Longleftrightarrow \quad \lim_{N \to \infty} \sum_{n=1}^{N} | z_n | = S$

A series $\sum_{n=1}^{\infty} z_n(w)$ **converges uniformly** $\forall\, w \in D$

$\Longleftrightarrow \quad \forall\, w \in D,\ \epsilon > 0, \exists\, M$ independent of $w \ni$

$| S - S_N | < \epsilon \quad \forall\, N > M$

**Remainder function:**

$$\rho_N(w) \equiv \sum_{n=1}^{\infty} z_n(w) - \sum_{n=1}^{N} z_n(w) = \sum_{n=N+1}^{\infty} z_n(w)$$

- **Theorem**

$$\lim_{N\to\infty}\sum_{n=1}^{N} z_n = S \Longrightarrow \quad \lim_{n\to\infty} z_n = 0$$

$$\Longrightarrow \quad z_n \text{ bounded}$$

**Absolute convergence ⟹ convergence**

## Taylor Series

- **Taylor's Theorem**

$f$ analytic in $N_{R_0}(z_0)$

$$\Longrightarrow \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \;\; \forall\, z \in N_{R_0}(z_0) \;\text{ or }\; |z - z_0| < R_0$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i}\oint_c ds\, \dfrac{f(s)}{(s - z_0)^{n+1}}$ ( **Taylor Series** )

and C is a closed contour enclosing $z_0$ and lying entirely within $N_{R_0}(z_0)$.

- **Proof**

Consider a small circle $\gamma$ centered at $z$ & lying entirely in $N_{R_0}(z_0)$.

CIF $\longrightarrow$ $$f(z) = \frac{1}{2\pi i}\oint_\gamma ds\, \frac{f(s)}{s-z}$$

Since $\frac{f(s)}{s-z}$ is analytic $\forall\ s \neq z$, $\gamma$ can be deformed into a circle C centered at $z_0$ with radius $R \ni\ |z - z_0| < R < R_0$ , ie C encloses both $z$ and $z_0$:

$$f(z) = \frac{1}{2\pi i}\oint_c ds\, \frac{f(s)}{s-z}$$

Let $t = s - z_0$, $w = z - z_0$, we have

$$f(z) = \frac{1}{2\pi i}\oint_c dt\, \frac{f(t+z_0)}{t-w} \qquad \text{where} \quad |t| > |w|$$

Now

$$\sum_{n=0}^{N} z^n \frac{1-z^{N+1}}{1-z} \quad \longrightarrow \quad \frac{1}{1-z} = \sum_{n=0}^{N} z^n + \frac{z^{N+1}}{1-z}$$

$$\therefore \qquad \frac{1}{t-w} = \frac{1}{t}\cdot\frac{1}{1-\frac{w}{t}} = \sum_{n=0}^{N}\frac{w^n}{t^{n+1}} + \frac{\left(\frac{w}{t}\right)^{N+1}}{t-w}$$

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{N} w^n \oint_c dt\, \frac{f(t+z_0)}{t^{n+1}} + \frac{1}{2\pi i}\oint_c dt\, \frac{f(t+z_0)}{t-w}\left(\frac{w}{t}\right)^{N+1}$$

DF $\longrightarrow$ $$= \sum_{n=0}^{N}\frac{f^{(n)}(z_0)}{n!} w^n + \rho_{N+1}(w)$$

where the remainder $\rho_{N+1}(w) = \frac{1}{2\pi i}\oint_c dt\, \frac{f(t+z_0)}{t-w}\left(\frac{w}{t}\right)^{N+1}$ .

Now:

$$|\rho_{N+1}(w)| = \frac{1}{2\pi}\left|\oint_c dt\, \frac{f(t+z_0)}{t-w}\left(\frac{w}{t}\right)^{N+1}\right| \leq \frac{1}{2\pi} 2\pi R\, K_{N+1} \qquad *$$

where

$$K_{N+1} = \max\left|\frac{f(t+z_0)}{t-w}\left(\frac{w}{t}\right)^{N+1}\right| = \frac{F}{T}\left|\frac{w}{t}\right|^{N+1}$$

where $F = \max|f|$ & $T = \min|t\text{-}w| \neq 0$ on C.

Since $|t| > |w|$ while $F$, $T$ are independent of $N$

$K_{N+1}$ and hence $\rho_{N+1} \longrightarrow 0$ as $N \longrightarrow \infty$

$$\therefore \qquad f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!} w^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

The restriction on C to enclose both $z$ and $z_0$ can be lifted by noting that $\frac{f(s)}{(s-z_0)^{n+1}}$ is analytic $\forall\ s \neq z_0$ so that $\oint_c ds\, \frac{f(s)}{(s-z_0)^{n+1}}$ has the same value for all C that lies within $N_{R_0}(z_0)$ and encloses $z_0$.

## Laurent Series

### Laurent's Theorem

$f$ analytic in $D$ : $R_1 < | z - z_0 | < R_2$

$$\Longrightarrow \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad \forall z \in D$$

where $a_n = \dfrac{1}{2\pi i} \oint_c ds \, \dfrac{f(s)}{(s - z_0)^{n+1}}$ ( **Laurent Series** )

and C is a closed contour enclosing $z_0$ and lying entirely within $D$.

### Proof

As in the Taylor series case, we begin with a small circle $\gamma$ centered at $z$ & lying entirely in $N_{R_0}(z_0)$.

$$\text{CIF} \longrightarrow \qquad f(z) = \frac{1}{2\pi i} \oint_\gamma ds \, \frac{f(s)}{s - z}$$

Unlike the Taylor series case, $\gamma$ can only be deformed into a pair of circles $C_<$ and $C_>$ centered at $z_0$ with radius $R_>$ & $R_<$ ∍

$$R_1 < R_< < | z - z_0 | < R_> < R_2$$

$$\longrightarrow \quad f(z) = \frac{1}{2\pi i} \left( \oint_{c_>} - \oint_{c_<} \right) ds \, \frac{f(s)}{s - z}$$

Let $t = s - z_0$, $w = z - z_0$, we have

$$f(z) = \frac{1}{2\pi i} \left( \oint_{c_>} - \oint_{c_<} \right) dt \, \frac{f(t + z_0)}{t - w}$$

where $| t | > | w |$ for $c_>$ & $| t | < | w |$ for $c_<$

For $c_>$, the result is the same as the Taylor series.

For $c_<$, we write the series in terms of powers of $\frac{t}{w}$:

$$\longrightarrow \qquad f(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} (z - z_0)^n \oint_c ds \, \frac{f(s)}{(s - z_0)^{n+1}}$$

where $c = c_>$ and $c_<$ for $n \geq$ and $< 0$, respectively.

Note that even for $n \geq 0$, the integrals do not equal to derivatives of $f$ because $f$ is not analytic at $z_0$.

### Power Series

$$P(z - z_0) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

## Absolute & Uniform Convergence

### Theorem

$P(z - z_0)$ converges at $z = z_1$

$\Longrightarrow \quad P(z - z_0)$ converges absolutely $\forall z$ ∍ $| z - z_0 | < | z_1 - z_0 |$

- **Proof**

Let $w = z - z_0$, $W = z_1 - z_0 \longrightarrow |w| < |W|$

$$P(W) = \sum_{n=0}^{\infty} a_n W^n \text{ converges} \longrightarrow \exists M \ni |a_n W^n| \le M \quad \forall n.$$

$$\left|a_n w^n\right| = \left|a_n W^n \left(\frac{w}{W}\right)^n\right| \le M \rho^n \qquad \text{where } \rho = \left|\frac{w}{W}\right| < 1$$

Now: $\sum_{n=0}^{\infty} M \rho^n$ converges for $\rho < 1$.

Comparison test $\longrightarrow \sum_{n=0}^{\infty} \left|a_n w^n\right|$ converges, ie. $P(w)$ converges absolutely.

- **Theorem**

$P(z - z_0)$ converges $\forall z \ni |z - z_0| < R$

$\Longrightarrow \quad P(z - z_0)$ converges uniformly $\forall z \ni |z - z_0| < R$

- **Proof**

Let $w = z - z_0$, $w_1 = z_1 - z_0$, $w_2 = z_2 - z_0 \ni |w| \le |w_1| < |w_2| < R$

$P(w_2)$ converges $\longrightarrow$ $P(w_1)$ converges absolutely

$\therefore \quad \forall \epsilon > 0, \exists M \ni |\rho_N(w_1)| < \epsilon \quad \forall N > M$

where $$\rho_N(w_1) = \left|\rho_N(w_1)\right| = \sum_{n=N+1}^{\infty} |a_n w^n|$$

Now:

$$\left|\rho_N(w)\right| \le \sum_{n=N+1}^{\infty} \left|a_n w^n\right| = \sum_{n=N+1}^{\infty} |a_n w_1^n| \left|\frac{w}{w_1}\right|^n \le \sum_{n=N+1}^{\infty} \left|a_n w_1^n\right| \le \rho_N(w_1)$$

Hence: $\forall \epsilon > 0, \exists M \ni |\rho_N(w)| \le \rho_N(w_1) < \epsilon \quad \forall N > M$

where $M$ is independent of $w$

$\longrightarrow \quad P(w)$ converges uniformly

- **Corollary**

$P(z - z_0)$ converges $\forall z \ni |z - z_0| < R$

$\Longrightarrow \quad P(z - z_0)$ is a continuous function $\forall z \ni |z - z_0| < R$

- **Proof**

Let $w = z - z_0$, $w_1 = z_1 - z_0$.

$$P(w) = \sum_{n=0}^{N} a_n w^n + \rho_N(w) = P_N(w) + \rho_N(w)$$

$$P(w_1) = \sum_{n=0}^{N} a_n {w_1}^n + \rho_N(w_1) = P_N(w_1) + \rho_N(w_1)$$

Now: $P_N(w)$ is a polynomial & therefore continuous

$\longrightarrow \quad \forall\, \epsilon > 0, \exists\, \delta > 0 \ni \ |P_N(w_1) - P_N(w)| < \epsilon \quad \forall \quad |w_1 - w| < \delta$

$\therefore \quad |P(w_1) - P(w)| = |P_N(w_1) - P_N(w) + \rho_N(w_1) - \rho_N(w)|$

$\leq |P_N(w_1) - P_N(w)| + |\rho_N(w_1) - \rho_N(w)|$

$\longrightarrow \quad \forall\, \epsilon > 0, \exists\, \delta > 0 \ni |P(w_1) - P(w)| < \epsilon + |\rho_N(w_1) - \rho_N(w)| \quad \forall \quad |w_1 - w| < \delta$

Since $P(w)$ & $P(w_1)$ converges uniformly

$\longrightarrow \quad \forall\, \beta > 0, \exists\, M \ni \ |\rho_N(w)| < \beta \ \& \ |\rho_N(w_1)| < \beta \quad \forall\, N > M$

$\longrightarrow \quad |\rho_N(w_1) - \rho_N(w)| \leq |\rho_N(w_1)| + |\rho_N(w)| < 2\beta$

$\longrightarrow \quad \forall\, \epsilon + 2\beta > 0, \ \exists\, \delta > 0 \ni |P(w_1) - P(w)| < \epsilon + 2\beta \quad \forall \quad |w_1 - w| < \delta$

Hence: $P(w)$ is continuous.

## Integration & Differentiation

- **Theorem**

$P(z - z_0)$ converges on & within C.

$g(z)$ continuous on C.

$$\Longrightarrow \quad \int_c \mathrm{dz}\, g(z)\, P(z - z_0) = \sum_{n=0}^{\infty} a_n \int_c \mathrm{dz}\, g(z)\,(z - z_0)^n$$

- **Proof**

Each term in $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is continuous $\longrightarrow$ term by term integration is valid.

The question is whether the infinite sum converges.

Writing: $$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{N} a_n (z - z_0)^n + \rho_{N+1}(z),$$

we need to show $$\int_c d\,z\, g(z)\, \rho_{N+1}(z) \xrightarrow[N \to \infty]{} 0 .$$

Now, $P(z - z_0)$ converges on $C$

$\longrightarrow \quad \left|\rho_{N+1}(z)\right| \xrightarrow[N \to \infty]{} 0$

$\therefore \quad \left|\int_c d\,z\, g(z)\, \rho_{N+1}(z)\right| \leq \|c\| \cdot \max |g| \cdot |\rho_{N+1}(z)| \xrightarrow[N \to \infty]{} 0$

QED.

- **Corollary**

$P(z - z_0)$ converges $\forall\, z \ni \ |z - z_0| < R$

$\Longrightarrow \quad P(z - z_0)$ is analytic $\forall\, z \ni |z - z_0| < R$

**Proof**

Each term in $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is analytic ⟶ so is the sum if it converges.

- **Theorem**

$P(z-z_0)$ converges $\forall\, z \ni |z-z_0| < R$

$$\Longrightarrow \quad \frac{d\,P(z)}{d\,z} = \sum_{n=1}^{\infty} a_n\, n\,(z-z_0)^{n-1}$$

- **Proof**

Each term in $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is continuous ⟶ term by term differentiation is valid.

The issue is once again convergence.

This can be done in a number of ways. Here, we shall show that it is absolutely convergent by the ratio test which states that ( see Arfken )

$$\sum_{n=0}^{\infty} u_n \text{ converges if } \lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$$

In our case, $u_n = |a_n|\, n\, r^{n-1}$ where $r = |z-z_0|$

Since $P$ converges absolutely, we have $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|\; r} > 1$

$$\therefore \quad \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|\; r} \cdot \frac{n}{n+1} = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|\; r} > 1$$

QED.

Another proof is to make use of the Cauchy derivative formula:

$$f\,'(z_0) = \frac{1}{2\pi i} \oint d\,z\, \frac{f(z)}{(z-z_0)^2}$$

Thus:

$$\frac{d\,P(z)}{d\,z} = \frac{1}{2\pi i} \oint d\,s\, \frac{P(s)}{(s-z)^2}$$

$$= \sum_{n=0}^{\infty} a_n \frac{1}{2\pi i} \oint d\,s\, \frac{(z-z_0)^n}{(s-z)^2} \qquad \text{( term by term integration )}$$

$$= \sum_{n=0}^{\infty} a_n \frac{d}{d\,z} (z-z_0)^n = \sum_{n=1}^{\infty} a_n\, n\,(z-z_0)^{n-1}$$

QED

## Uniqueness

- **Theorem**

$f(z) = P(z-z_0) \qquad \forall\, z \ni |z-z_0| < R$

$\Longrightarrow \quad P(z-z_0)$ is the Taylor's series representation of $f(z)$.

**Proof**

$$f(z-z_0) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$\longrightarrow \quad \oint dz\,(z-z_0)^{-m} f(z-z_0) = \sum_{n=0}^{\infty} a_n \oint dz\,(z-z_0)^{n-m} = 2\pi i\, a_{m-1}$$

$$\therefore \quad a_m = \frac{1}{2\pi i}\oint dz\,(z-z_0)^{-(m+1)} f(z-z_0) = \frac{1}{m!} f^{(m)}(z-z_0)$$

ie. it's a Taylor series.

- **Theorem**

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \forall\ z \ni R_1 < |z - z_0| < R_2$

$\Longrightarrow \quad \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is the Laurent series representation of $f(z)$.

- **Proof**

Same as the Taylor series case.

- **Multiplication & Division**

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \sum_{m=0}^{\infty} b_m (z-z_0)^m = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

where $\quad c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$

- **Leibniz's Rule**

$$[f\,g]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

- ## Summary

For a power series:

1. Convergence at point $\longrightarrow$ absolute convergence inside disk
   $\longrightarrow$ uniform convergence
   $\longrightarrow$ continuous
2. Term by term integration & differentiation legal.

1+2. $\longrightarrow$ Unique & equals to Taylor series $\longleftrightarrow$ analytic.

**Importance of Uniform Convergence**:

ref: R.G.Bartle, "The Elements of Real Analysis", 2nd ed., sec 17.
T.M.Apostle, "Calculus", Vol 1, 2nd ed., Chap 11.

1. The limit of a convergent sequence of continuous functions need not be continuous.
   eg. $f_n(x) = x^n$ for $x \in [0, 1]$
   $f_n$ is continous & convergent $\forall\ x \in [0, 1]$ but
   $$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{for } 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$
   is not continuous at $x = 1$.
2. uniform convergence $\Longrightarrow$ continuity $\Longrightarrow$ Term by term differentiation & integration. ( interchange of sum & differentiation / integration operations ).
3. Not every
4. **lemma:**
   A sequence $f_n(x)$ does not converge uniformly in $D$
   $\Longleftrightarrow\ \exists\, \epsilon > 0 \ni \left| f_{n_k}(x_k) - f(x_k) \right| \ge \epsilon$
   where $x_k$ is a sequence in $D$ & $f_{n_k}$ is a subsequence of $f_n$.

## Analytic Continuation

### Definition

Consider 2 domains $D_1$ and $D_2$ with $D_1 \cap D_2 \neq \Phi$
Let $f_1$ & $f_2$ be analytic in $D_1$ & $D_2$, respectively.
If $f_1 = f_2$ in $D_1 \cap D_2$
$\Longleftrightarrow$ $f_2$ is the **analytic continuation** of $f_1$ into $D_2$
$$\Longleftrightarrow \quad F(z) = \begin{cases} f_1 & z \in D_1 \\ f_2 & z \in D_2 \end{cases}$$
is the **analytic continuation** of either $f_1$ or $f_2$ into $D_1 \cup D_2$.
$f_1$ and $f_2$ are the **elements** of $F$.