6. Residues & Poles

■ Singularities

■ Definitions

Singularities

Points at which functions are not analytic.

■ Singular point

singular point

z_0 is a **singular point** of $f(z)$

$\Longleftrightarrow$ **1.** $f(z_0)$ is not analytic.

2. $\forall\ N(z_0)\ \exists\ z \in N(z_0)\ \ni\ f(z)$ is analytic.

■ Isolated singular point

isolated sinular point

z_0 is an **isolated singular point** of $f(z)$

$\Longleftrightarrow$ **1.** $f(z_0)$ is a singular point.

2. $\exists\ N(z_0)\ \ni\ f(z)$ is analytic in $N(z_0)\setminus\{z_0\}$

$$\Longleftrightarrow \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad 0 < |z - z_0| < R$$

■ Types of singular points

Non-Isolated Singular Point (Branch Point; Accumualtion point of a set of isolated singular points)

Isolated singular point (Pole; Removable singular point; Essential singular point)

■ Poles

pole

z_0 is a **pole of order** m of $f(z)$

$$\Longleftrightarrow \quad f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n \qquad a_{-m} \neq 0$$

z_0 is a **simple pole** $\Longleftrightarrow$ $m = 1$

Examples

1. $\dfrac{z+1}{z^3\left(z^2+1\right)}$ has 3 simple poles at $z = 0,\ \pm i$

2. $\dfrac{1}{\sin \frac{\pi}{z}}$

Singular points (simple poles) at $\sin \dfrac{\pi}{z} = 0$, i.e. $\dfrac{\pi}{z} = n\,\pi \rightarrow\ z = \dfrac{1}{n}$.

Since $n = 0$ is the accumulation point of $\left\{\dfrac{1}{n}\right\}$, all singular points except $n = 0$ are isolated

- **Removable singular point**

removable singular point

z_0 is a **removable singular point** of $f(z)$

$\Longleftrightarrow$ **1.** $f(z_0)$ is an isolated singular point.

2. $f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n$

Example:

$z_0 = 0$ is a **removable singular point** of $f(z) = \dfrac{1 - \cos z}{z^2}$

proof:

Using $\cos z = \sum_{n=0}^{\infty} (-)^n\,\dfrac{z^{2n}}{(2n)!}$, we have

$$
\begin{aligned}
f(z) &= \frac{1}{z^2}\left\{ 1 - \sum_{n=0}^{\infty} (-)^n\,\frac{z^{2n}}{(2n)!}\right\} \\
&= -\frac{1}{z^2}\sum_{n=1}^{\infty} (-)^n\,\frac{z^{2n}}{(2n)!} \\
&= \sum_{n=1}^{\infty} (-)^{n-1}\,\frac{z^{2(n-1)}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-)^n\,\frac{z^{2n}}{(2n+2)!}
\end{aligned}
$$

- **Essential singular point**

essential singular point

z_0 is an **essential singular point** of $f(z)$

$\Longleftrightarrow$ **1.** $f(z_0)$ is an isolated singular point.

2. $f(z) = \sum_{n=-\infty}^{\infty} a_n\,(z - z_0)^n$ $\qquad a_{-\infty} \neq 0$

Example:

$z = 0$ is an essential singular point of $\exp \dfrac{1}{z}$

proof:

$$\exp\frac{1}{z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n \qquad 0 < \left|\frac{1}{z}\right| < \infty \quad \text{or} \quad 0 < |z| < \infty$$

$$= \sum_{n=-\infty}^{0} \frac{1}{|n|\,!} z^n$$

- **Principal Part**

Let z_0 be an isolated singular point of $f(z)$

ie. $\quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad 0 < |z - z_0| < R$

The **singular part** of f is defined as

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n = \sum_{n=1}^{\infty} a_{-n} \frac{1}{(z - z_0)^n}$$

- **Meromorphic Functions**

meromorphic

f is **meromorphic** in D

$\Longleftrightarrow \quad f$ is analytic in D except for a finite number of poles.

- **Residues**

- **Definition**

residues def

Let z_0 be an isolated singularity of f.

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ converges for $0 < |z - z_0| < R$

$\longrightarrow \quad \operatorname*{Res}_{z=z_0} f(z) \equiv a_{-1}$

$= $ coefficient of $\frac{1}{z - z_0}$ of the Laurent series expansion about z_0.

$$= \frac{1}{2\pi i} \oint_c \mathrm{dz}\, f(z)$$

where c encloses z_0 & lies entirely within $0 < |z - z_0| < R$

- **Example**

eg 1

Let $f(z) = \frac{1}{z(z-2)^4}$. Find $\operatorname*{Res}_{z=2} f$.

Solution:

A) 1st, expand $\frac{1}{z}$ about $z=2$:

$$\frac{1}{z}=\frac{1}{z-2+2}=\frac{1}{z-2}\cdot\frac{1}{1+\frac{2}{z-2}}=\frac{1}{z-2}\sum_{n=0}^{\infty}(-)^n\left(\frac{2}{z-2}\right)^n$$

where $0\le\left|\frac{2}{z-2}\right|<1$ or $\infty>|z-2|>2$

$$\longrightarrow \quad f(z)=\sum_{n=0}^{\infty}(-)^n\frac{2^n}{(z-2)^{n+5}}=\sum_{n=5}^{\infty}(-)^{n-5}\frac{2^{n-5}}{(z-2)^n}$$

The residue cannot be calculated from this expansion since the convergent region is not of the form $0<|z-z_0|<R$.

B) 1st, expand $\frac{1}{z}$ about $z=2$:

$$\frac{1}{z}=\frac{1}{z-2+2}=\frac{1}{2}\cdot\frac{1}{1+\frac{z-2}{2}}=\frac{1}{2}\sum_{n=0}^{\infty}(-)^n\left(\frac{z-2}{2}\right)^n$$

where $0\le\left|\frac{z-2}{2}\right|<1$ or $0\le|z-2|<2$

$$\longrightarrow \quad f(z)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-)^n}{2^n}(z-2)^{n-4}=\sum_{n=-4}^{\infty}\frac{(-)^{n+4}}{2^{n+5}}(z-2)^n$$

where $0<|z-2|<2$

$$\therefore \quad \operatorname*{Res}_{z=2} f=\frac{(-)^3}{2^4}=-\frac{1}{16}$$

C) Let C be $|z-2|=r\longrightarrow 0$; $z=2+r\,e^{i\theta}$

$$\operatorname*{Res}_{z=2} f=\frac{1}{2\pi i}\oint_c dz\,\frac{1}{z(z-2)^4}$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}d\theta\; i e^{i\theta} r\,\frac{1}{\left(2+r e^{i\theta}\right)r^4 e^{4i\theta}}$$

$$=\frac{1}{2\pi}r^{-3}\int_0^{2\pi}d\theta\;\frac{e^{-3i\theta}}{2+r e^{i\theta}}$$

$$=\frac{1}{4\pi}r^{-3}\int_0^{2\pi}d\theta\; e^{-3i\theta}\sum_{n=0}^{\infty}\left(\frac{-r e^{i\theta}}{2}\right)^n \qquad (r/2<1)$$

$$=\frac{1}{4\pi}\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n r^{n-3}\int_0^{2\pi}d\theta\; e^{(n-3)i\theta}$$

$$=\frac{1}{4\pi}\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n r^{n-3}\,2\pi\,\delta_{n,3}$$

$$=-\frac{1}{16}$$

D) Using the formula for poles of order 4:

$$\operatorname*{Res}_{z=2} f=\frac{1}{3!}\frac{d^3}{dz^3}\left(\frac{1}{z}\right)_{z=2}=\frac{1}{3!}(-)^3\,3!\,\frac{1}{2^4}=-\frac{1}{16}$$

Cauchy's Residue Theorem

Cauchy's residue thm

$$\oint_C dz\, f(z) = 2\pi i \sum_{n=1}^{N} \operatorname*{Res}_{z=z_n} f(z)$$

where f is meromorphic , all singular points z_n lie inside c & N is finite.

proof

Let c_n be simple closed contour enclosing only one singular point z_n.

C-GT $\longrightarrow$

$$\oint_C dz\, f(z) = \sum_{n=1}^{N} \oint_{C_n} dz\, f(z) = 2\pi i \sum_{n=1}^{N} \operatorname*{Res}_{z=z_n} f(z)$$

Example

eg 2

1. Let C be $|z| = 2$, $f = \dfrac{5z-2}{z(z-1)}$

$$\longrightarrow \quad \oint_c dz\, \frac{5z-2}{z(z-1)} = 2\pi i \left(\operatorname*{Res}_{z=0} f + \operatorname*{Res}_{z=1} f \right)$$

$$= 2\pi i \left\{ \left(\frac{5z-2}{z-1} \right)_{z=0} + \left(\frac{5z-2}{z} \right)_{z=1} \right\}$$

$$= 2\pi i\,(2+3) = 10\pi i$$

2. Let C be $|z-2| = 1$, $f(z) = \dfrac{1}{z(z-2)^4}$

Since only pole $z = 2$ is inside C, we have

$$\oint_c dz\, \frac{1}{z(z-2)^4} = 2\pi i \operatorname*{Res}_{z=2} f = -\frac{\pi i}{8}$$

Theorem $\left(z \to \frac{1}{z}\right)$

residue thm 2

$$\oint_C dz\, f(z) = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right]$$

where f is meromorphic & **all** singular points of f are inside C.

- **proof**

Let β : $|z| = R$ be a circle which encloses C.

Taylor ' s theorem $\longrightarrow$

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad \text{where } R < |z| < \infty \quad (f \text{ analytic})$$

and $\quad c_n = \dfrac{1}{2\pi i} \oint_{\gamma} dz \, \dfrac{f(z)}{z^{n+1}}$

where γ is any contour that encloses β.

Let $w = \dfrac{1}{z}$

$\longrightarrow \qquad dz = \dfrac{-1}{w^2} dw$

$$f(z) = f\left(\frac{1}{w}\right) = F(w) = \sum_{n=0}^{\infty} c_n \left(\frac{1}{w}\right)^n \qquad \text{where } 0 < \left|w\right| < \frac{1}{R}$$

with $\quad c_n = \dfrac{-1}{2\pi i} \oint_{\gamma} dw \, w^{n-1} f\left(\dfrac{1}{w}\right) = \dfrac{-1}{2\pi i} \oint_{\gamma} dw \, w^{n-1} F(w)$

Now:

$$\oint_c dz\, f(z) = -\oint_c dw \frac{1}{w^2} F(w)$$

$$= 2\pi i \operatorname*{Res}_{w=0} \left\{ \frac{1}{w^2} F(w) \right\} = 2\pi i \operatorname*{Res}_{w=0} \left\{ \frac{1}{w^2} f\left(\frac{1}{w}\right) \right\}$$

Note: the contour that encloses the region $R < |z| < \infty$ or $0 < \left|w\right| < \dfrac{1}{R}$ is $-c$. Also, F is analytic except at $w = 0$. Since w is a dummy variable, we can write

$$\oint_C dz\, f(z) = 2\pi i \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right]$$

- **Example**

Let C be $|z| = 2$, $f = \dfrac{5z-2}{z(z-1)}$

Poles $z = 0, 1$ are all inside C.

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{\frac{5}{z} - 2}{\frac{1}{z} \cdot \left(\frac{1}{z} - 1\right)} = \frac{5 - 2z}{z(1-z)}$$

$\longrightarrow \qquad \operatorname*{Res}_{z=0} \left[\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right) \right] = 5$

$$\oint_C dz \frac{5z-2}{z(z-1)} = 10\pi i \qquad \text{(cf example)}$$

- **Theorem (multiple-order poles)**

residue thm 3

z_0 is a pole of order m of f

$$\Longleftrightarrow \quad f(z) = \frac{\phi(z)}{(z - z_0)^m} \qquad \text{where } \phi \text{ is analytic \& } \phi(z_0) \neq 0$$

$$\Longrightarrow \quad \operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

- **proof**

ϕ analytic & $\phi(z_0) \neq 0$

Taylor Theorem $\longrightarrow \quad \phi(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with $a_0 \neq 0$

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-m} = \sum_{n=-m}^{\infty} c_n (z - z_0)^n$$

where $c_n = a_{n+m}$ & $c_{-m} = a_0 \neq 0$.

which is simply the definition for z_0 to be a pole of order m of f.

$$\operatorname*{Res}_{z=z_0} f(z) = c_{-1} = a_{m-1} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

- **Example**

1. Let $f(z) = \dfrac{z^3 + 2z}{(z - i)^3}$

$\longrightarrow \quad z = i$ is a pole of order 3

$$\therefore \quad \operatorname*{Res}_{z=i} f(z) = \frac{1}{2!} \frac{d^2}{dz^2} \left(z^3 + 2z\right) \Big|_{z=i} = 3i$$

2. Let $f(z) = \dfrac{\sinh z}{z^4}$

Since $\sinh 0 = 0$, $z = 0$ is not a pole of order 4.

Using

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

we have

$$f = \sum_{n=0}^{\infty} \frac{z^{2n-3}}{(2n+1)!}$$

which means $z = 0$ is pole of order 3.

However: $\quad \dfrac{1}{3!} \dfrac{d^3}{dz^3} \sinh z \Big|_{z=0} = \dfrac{1}{6}$

while $\quad c_{-1} = \dfrac{1}{3!} = \dfrac{1}{6}$

The incorrect usage of the derivative formula gives the correct residue here.

3. Let $f(z) = \dfrac{1}{z\,(e^z - 1)}$

Poles are at $z = 0$ & solutions of $e^z = 1$ ($z = 2\,n\,\pi\,i$).

Since $e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}$

$$\therefore \quad f(z) = \frac{1}{\sum_{n=1}^{\infty} \frac{z^{n+1}}{n!}} = \frac{1}{z^2} \cdot \frac{1}{\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}}$$

Hence $z = 0$ is a pole of order 2.

Now:

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = 1 + \frac{z}{2!} + \ldots$$

$$\frac{1}{\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}} = 1 - \frac{z}{2!} + \ldots$$

$$\therefore \quad \operatorname*{Res}_{z=0} f(z) = -\frac{1}{2}$$

Another way to do it is by differentiation:

$$\frac{d}{d\,z} \frac{1}{\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}} = -\frac{1}{\left(\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}\right)^2} \sum_{m=1}^{\infty} \frac{m\,z^{m-1}}{(m+1)!}$$

$$\operatorname*{Res}_{z=0} f(z) = \left(\frac{d}{dz} \frac{1}{\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}} \right)_{z=0} = -\frac{1}{1} \cdot \frac{1}{2!} = -\frac{1}{2}$$

- **Theorem (zeros-poles)**

residue thm 4

Let p, q be analytic at z_0.

Let z_0 be a zero of order m of q & $p(z_0) \neq 0$

$\Longrightarrow \quad \frac{p}{q}$ has a pole of order m at z_0

- **proof**

z_0 is a zero of order m of q

$\longrightarrow \quad q(z) = (z - z_0)^m\,\alpha(z)$ where α is analytic & $\alpha(z_0) \neq 0$

Hence $\frac{p}{q} = \frac{\phi}{(z - z_0)^m}$ where $\phi = \frac{p}{\alpha}$ & $\phi(z_0) \neq 0$

$\Longrightarrow \quad \frac{p}{q}$ has a pole of order m at z_0

- **Example**

$f(z) = z\,(e^z - 1)$ has a zero of order 2 at $z = 0$

$\longrightarrow \quad \dfrac{1}{f} = \dfrac{1}{z\,(e^z - 1)}$ has a pole of order 2 at $z = 0$

- **Corollary**

residue cor4

Let $p,\ q$ be analytic at z_0.

$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \& \quad q'(z_0) \neq 0$

$\Longrightarrow$ z_0 is a simple pole of p/q with

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

- **proof**

This is simply the $m = 1$ case of theorem.

- **Example**

1. Let $f(z) = \dfrac{\cos z}{\sin z}$

Poles are at $z = n\,\pi$.

$(\sin z)' = \cos z \quad \longrightarrow \quad (\sin z)'|_{z=n\pi} = (-1)^n \neq 0$

$\therefore$ these poles are simple.

$$\operatorname*{Res}_{z=n\pi} f(z) = \left(\frac{\cos z}{(\sin z)'}\right)_{z=n\pi} = \left(\frac{\cos z}{\cos z}\right)_{z=n\pi} = 1$$

2. Let $f(z) = \dfrac{z}{z^4 + 4}$

Poles are at $z^4 + 4 = 0$

$\longrightarrow \quad z = (-4)^{\frac{1}{4}} = \sqrt{2}\; e^{i\frac{\pi}{4} + i\,n\frac{\pi}{2}}$

For $n = 0$ $\quad z = \sqrt{2}\left(\dfrac{\sqrt{2}}{2} + i\,\dfrac{\sqrt{2}}{2}\right) = 1 + i$

$$\operatorname*{Res}_{z=1+i} f(z) = \left(\frac{z}{4\,z^3}\right)_{z=1+i} = \frac{1}{4\,(1+i)^2} = \frac{(1-i)^2}{16} = -\frac{i}{8}$$

- **Riemann's Theorem**

Riemann thm

Let f be analytic & bounded in $N(z_0)\backslash\{z_0\}$ & $f(z_0)$ is not analytic.

$\Longrightarrow$ z_0 is a removable singularity.

- **proof**

f has an isolated singularity at z_0 $\longrightarrow$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad \forall\, z \in N_\epsilon(z_0) \backslash \{z_0\}$$

where $\quad b_n = \dfrac{1}{2\pi i} \oint_c d z \dfrac{f(z)}{(z - z_0)^{-n+1}} \qquad c \subset N_\epsilon(z_0)$

f bounded $\longrightarrow$ $\exists\, M \ni |f| \leq M \qquad \forall\, z \in N_\epsilon(z_0) \backslash \{z_0\}$

Choosing c to be a circle centered at z_0 with radius $\rho < \epsilon$

$$\longrightarrow \quad b_n \leq \frac{1}{2\pi} 2\pi\rho \frac{M}{\rho^{-n+1}} = \rho^n M$$

Letting $\rho \longrightarrow 0$, we have $b_n = 0$, which means z_0 is a removable singularity.

- **Picard's Theorem**

Let z_0 be an essential singularity of f

$\Longrightarrow$ In every neighborhood of z_0, f assumes every finite value, except for one possible value, an infinite number of times.

- **Example**

Let $f(z) = e^{\frac{1}{z}}$

$\longrightarrow$ $z = 0$ is an essential singularity

Points at which $f = -1$ are solutions of

$$\frac{1}{z} = i(2n + 1)\pi$$

ie. $\quad z = \dfrac{-i}{(2n + 1)\pi}$

There are infinitely many of these points in any neighborhood of $z = 0$.

- **Casorati - Weierstrass Theorem**

Casorati - Weierstrass Theorem

Let z_0 be an essential singularity of f

w_0 is any complex number.

$\Longrightarrow$ $\forall\, \epsilon > 0 \ \exists\, z \in N(z_0) \backslash \{z_0\} \ni |f(z) - w_0| < \epsilon$

- **proof**

z_0 is an isolated singularity
$\longrightarrow \quad \exists\, N_\delta(z_0) \ni \quad f$ is analytic in $N_\delta(z_0)\backslash\{z_0\}$
Assume $\quad |\, f(z) - w_0 \,| \geq \epsilon$
$\longrightarrow \quad g(z) = \dfrac{1}{f(z) - w_0}$ is analytic & bounded in $N_\delta(z_0)\backslash\{z_0\}$
$\therefore \quad z_0$ is a removable singularity of g
$\longrightarrow \quad g$ can be defined to be analytic at z_0
Now: $f(z) = \dfrac{1}{g(z)} + w_0$ in $N_\delta(z_0)$
If $g(z_0) \neq 0 \longrightarrow \quad f(z_0) = \dfrac{1}{g(z_0)} + w_0$ is finite
$\longrightarrow \quad z_0$ is a removable singularity of f
If $g(z_0) = 0 \longrightarrow \quad z_0$ must be an isolated zero of finite order of g
$\longrightarrow \quad z_0$ is a pole of finite order of f.
$\therefore \quad |\, f(z) - w_0 \,| < \epsilon$ if z_0 is an essential singularity of f

Zeros

- **Definition**

zero def

z_0 is a **zero of order *m*** of an analytic function f
$\Longleftrightarrow \quad f^{(n)}(z_0) = 0 \;\; \forall\, n = 0, 1. .. m - 1 \qquad f^{(m)}(z_0) \neq 0$

- **Lemma**

zeroes lem1

z_0 is a **zero of order *m*** of an analytic function f
$\Longleftrightarrow \quad f(z) = (z - z_0)^m\, g(z)$ where g is analytic & $g(z_0) \neq 0$

- **proof**

$\Longrightarrow$

f is analytic at z_0 $\longrightarrow$

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(z_0)\,\frac{(z-z_0)^n}{n!} \qquad \text{(Taylor's Theorem)}$$

z_0 is a zero of order ***m*** of f

$$\longrightarrow \qquad f(z) = \sum_{n=m}^{\infty} f^{(n)}(z_0)\,\frac{(z-z_0)^n}{n!}$$

Let $k = n - m \longrightarrow \qquad n = k + m \qquad k = 0 \,..\infty$

$$f(z) = \sum_{k=0}^{\infty} f^{(k+m)}(z_0)\,\frac{(z-z_0)^{k+m}}{(k+m)!}$$

$$= (z-z_0)^m \sum_{k=0}^{\infty} f^{(k+m)}(z_0)\,\frac{(z-z_0)^k}{n!}$$

$$= (z-z_0)^m\, g(z)$$

where $\qquad g(z) = \sum_{k=0}^{\infty} f^{(k+m)}(z_0)\,\frac{(z-z_0)^k}{n!} = \frac{f(z)}{(z-z_0)^m}$

g is analytic $\forall\, z \neq z_0$ because it's the ratio of analytic functions f and $(z-z_0)^m$. Furthermore, $g(z_0) = f^{(m)}(z_0)$ exists & is non-zero. Hence, g is analytic wherever f is.

$\Longleftarrow$

g is analytic at z_0 $\longrightarrow$

$$g(z) = \sum_{n=0}^{\infty} g^{(n)}(z_0)\,\frac{(z-z_0)^n}{n!} \qquad \text{(Taylor's Theorem)}$$

$$\therefore \qquad f(z) = (z-z_0)^m\, g(z) = \sum_{n=0}^{\infty} g^{(n)}(z_0)\,\frac{(z-z_0)^{n+m}}{n!}$$

Using $\dfrac{d^k}{dz^k} z^l = \dfrac{l\,!}{(l-k)!} z^{l-k}$

$$\longrightarrow \qquad f^{(k)}(z) = \sum_{n=0}^{\infty} g^{(n)}(z_0)\,\frac{(z-z_0)^{n+m-k}}{n!} \cdot \frac{(n+m)!}{(n+m-k)!}$$

$$= (z-z_0)^{m-k} \sum_{n=0}^{\infty} g^{(n)}(z_0)\,\frac{(z-z_0)^n}{n!} \cdot \frac{(n+m)!}{(n+m-k)!}$$

Hence: $\qquad f^{(k)}(z_0) = 0 \qquad \forall\, k < m$

$\qquad f^{(m)}(z_0) = g(z_0) \neq 0$

$\longrightarrow \qquad z_0$ is a zero of order ***m*** of f

- **Example**

eg 3

Let $f(z) = z\,(e^z - 1)$

$\longrightarrow \quad f(0) = 0$

$\quad f'(z) = e^z - 1 + z\,e^z \quad \longrightarrow \quad f'(0) = 0$

$\quad f''(z) = 2\,e^z + z\,e^z \quad \longrightarrow \quad f''(0) = 2 \neq 0$

$\therefore \quad z = 0$ is a zero of order 2.

Writing $\quad f(z) = z^2\,g(z)$

$$\longrightarrow \quad g(z) = \frac{e^z - 1}{z}$$

where $\quad g(0) = 1 \neq 0$

- **Theorem**

zero thm1

Let f be continuous & $f(z_0) \neq 0$

$\Longrightarrow \quad \exists\, N(z_0) \ni f(z) \neq 0 \quad \forall\, z \in N(z_0)$

- **proof:**

f continuous at z_0

$$\longrightarrow \quad \exists\, \delta > 0 \ni \left| f(z) - f(z_0) \right| < \frac{1}{2} \left| f(z_0) \right| \qquad \forall\, z \in N_\delta(z_0)$$

if $\exists\, z_1 \in N_\delta(z_0) \ni f(z_1) = 0$ we have $\left| f(z_0) \right| < \frac{1}{2} \left| f(z_0) \right|$

This is impossible if $f(z_0) \neq 0$.

Hence $\quad f(z) \neq 0 \qquad \forall\, z \in N_\delta(z_0)$

- **Theorem**

zeroes lem2

Let f be analytic & $f(z_0) = 0$

If $\nexists\, N(z_0) \ni f(z) \equiv 0 \quad \forall\, z \in N(z_0)$

$\Longrightarrow$
1. z_0 is an zero of finite order.
2. $\exists\, N(z_0) \ni f(z) \neq 0 \qquad \forall\, z \in N(z_0) \backslash \{z_0\}$

(z_0 is an **isolated zero** of finite order)

- **proof:**

See ex 58.13

- **Corollary**

zero cor1

Let f be analytic & $f(z_0) = 0$

If $\nexists\, N(z_0) \ni f(z) \neq 0 \qquad \forall\, z \in N(z_0) \backslash \{z_0\}$

$\Longrightarrow \quad \exists\, N(z_0) \ni f(z) \equiv 0 \quad \forall\, z \in N(z_0)$

- **proof:**

Apply the rule: $A \longrightarrow B \quad \Longrightarrow \quad !B \longrightarrow !A$
to the last theorem.

- **Corollary**

zero cor2
Let f be analytic in D.
Let $f(z) = 0 \ \forall z \in A \subset D$ where A can be an arc
$\Longrightarrow \quad f(z) = 0 \ \forall z \in D$

- **proof**

This is just a re-phrasing of the last corollary

- **Theorem**

zero thm2
Let f be analytic in D & $f(z_0) = 0$ where $z_0 \in D$.
$\Longrightarrow$ either $f(z_0)$ is an **isolated zero** of finite order.
or $f(z) \equiv 0 \quad \forall z \in D$

- **proof:**

This is simply a summary of the foregoing results.

- **Corollary**

zero cor3
Let f be analytic in D
$\Longrightarrow \quad f$ is uniquely determined by its values in a sub-domain of D

- **proof:**

Consider 2 analytic functions f, g & their difference $h = f - g$.
If $h = 0$ in $A \subset D$ then $h = 0$ in D.

- **Example**

Let $f(z) = \cos^2 z + \sin^2 z - 1$ (f is entire)
Since $\cos^2 x + \sin^2 x = 1$ for real x
$\therefore \quad f(z) = 0$ for the entire real axis
$\Longrightarrow \quad f(z) = 0$ for all z.
ie. $\cos^2 z + \sin^2 z = 1$

Exercises