7. Applications of Residues

Improper Integrals

■ Improper Integrals of the 1st kind

■ Definition

improper integral 1st kind

Let a be a finite constant.

The following integrals are all **improper integrals of the 1st kind :**

$$\int_a^\infty d\,x\; f(x) \equiv \lim_{R\longrightarrow\infty} \int_a^R d\,x\; f(x)$$

$$\int_{-\infty}^a d\,x\; f(x) \equiv \lim_{R\longrightarrow\infty} \int_{-R}^a d\,x\; f(x)$$

$$\int_{-\infty}^\infty d\,x\; f(x) \equiv \lim_{R\longrightarrow\infty} \int_a^R d\,x\; f(x) + \lim_{R\longrightarrow\infty} \int_{-R}^a d\,x\; f(x)$$

PV 1st kind

The **Cauchy principal value** of the last integral is defined as:

$$\text{PV} \int_{-\infty}^\infty d\,x\; f(x) \equiv \lim_{R\longrightarrow\infty} \int_{-R}^R d\,x\; f(x)$$

These integrals are said to **converge** if the corresponding limits exist, otherwise, they **diverge**.

Note: It's possible that $\text{PV} \int_{-\infty}^\infty d\,x\; f(x)$ exists but $\int_{-\infty}^\infty d\,x\; f(x)$ diverges. eg. $f(x) = x$.

■ Theorem

Let f be even, ie, $f(-x) = f(x)$

$$\Longrightarrow \quad \text{PV} \int_{-\infty}^\infty d\,x\; f(x) = 2 \int_0^\infty d\,x\; f(x)$$

- **proof**

Assuming the integrals involved exist, we have:

$$\int_{-R}^{R} d\,x\; f(x) = \int_{0}^{R} d\,x\; f(x) + \int_{-R}^{0} d\,x\; f(x)$$

$$f(-x) = f(x) \quad \longrightarrow \qquad \int_{-R}^{0} d\,x\; f(x) \underset{x\to -x}{=} \int_{R}^{0} d(-x)\; f(-x) = \int_{0}^{R} d\,x\; f(x)$$

$$\therefore \qquad \int_{-R}^{R} d\,x\; f(x) = 2\int_{0}^{R} d\,x\; f(x)$$

$$\text{PV}\int_{-\infty}^{\infty} d\,x\; f(x) \equiv \lim_{R\longrightarrow\infty}\int_{-R}^{R} d\,x\; f(x) = \lim_{R\longrightarrow\infty} 2\int_{0}^{R} d\,x\; f(x) = 2\int_{0}^{\infty} d\,x\; f(x) = \int_{-\infty}^{\infty} d\,x\; f(x)$$

- **Theorem 1**

thm 1

Let f be meromorphic & has no poles on the real axis.

Let C_R be the upper half circle of radius R & centered at $z = 0$.

If $\lim_{R\to\infty}\left|\int_{C_R} d\,z\; f(z)\right| = 0$

$$\Longrightarrow \qquad \text{PV}\int_{-\infty}^{\infty} d\,x\; f(x) = 2\,\pi\,i\sum_{k}\text{Res}\; f(z_k)$$

where the sum includes all poles in the upper plane.

- **proof**

thm 1 proof

For R finite, we have

$$\int_{-R}^{R} d\,x\; f(x) + \int_{C_R} d\,z\; f(z) = 2\,\pi\,i\sum_{k}\text{Res}\; f(z_k)$$

where the sum includes only poles inside the half disk.

$$\lim_{R\to\infty}\left|\int_{C_R}\text{dz}\; f(z)\right| = 0 \longrightarrow \qquad \lim_{R\to\infty}\int_{C_R}\text{dz}\; f(z) = 0$$

$$\therefore \qquad \text{PV}\int_{-\infty}^{\infty} d\,x\; f(x) = \lim_{R\longrightarrow\infty}\int_{-R}^{R} d\,x\; f(x) = 2\,\pi\,i\sum_{k}\text{Res}\; f(z_k)$$

where the sum includes all poles in the upper plane.

- **Example**

Find $\quad I = \int_{0}^{\infty} d\,x\; f(x) \qquad$ where $f(x) = \dfrac{x^2}{x^6 + 1}$

Solution:

$$f(-x) = f(x) \quad \longrightarrow \qquad I = \frac{1}{2} \int_{-\infty}^{\infty} d\,x\, f(x) = \frac{1}{2}\,\text{PV} \int_{-\infty}^{\infty} d\,x\, f(x)$$

Let $\quad f(z) = \dfrac{z^2}{z^6 + 1}$

$$\left| f(z) \right| \underset{|z|>>1}{\longrightarrow} \frac{1}{|\,z\,|^4} \underset{|z|\to\infty}{\longrightarrow} 0$$

Poles of f are at $\qquad z^6 + 1 = 0$

$$\longrightarrow \qquad z_n = (-1)^{\frac{1}{6}} = e^{\frac{(i\pi + i\,2\,n\,\pi)}{6}}$$

$$z_n = e^{i\frac{\pi}{6}},\ e^{i\frac{\pi}{6}+i\frac{\pi}{3}},\ e^{i\frac{\pi}{6}+i\frac{2\pi}{3}}, \qquad (\text{ upper plane })$$

$$e^{i\frac{\pi}{6}+i\pi},\ e^{i\frac{\pi}{6}+i\frac{4\pi}{3}},\ e^{i\frac{\pi}{6}+i\frac{5\pi}{3}} \qquad (\text{ lower plane })$$

Let $\quad J = \oint_c \text{dz}\, f(z)$

where c is the infinite half circle in the upper half plane.

$$\longrightarrow \qquad J = 2\,I + \int_{c_R} \text{dz}\, f(z) = 2\,\pi\,i \sum_n \text{Res}\, f(z_n)$$

where n runs over poles in the upper half plane &
the c_R is the arc of the infinite half circle.

$$\left| f(z) \right| \underset{|z|\to\infty}{\longrightarrow} 0 \qquad \Longrightarrow \qquad \int_{c_R} d\,z\, f(z) = 0$$

Since all poles are simple:

$$\text{Res}\, f(z_n) = \left(\frac{z^2}{6\,z^5} \right)_{z=z_n} = \frac{1}{6\,{z_n}^3}$$

$$\therefore \qquad I = \frac{1}{6}\,\pi\,i \left\{ e^{-i\frac{\pi}{2}} + e^{-i\frac{\pi}{2} - i\pi} + e^{-i\frac{\pi}{2} - 2\,i\,\pi} \right\}$$

$$= \frac{1}{6}\,\pi\,i\,\{-i + i - i\} = \frac{\pi}{6}$$

■ Improper Integrals of the 2nd kind

■ Definition

Let f be singular at a, the following integrals are all **improper integrals of the 2nd kind:**

$$\int_a^b d\,x\, f(x) \equiv \lim_{t \longrightarrow a_+} \int_t^b d\,x\, f(x) \qquad \text{where } b > a$$

$$\int_c^a d\,x\, f(x) \equiv \lim_{t \longrightarrow a_-} \int_c^t d\,x\, f(x) \qquad \text{where } a > c$$

$$\text{PV} \int_c^b d\,x\, f(x) \equiv \lim_{t \longrightarrow a_+} \int_t^b d\,x\, f(x) + \lim_{t \longrightarrow a_-} \int_c^t d\,x\, f(x) \qquad \text{where } b > a > c$$

These integrals are said to **converge** if the corresponding limits exist, otherwise, they **diverge**.

Indented Path

indented path

Consider a simple pole at x_0 between 2 points a & b on the real axis.
The **upper indented path** $I_U(a, x_0, b; \rho)$ from a to b goes along the real axis from a to $x_0 - \rho$, along the upper half circle centered at x_0 to $x_0 + \rho$, and finally, along the real axis to b.
Likewise, **the lower indented path** $I_L(a, x_0, b; \rho)$ goes from a to b via the lower half circle.

Hence:

$$\int_{I_U} d\,z\; f(z) = \int_a^{x_0-\rho} d\,x\; f(x) + \int_{C_{\rho U}} d\,z\; f(z) + \int_{x_0+\rho}^{b} d\,x\; f(x)$$

$$\underset{\rho\longrightarrow 0}{\longrightarrow} \text{PV} \int_a^b d\,x\; f(x) + \int_{C_{\rho U}} d\,z\; f(z)$$

where $C_{\rho U}$ denotes the upper half circle of radius ρ centered at x_0.

x_0 is a simple pole

$$\longrightarrow \quad f(z) = \sum_{n=-1}^{\infty} a_n\,(z - x_0)^n = \sum_{n=-1}^{\infty} a_n\,\rho^n\,e^{i\,n\,\phi}$$

where $\quad z - x_0 = \rho\,e^{i\,\phi}$

$$\longrightarrow \quad \int_{C_{\rho U}} d\,z\; f(z) = \sum_{n=-1}^{\infty} a_n\,i\,\rho^{n+1} \int_\pi^0 d\,\phi\; e^{i\,(n+1)\,\phi}$$

$$= \sum_{n=-1}^{\infty} a_n\,\frac{\rho^{n+1}}{n+1}\left\{\,1 - e^{i\,(n+1)\,\pi}\,\right\}$$

$$\underset{\rho\longrightarrow 0}{\longrightarrow} a_{-1}\,i \int_\pi^0 d\,\phi = -\,i\,\pi\,a_{-1} = -\,i\,\pi \underset{z\longrightarrow x_0}{\text{Res}}\; f(z)$$

Hence $\quad \int_{I_U} d\,z\; f(z) \underset{\rho\longrightarrow 0}{\longrightarrow} \text{PV} \int_a^b d\,x\; f(x) - \,i\,\pi \underset{z\longrightarrow x_0}{\text{Res}}\; f(z)$

Similarly, one can show that

$$\int_{I_L} d\,z\; f(z) \underset{\rho\longrightarrow 0}{\longrightarrow} \text{PV} \int_a^b d\,x\; f(x) + \,i\,\pi \underset{z\longrightarrow x_0}{\text{Res}}\; f(z)$$

Theorem 2

thm 2

Let f be meromorphic with simple poles on the real axis.
Let C_R be the upper half circle of radius R & centered at $z = 0$.
If $\lim_{R\to\infty} \left| \int_{C_R} \text{dz}\; f(z) \right| = 0$

$$\Longrightarrow \quad \text{PV} \int_{-\infty}^{\infty} d\,x\; f(x) = 2\,\pi\,i \sum_k \text{Res}\; f(z_k) + \pi\,i \sum_l \text{Res}\; f(z_l)$$

where the k sum includes all poles in the upper plane excluding the real axis.
& the l sum includes all simple poles on the real axis.

proof

Apply the same procedure as in the proof of theorem1 using either the upper or lower indented path.

- **Example 1**

Let $f(x) = \frac{\sin x}{x}$

Find $\quad I = \int_0^\infty \text{dx}\, f(x)$

Solution:

$f(-x) = f(x) \quad \longrightarrow \quad I = \frac{1}{2}\,\text{PV} \int_{-\infty}^{\infty} d\,x\, f(x)$

Let $\quad g(x) = \frac{1}{x}\, e^{i\,x}$

$\longrightarrow \quad f(x) = \text{Im}\, g(x)$

Let $\quad g(z) = \frac{1}{z}\, e^{i\,z}$

Consider contour c composed of the real axis & the upper half circle C_R $(R \to \infty)$.

$$\oint_c d\,z\;\, g(z) = \int_{C_R} d\,z\;\, g(z) + \int_{-R}^{R} d\,x\, g(x)$$

Now:

$$|g(z)| \underset{|z|>>1}{\longrightarrow} \frac{1}{|\,z\,|} \underset{|z|\to\infty}{\longrightarrow} 0 \quad \Longrightarrow \quad \int_{C_R} d\,z\;\, g(z) \underset{R\to\infty}{\longrightarrow} 0$$

Since pole $z = 0$ of $g(z)$ is on the real axis segment of c :

$$\oint_c d\,z\;\, g(z) = \pi\, i\;\, \underset{z=0}{\text{Res}}\;\, \frac{1}{z}\, e^{i\,z} = \pi\, i$$

$$= \;\text{PV} \int_{-\infty}^{\infty} d\,x\, g(x)$$

$$\therefore \qquad I = \frac{1}{2}\,\text{Im}\;\,\text{PV} \int_{-\infty}^{\infty} d\,x\, g(x) = \frac{\pi}{2}$$

- **Example 2**

Let $f(x) = \frac{1}{\left(x^2 + 4\right)^2} \ln x$

Find $\quad I = \int_0^\infty d\,x\, f(x)$

Solution:

Let $f(z) = \dfrac{1}{\left(z^2 + 4\right)^2} \operatorname{Log} z$ (Principal branch $-\pi < \theta < \pi$)

Consider contour c composed of 2 upper half-circles C_R ($R \to \infty$) & C_ϵ ($\epsilon \to 0$) , which are centered at the origin & joined along the real axis.

On C_R:

$$\left|\int_{C_R} dz\, f(z)\right| \underset{R\to\infty}{\longrightarrow} \pi R \cdot \frac{1}{R^4} \ln R \underset{R\to\infty}{\longrightarrow} 0$$

On C_ϵ:

$$\left|\int_{C_\epsilon} dz\, f(z)\right| \underset{\epsilon\to 0}{\longrightarrow} \left|\int_{C_\epsilon} dz\, \frac{1}{16} \operatorname{Log} z\right| \underset{\epsilon\to 0}{\longrightarrow} \frac{\pi}{16} \epsilon \cdot \ln \epsilon \underset{\epsilon\to 0}{\longrightarrow} 0$$

On the left leg of the real axis joining C_R & C_ϵ: $\theta = \pi$, $z = r\, e^{i\pi}$

$$\int_L dz\, f(z) = \int_R^\epsilon dr\, e^{i\pi} \cdot \frac{1}{\left(r^2+4\right)^2} \{\ln r + i\pi\}$$

$$= \int_\epsilon^R dr\, \frac{1}{\left(r^2+4\right)^2} \{\ln r + i\pi\}$$

On the right leg of the real axis joining C_R & C_ϵ: $\theta = 0$, $z = r$

$$\int_R dz\, f(z) = \int_\epsilon^R dr\, \frac{1}{\left(r^2+4\right)^2} \ln r$$

$$\longrightarrow \quad \left\{\int_L + \int_R\right\} dz\, f(z) = 2\int_\epsilon^R dr\, \frac{1}{\left(r^2+4\right)^2} \ln r + \int_\epsilon^R dr\, \frac{1}{\left(r^2+4\right)^2}\, i\pi$$

$$\underset{R\to\infty,\, \epsilon\to 0}{\longrightarrow} 2I + i\pi \frac{J}{2}$$

where $\quad J = \displaystyle\int_{-\infty}^{\infty} dr\, \frac{1}{\left(r^2+4\right)^2}$

J can be evaluated by contour integral with c' being the upper half circle:

$$\oint_{c'} dz\, \frac{1}{\left(z^2+4\right)^2} = J + \int_{C_R} dz\, \frac{1}{\left(z^2+4\right)^2}$$

$$= \mathrm{J}$$

$$= 2\pi i \operatorname*{Res}_{z=2i} \frac{1}{\left(z^2+4\right)^2}$$

$$= 2\pi i \left(\frac{d}{dz} \frac{1}{(z+2i)^2}\right)_{z=2i}$$

$$= \ 2\pi i\,(-2)\,\frac{1}{(4i)^3}$$

$$= \ \frac{\pi}{16}$$

$$\therefore \qquad \oint_c dz\, f(z) = 2I + i\,\frac{\pi^2}{32}$$

$$= \ 2\pi i \operatorname*{Res}_{z=2i} \frac{1}{\left(z^2 + 4\right)^2} \operatorname{Log} z$$

$$= \ 2\pi i \left(\frac{d}{dz}\,\frac{\operatorname{Log} z}{(z+2i)^2}\right)_{z=2i}$$

$$= \ 2\pi i \left\{(-2)\,\frac{1}{(4i)^3}\operatorname{Log} 2i + \frac{1}{(4i)^2}\,\frac{1}{2i}\right\}$$

$$= \ \frac{\pi}{16}\operatorname{Log} 2i - \frac{\pi}{16}$$

Using $\quad \operatorname{Log} 2i = \ln 2 + \operatorname{Log} i$

$$\operatorname{Log} i = \ \operatorname{Log} e^{i\frac{\pi}{2}} = i\,\frac{\pi}{2}$$

we have $\quad 2I + i\,\dfrac{\pi^2}{32} = \dfrac{\pi}{16}\left(\ln 2 + i\,\dfrac{\pi}{2}\right) - \dfrac{\pi}{16}$

$\longrightarrow \quad I = \dfrac{\pi}{32}\,(\ln 2 - 1)$

■ Improper Integrals involving Sines & Cosines

$$\int_{-\infty}^{\infty} dx\, f(x)\sin a x \qquad \text{or} \qquad \int_{-\infty}^{\infty} dx\, f(x)\cos a x \qquad (a > 0)$$

where f is meromorphic with no poles on the real axis.

■ Technique

Start with $\quad \displaystyle\int_{-\infty}^{\infty} d\,x\, f(x)\, e^{i a x}$ & apply theorem 1

For $a > 0$, $\left|e^{i a z}\right|^2 = e^{i a z - i a z^*} = e^{-2 a y} \xrightarrow[y\to\infty]{} 0$,

except for the real axis where $y = 0$.

Hence, for $\left| f \right| \xrightarrow[|z|\to\infty]{} 0$, theorem 1 is applicable in the upper plane.

A more precise statement is provided by the Jordan's lemma.

■ Jordan's Inequality

jordan's ineq

$$\int_0^{\pi} d\theta\, e^{-R\sin\theta} < \frac{\pi}{R} \qquad\qquad R > 0$$

- **proof**

By comparing the graphs of the curves $y = \sin\theta$ & $y = 2\,\frac{\theta}{\pi}$, one sees that

$$\sin\theta \geq 2\,\frac{\theta}{\pi} \qquad \forall \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\longrightarrow \qquad e^{-R\sin\theta} \leq e^{-2R\frac{\theta}{\pi}} \qquad \text{if} \quad R > 0$$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} d\,\theta\, e^{-R\sin\theta} \leq \int_0^{\frac{\pi}{2}} d\,\theta\, e^{-2R\frac{\theta}{\pi}} = \frac{-\pi}{2R}\left(e^{-R} - 1\right) < \frac{\pi}{2R}$$

Using $\phi = \theta - \frac{\pi}{2}$, we have

$$\int_{\frac{\pi}{2}}^{\pi} d\theta\, e^{-R\sin\theta} = \int_0^{\frac{\pi}{2}} d\phi\, e^{-R\cos\phi}$$

Since $\cos\phi \geq 1 - 2\,\frac{\phi}{\pi} \qquad \forall \quad 0 \leq \phi \leq \frac{\pi}{2}$

$$\longrightarrow \qquad e^{-R\cos\phi} \leq e^{-R+2R\frac{\phi}{\pi}} \qquad \text{if} \quad R > 0$$

$$\therefore \qquad \int_0^{\frac{\pi}{2}} d\,\theta\, e^{-R\cos\theta} \leq \int_0^{\frac{\pi}{2}} d\,\theta\, e^{-R+2R\frac{\theta}{\pi}} = \frac{\pi}{2R}\, e^{-R}\left(e^{R} - 1\right) < \frac{\pi}{2R}$$

Hence $$\int_0^{\pi} d\,\theta\, e^{-R\sin\theta} = \int_0^{\frac{\pi}{2}} d\,\theta\, e^{-R\sin\theta} + \int_{\frac{\pi}{2}}^{\pi} d\,\theta\, e^{-R\sin\theta} < \frac{\pi}{R}$$

- **Jordan's Lemma**

jordan's lemma

Let f be analytic in the upper half plane above a semi half circle $z = R_0\, e^{i\theta}$, $0 \leq \theta \leq \pi$.

Let C_R be any semi half circle $z = R\, e^{i\theta}$, $0 \leq \theta \leq \pi$ & $R > R_0$.

If, for given R, $\exists\, M_R > 0 \ni |f(z)| \leq M_R \quad \forall \; z \in C_R$

& $\quad M_R \longrightarrow 0 \qquad$ as $R \longrightarrow \infty$

$$\Longrightarrow \qquad \lim_{R\longrightarrow\infty} \int_{C_R} dz\, f(z)\, e^{i a z} = 0$$

- **proof**

On C_R: $e^{i a z} = \exp\left(i a R e^{i\theta}\right)$

$$\longrightarrow \qquad \left| e^{i a z}\right|^2 = \exp\left(i a R e^{i\theta} - i a R e^{-i\theta}\right) = \exp(-2 a R \sin\theta)$$

$$\left| e^{i a z} \right| = e^{-a R \sin\theta}$$

$$\therefore \qquad \left|\int_{C_R} d z\, f(z)\, e^{i a z}\right| = \left| \int_0^\pi d\theta\; i R e^{i\theta}\, f\left(R e^{i\theta}\right) \exp\left(i a R e^{i\theta}\right) \right|$$

$$\leq \left| \int_0^\pi d\theta \;\; R \;\; M_R\, e^{-a R \sin\theta} \right|$$

$$< R\, M_R \frac{\pi}{a R} = M_R \frac{\pi}{a}$$

Since $M_R \longrightarrow 0$ as $R \longrightarrow \infty$, we have

$$\lim_{R\longrightarrow\infty} \int_{C_R} d z\, f(z)\, e^{i a z} = 0$$

- **Example 1**

Let $f(x) = \dfrac{\cos 3 x}{\left(x^2+1\right)^2}$. Find $I = \displaystyle\int_{-\infty}^{\infty} d x\, f(x)$

Solution:

$\cos 3 x = \operatorname{Re} e^{i 3 x}$

$$\longrightarrow \qquad I = \operatorname{Re} \int_{-\infty}^{\infty} d x \frac{e^{i 3 x}}{\left(x^2+1\right)^2} = \operatorname{Re} \int_{-\infty}^{\infty} d x\, g(x)$$

where $g(x) = \dfrac{e^{i 3 x}}{\left(x^2+1\right)^2}$

Analytic continuation: $g(z) = \dfrac{e^{i 3 z}}{\left(z^2+1\right)^2} = \dfrac{e^{i 3 z}}{(z+i)^2\,(z-i)^2}$

Consider contour c composed of the real axis & the upper half circle C_R ($R \to \infty$).

$$\oint_c d z\;\; g(z) = \int_{C_R} d z\;\; g(z) + \int_{-R}^{R} d x\, g(x)$$

Now:

$$\left| g(z)\right| \longrightarrow \frac{1}{|z|^4} \underset{|z|\to\infty}{\longrightarrow} 0 \qquad \Longrightarrow \qquad \int_{C_R} d z\;\; g(z) \underset{R\to\infty}{\longrightarrow} 0$$

$$\oint_c dz\ g(z) = 2\pi i \operatorname*{Res}_{z=i} \frac{e^{i3z}}{(z^2+1)^2} = 2\pi i \frac{d}{dz}\left(\frac{e^{i3z}}{(z+i)^2}\right)_{z=i}$$

$$= 2\pi i \left\{3i \frac{e^{i3z}}{(z+i)^2} - 2\frac{e^{i3z}}{(z+i)^3}\right\}_{z=i}$$

$$= 2\pi i\ e^{-3}\left\{3i\frac{1}{(2i)^2} - 2\frac{1}{(2i)^3}\right\}$$

$$= 2\pi i\ e^{-3}\left\{-3i\frac{1}{4} - \frac{1}{4}i\right\}$$

$$= 2\pi\ e^{-3}$$

$$= \int_{-\infty}^{\infty} dx\, g(x)$$

$$\therefore \qquad I = \operatorname{Re}\int_{-\infty}^{\infty} dx\, g(x) = 2\pi\ e^{-3}$$

- **Example 2**

Let $f(x) = \dfrac{x\sin x}{x^2+2x+2}$. Find $I = \displaystyle\int_{-\infty}^{\infty} dx\ f(x)$

Solution:

Let $\quad g(x) = \dfrac{x\,e^{ix}}{x^2+2x+2} \qquad g(z) = \dfrac{z\,e^{iz}}{z^2+2z+2}$

$\quad f(x) = \operatorname{Im} g(x)$

Consider contour c composed of the real axis & the upper half circle C_R $(R\to\infty)$.

$$\oint_c dz\ g(z) = \int_{C_R} dz\ g(z) + \int_{-R}^{R} dx\, g(x)$$

Now:

$$|g(z)| \xrightarrow[|z|>>1]{} \frac{1}{|z|} \xrightarrow[|z|\to\infty]{} 0 \qquad \Longrightarrow \qquad \int_{C_R} dz\ g(z) \xrightarrow[R\to\infty]{} 0$$

Poles of $g(z)$ are at

$$z = -1 \pm \sqrt{1-2} = -1 \pm i$$

$$\longrightarrow \qquad \oint_c dz\ g(z) = 2\pi i \operatorname*{Res}_{z=-1+i} g(z) = 2\pi i \cdot \left(\frac{z\,e^{iz}}{2z+2}\right)_{z=-1+i}$$

$$= \; 2\pi i \cdot \frac{-1+i}{2i} e^{-i-1}$$
$$= \; \pi\,(-1+i)\; e^{-i-1}$$
$$= \; \frac{\pi}{e}(-1+i)(\cos 1 - i \sin 1)$$
$$= \; \frac{\pi}{e}\{-\cos 1 + \sin 1 + i(\cos 1 + \sin 1)\}$$
$$= \; \int_{-\infty}^{\infty} d\,x\, g(x)$$

$$\therefore \qquad I = \operatorname{Im} \int_{-\infty}^{\infty} d\,x\;\; g(x) = \frac{\pi}{e}(\cos 1 + \sin 1)$$

■ Definite Integrals involving Sines & Cosines

$$\int_0^{2\pi} d\,\theta\, F(\sin\theta,\; \cos\theta)$$

■ Technique

Set $z = e^{i\theta}$

$$\longrightarrow \qquad d\,\theta = \frac{d\,z}{i\,z}, \qquad \sin\theta = \frac{z - \frac{1}{z}}{2\,i}, \qquad \cos\theta = \frac{z + \frac{1}{z}}{2}$$

$$\int_0^{2\pi} d\,\theta\, F(\sin\theta,\; \cos\theta) = \oint_C \frac{d\,z}{i\,z} F\left(\frac{z - \frac{1}{z}}{2\,i},\; \frac{z + \frac{1}{z}}{2}\right)$$

where C is the unit circle centered at the origin.

■ Example

Let $\;f(\theta) = \dfrac{1}{1 + a \sin\theta} \qquad\qquad -1 < a < 1$

Find $\;I = \displaystyle\int_0^{2\pi} d\,\theta\; f(\theta)$

Solution:

Let $\quad z = e^{i\theta}$

$\longrightarrow \quad dz = i\, e^{i\theta}\, d\theta = i\, z\, d\theta \qquad d\theta = -i\,\frac{dz}{z}$

$$\sin\theta = \frac{1}{2\,i}\left(z - z^{-1}\right)$$

$$I = \oint_c -i\, z^{-1}\, d\, z\, \frac{1}{1 + \frac{a}{2\,i}\left(z - z^{-1}\right)} \qquad \text{where } c: \quad |z| = 1$$

$$= \oint_c dz\, \frac{2}{2\,i\,z + a\left(z^2 - 1\right)}$$

Poles are at:

$$a\,z^2 + 2\,i\,z - a = 0$$

or $\quad z_\pm = \frac{1}{a}\left\{-i \pm \sqrt{-1 + a^2}\right\}$

$$= \frac{i}{a}\left\{-1 \pm \sqrt{1 - a^2}\right\} \qquad \text{where} \quad |a| < 1$$

$$\left|z_\pm\right| = \frac{1}{|a|}\left\{1 \mp \sqrt{1 - a^2}\right\}$$

For poles inside c, we need

$$\frac{1}{|a|}\left\{1 \mp \sqrt{1 - a^2}\right\} < 1$$

or $\quad 1 \mp \sqrt{1 - a^2} < \left|a\right|$

$$\mp\sqrt{1 - a^2} < -(1 - |a|)$$

For z_+: $\quad \sqrt{1 - a^2} > 1 - \left|a\right|$

$\longrightarrow \quad 1 - a^2 > 1 - 2\left|a\right| + a^2$

$$-a^2 > -|a|$$

$$|a| < 1$$

which is by definition automatically satisfied.

For z_-: $\quad \sqrt{1 - a^2} < -(1 - |a|)$

which is by definition impossible.

$\therefore \quad$ only z_+ contributes to the integral.

$$I = 2\,\pi\,i\,\underset{z=z_+}{\text{Res}}\, \frac{2}{2\,i\,z + a\left(z^2 - 1\right)}$$

$$= 4\,\pi\,i\left(\frac{1}{2\,i + 2\,a\,z}\right)_{z=z_+}$$

$$= 2\pi i \frac{1}{i + i\left\{-1 + \sqrt{1-a^2}\right\}}$$

$$= 2 \frac{\pi}{\sqrt{1-a^2}}$$

Multi-Valued Functions

The foregoing theorems all apply to single valued functions. In dealing with many valued functions, the only additional rule is that no contour should cut across any branch cut.

Improper Integrals involving $x^{\mu-1}$

$$\int_0^\infty d\,x\; f(x)\, x^{\mu-1} = \pi \csc(\pi\mu) \sum_{\text{all}} \text{Res}\left\{(-z)^{\mu-1} f(z)\right\}$$

where

1. f is analytic at $z = 0$.

2. f has only simple poles on real axis.

3. $z^{\mu} f(z) \underset{z\to 0}{\longrightarrow} 0$ $\qquad$ $z^{\mu} f(z) \underset{z\to\infty}{\longrightarrow} 0$

4. $-1 = e^{-i\pi}$, $0 < \theta < 2\pi$.

Reference:
P.M.Morse, H.Feshbach, "Methods of Theoretical Physics", pp.410-1.

- **proof**

Consider contour c composed of 2 circles C_R ($R \to \infty$) & C_ϵ ($\epsilon \to 0$), which are centered at the origin & joined by 2 straight lines on either side of the branch cut.
Let the branch $\alpha < \theta < \alpha + 2\pi$ of z^ν be used.
On the upper & lower side of the branch cut:

$$z = r\, e^{i\alpha}, \quad r\, e^{i\alpha + i 2\pi}, \text{ respectively.}$$

$$\longrightarrow \quad dz = e^{i\alpha}\, dr, \quad e^{i\alpha + i 2\pi}\, dr$$

Using $-1 = e^{-i\pi}$, we have
for the upper side,

$$(-z)^{\mu-1} = \left(r\, e^{i\alpha - i\pi}\right)^{\mu-1} = e^{i(\alpha-\pi)(\mu-1)} \cdot r^{\mu-1}$$

for the lower side,

$$(-z)^{\mu-1} = \left(r\, e^{i\alpha + i\pi}\right)^{\mu-1} = e^{i(\alpha+\pi)(\mu-1)} \cdot r^{\mu-1}$$

$$\therefore \quad \int_U dz\, (-z)^{\mu-1} f(z) = \int_\epsilon^R dr \cdot e^{i\alpha} \cdot e^{i(\alpha-\pi)(\mu-1)} \cdot r^{\mu-1} f\left(r\, e^{i\alpha}\right)$$

$$\int_L dz\, (-z)^{\mu-1} f(z) = \int_R^\epsilon dr \cdot e^{i\alpha} \cdot e^{i(\alpha+\pi)(\mu-1)} \cdot r^{\mu-1} \cdot f\left(r\, e^{i\alpha}\right)$$

Setting $x = r\, e^{i\alpha}$, $dx = e^{i\alpha}\, dr$, $\& \epsilon \to 0$

$$\int_U dz\, (-z)^{\mu-1} f(z) = e^{-i\pi(\mu-1)} \cdot \int_{\epsilon e^{i\alpha}}^{R e^{i\alpha}} dx\; x^{\mu-1} f(x)$$

$$\int_L dz\, (-z)^{\mu-1} f(z) = e^{i\pi(\mu-1)} \cdot \int_{R e^{i\alpha}}^{\epsilon e^{i\alpha}} dx\; x^{\mu-1} f(x)$$

$$\longrightarrow \quad \left(\int_U + \int_L\right) dz\, (-z)^{\mu-1} f(z) = \left(-e^{-i\pi\mu} + e^{i\pi\mu}\right) \cdot \int_{\epsilon e^{i\alpha}}^{R e^{i\alpha}} dx\; x^{\mu-1} f(x)$$

$$= \; 2\, i \sin(\pi\mu) \cdot \int_{\epsilon e^{i\alpha}}^{R e^{i\alpha}} dx\; x^{\mu-1} f(x)$$

Now:

$$\left|\int_{C_R} dz\, (-z)^{\mu-1} f(z)\right| \le 2\pi R \cdot R^{-1} \cdot |(-z)^\mu f(z)| \underset{R\to\infty}{\longrightarrow} 0$$

where $\left|(-z)^\mu f(z)\right| \underset{R\to\infty}{\longrightarrow} 0.$

Similarly, $\left|(-z)^{\mu} f(z)\right| \underset{\epsilon\to\infty}{\longrightarrow} 0$ implies

$$\left|\int_{C_\epsilon} dz\,(-z)^{\mu-1} f(z)\right| \underset{\epsilon\to 0}{\longrightarrow} 0$$

$$\therefore \quad \oint_c dz\,(-z)^{\mu-1} f(z) = \left\{\int_{C_R} + \int_{C_\epsilon} + \int_L + \int_U\right\} dz\,(-z)^{\mu-1} f(z)$$

$$= \left\{\int_L + \int_U\right\} dz\,(-z)^{\mu-1} f(z)$$

$$= 2\,i\sin(\pi\mu)\cdot\int_{\epsilon e^{i\alpha}}^{R e^{i\alpha}} \mathrm{dx}\; x^{\mu-1} f(x)$$

$$= 2\pi i \sum \text{Res}\;(-z)^{\mu-1} f(z)$$

where the sum is over poles inside c.

Setting $\alpha = 0$, $R\to\infty$, $\epsilon\to 0$:

$$\int_0^\infty dx\, f(x)\, x^{\mu-1} = \pi\csc(\pi\mu) \sum_{\text{all}} \text{Res}\,\left\{(-z)^{\mu-1} f(z)\right\}$$

Reminder: $-1 = e^{-i\pi}$, $\quad 0 < \theta < 2\pi$.

- **Example**

Let $f(x) = \dfrac{x^{-a}}{x+1}$ $\qquad 0 < a < 1$

Find $\quad I = \displaystyle\int_0^\infty dx\, f(x)$

Solution:

$$I = \pi\csc\pi(1-a)\,\underset{z=-1}{\text{Res}}\left\{(-z)^{-a}\cdot\frac{1}{z+1}\right\}$$

$$= \pi\csc\,(\pi a)$$

■ Argument Principle & Rouche's Theorem

■ Winding Number

winding number

Consider a simple closed contour in the z plane.

Let $f(z)$ be analytic & non-zero on C.

As z goes around C once, the argument of z changes by 2π :

$$\Delta_C \arg z = 2\pi$$

At the same time, $w = f(z)$ describes a closed contour Γ, not necessarily simple, in the w plane. The change of the argument of w is an integral multiple of 2π.

$$\Delta_C \arg w = \Delta_C \arg f(z) = 2\pi n$$

where n is called the **winding number** of Γ.

Note:

1. The condition $f(z) \neq 0$ on C guarantees that Γ doesn't pass through the origin; hence the uniqueness of n.
2. n can be negative, zero, or positive.

■ Calculation of Phase Change

phase change

Set $f(z) = \rho\, e^{i\phi}$ $\longrightarrow$ $\arg f = \phi$

Consider the function:

$$g(z) = \frac{1}{f(z)} \frac{df}{dz} = \frac{d}{dz} \ln f$$

$$\longrightarrow \quad \oint_c dz\, g(z) = \oint_c dz \frac{d}{dz} \ln f = (\ln \rho + i\phi)_a^a = i\,\Delta\phi = i\,\Delta_c \arg f$$

where a is an arbitrary point on C, which is **simple & closed**.

Note: the above calculation is valid only if $\ln f$ is analytic on C. This implies f must be **analytic & non-zero** on C.

■ Argument Principle

argument principle

Consider a simple closed contour C.

Let f be analytic & non-zero on C, but meromorphic inside C.

$$\Longrightarrow \quad \Delta_c \arg f(z) = 2\pi (Z - P)$$

where Z = number of **zeros**, & P = number of **poles**, inside C.

Note: each zero or pole of order m contributes m to Z or P, respectively.

■ proof

From the phase change formula:

$$i\,\Delta_c \arg f(z) = \oint_c dz\, g(z) = 2\pi i \sum_k \operatorname{Res} g(z_k)$$

where $g = \frac{1}{f} \frac{df}{dz}$

The poles of g are at $f = 0$ or $\frac{df}{dz} \to \infty$.

Let z_0 be a zero of order m of f

$\longrightarrow$ $f(z) = (z - z_0)^m\, h(z)$ where $h(z_0) \neq 0$ is analytic.

$$\therefore \quad g = \frac{1}{(z - z_0)^m\, h} \cdot \left\{ m\,(z - z_0)^{m-1}\, h + (z - z_0)^m \frac{dh}{dz} \right\}$$

$$= \frac{m}{z - z_0} + \frac{1}{h} \frac{dh}{dz}$$

Since h is analytic at z_0, $\frac{1}{h}\frac{dh}{dz}$ is also analytic there.

$\longrightarrow$ $\operatorname{Res} g(z_0) = m$

For z_0 a pole of order m of f, the above procedure applies if $m \longrightarrow -m$.

Hence: $\operatorname{Res} g(z_0) = -m$

Applying the foregoing to every zero & pole inside C, we have

$$\sum_k \operatorname{Res} g(z_k) = Z - P$$

The theorem is thus proved.

Example

Let $f(z) = \dfrac{1}{z^2}$

$\longrightarrow \quad Z = 0 \qquad P = 2$

$\therefore \quad \dfrac{1}{2\pi} \Delta_c \arg f(z) = -2$

- **Rouche's Theorem**

Let C be a simple closed contour.

Let f & g be analytic on & inside C.

If $\quad |f| > |g| \qquad \forall z$ on C

$\Longrightarrow \quad f$ & $f + g$ have, inside C, the same number of zeros Z, counting multiplicities.

- **proof**

Since

$|f| > |g| \geq 0 \quad$ on C

$\longrightarrow \quad f \neq 0 \qquad$ on C

Similarly,

$|f + g| \geq ||f| - |g|| > 0$

$\longrightarrow \quad f + g \neq 0 \qquad$ on C

$\Longrightarrow \quad$ all zeros of f & $f + g$ are inside C.

Since f & g are both analytic inside C, $P = 0$ for both.

$\longrightarrow \quad Z_f = \dfrac{1}{2\pi} \Delta_c \arg f \qquad Z_{f+g} = \dfrac{1}{2\pi} \Delta_c \arg(f + g)$

$\therefore \quad Z_{f+g} = \dfrac{1}{2\pi} \Delta_c \arg f\left(1 + \dfrac{g}{f}\right)$

$= \dfrac{1}{2\pi} \Delta_c \arg f + \dfrac{1}{2\pi} \Delta_c \arg\left(1 + \dfrac{g}{f}\right)$

$= Z_f + Z_h$

where $h = 1 + \dfrac{g}{f}$.

Now:

$|h - 1| = \left|\dfrac{g}{f}\right| < 1$

$\longrightarrow \quad h \neq 0 \quad \longrightarrow \quad Z_h = 0$

$\therefore \quad Z_{f+g} = Z_f$

- **Example**

Find the number of roots of

$z^7 - 4z^3 + z - 1 = 0$

inside circle $|z| = 1$.

Solution:

Let $f(z) = -4z^3$ $\quad g(z) = z^7 + z - 1$

On $|z| = 1$

$|f| = 4 \quad |g| \le |z^7| + |z| + 1 = 3$

$\longrightarrow \quad |f| > |g|$

Since $Z_f = 3$

$\therefore \quad Z_{f+g} = 3$

Laplace Transform

Definition

Given a **complex** function $f(t)$ of a **real** variable t.

Its **Laplace transform** $F(s)$ is defined, if the integral exists, as

$F = \mathfrak{L}[f]$

One-sided:

$$F(s) = \mathfrak{L}[f(t); s] = \int_0^\infty dt\, e^{-st} f(t) \qquad \text{Re}\, s > 0$$

Two-sided:

$$F(s) = \mathfrak{L}[f(t); s] = \int_{-\infty}^\infty dt\, e^{-st} f(t) \qquad \text{Re}\, s > 0$$

$\mathfrak{L}$ is **linear:** $\quad \mathfrak{L}[a f + b g] = a\,\mathfrak{L}[f] + b\,\mathfrak{L}[g]$

If $\exists\, s_0,\ t_0 \quad \ni \quad \left| e^{-s_0 t} f(t) \right| \le M \quad \forall\, t > t_0$

$\longrightarrow \quad F(s)$ exists $\forall\, s > s_0$

f is then said to be of **exponential order**.

For $t \longrightarrow 0$, $\ e^{-st} f(t) \longrightarrow f(t)$

$\therefore \quad F(s)$ does not exists if $f \sim t^n \quad \forall\, n \le -1$

Inverse Transform

Bromwich integral:

$$f(t) = \frac{1}{2\pi i} \text{PV} \int_{\gamma - i\infty}^{\gamma + i\infty} ds\, e^{st} F(s) \qquad t > 0, \text{ real}$$

$$= \sum_n \underset{s = s_n}{\text{Res}} \left[e^{st} F(s) \right] \qquad \left(F \underset{|s| \to \infty}{\longrightarrow} 0 \right)$$

where the sum is over all poles of F which are assumed to lie on the left side of the vertical line $\text{Re}\, s = \gamma$.

- **proof**

Consider the contour shown in fig 74.
The proof is complete if one can show

$$I = \int_{C_R} d\,s\ e^{s\,t}\, F(s) \underset{R\to\infty}{=} 0$$

Now:

$$|\,I\,| \le \max\,|\,F\,| \cdot R \cdot J$$

where $$J = \left| \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} d\,\theta\, e^{i\,\theta}\, e^{t\,R\,e^{i\,\theta}} \right|$$

$$= \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} d\,\theta\, e^{t\,R\cos\theta}$$

$$= \int_{0}^{\pi} \mathrm{d}\phi\, e^{-t\,R\sin\phi} \qquad \phi = \theta - \frac{\pi}{2}$$

$$< \frac{\pi}{R\,t} \qquad \text{(Jordan's inequality)}$$

$$\longrightarrow \qquad |\,I\,| < \max\,|\,F\,|\,\frac{\pi}{t}$$

$$\therefore \qquad I \underset{R\to\infty}{=} 0 \text{ if} \qquad F \xrightarrow[|\,s\,|\to\infty]{} 0$$

- **Example 1**

$$F(s) = \frac{s}{\left(s^2 + a^2\right)^2}$$

Poles of order 2 are at $s = \pm\, i\, a$.

$$\left|F\right| \longrightarrow \frac{1}{|\;s\;|^3} \to 0$$

$$\underset{s=\pm i\,a}{\mathrm{Res}}\left[\, e^{s\,t}\, F(s) \,\right] = \frac{d}{d\,s}\left(\frac{s\,e^{s\,t}}{(\,s\,\pm\,i\,a\,)^2} \right)_{s=\pm i\,a}$$

$$= \left. \frac{e^{s\,t}(\,s\,t\,+\,1\,)}{(\,s\,\pm\,i\,a\,)^2} - 2\,s\,\frac{e^{s\,t}}{(\,s\,\pm\,i\,a\,)^3} \right|_{s=\pm i\,a}$$

$$= e^{\pm i\,a\,t}\left\{ \frac{\pm\,i\,a\,t\,+1}{-4\,a^2} - \frac{1}{-4\,a^2} \right\}$$

$$= \mp\, i\,\frac{t}{4\,a}\, e^{\pm i\,a\,t}$$

$$\therefore \qquad f(t) = -\,i\,\frac{t}{4\,a}\left(e^{i\,a\,t} - e^{-i\,a\,t}\right) = \frac{t}{2\,a}\sin a\,t$$

- **Example 2**

$$F(s) = \frac{\tanh s}{s^2} = \frac{\sinh s}{s^2\cosh s}$$

$$\left| F \right| \longrightarrow \frac{1}{|\, s \,|^2} \to 0$$

Poles are at $s = 0, \quad (2\,n+1)\,\dfrac{\pi}{2}\,i.$

For $s = 0$:

$$e^{s\,t}\,F(s) = \frac{1}{s^2}\left(s - \frac{s^3}{3} + \ldots\right)(\,1 + s\,t\, + \ldots) = \frac{1}{s} + \ldots$$

$$\therefore \qquad \underset{s=0}{\text{Res}}\left[\, e^{s\,t}\,F(s)\,\right] = 1$$

$$\underset{s=(2\,n+1)\,\frac{\pi}{2}\,i}{\text{Res}}\left[\, e^{s\,t}\,F(s)\,\right] = \frac{e^{s\,t}}{s^2} = -\frac{4\,e^{(2\,n+1)\,\frac{\pi}{2}\,i\,t}}{(2\,n+1)^2\,\pi^2}$$

$$f(t) = 1 - \frac{4}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{e^{(2\,n+1)\,\frac{\pi}{2}\,i\,t}}{(2\,n+1)^2}$$

Now:

$$\sum_{n=-\infty}^{\infty}\frac{e^{(2\,n+1)\,\frac{\pi}{2}\,i\,t}}{(2\,n+1)^2} = \sum_{n=0}^{\infty}\frac{e^{(2\,n+1)\,\frac{\pi}{2}\,i\,t}}{(2\,n+1)^2} + \sum_{n=1}^{\infty}\frac{e^{-(2\,n-1)\,\frac{\pi}{2}\,i\,t}}{(2\,n-1)^2}$$

$$= \sum_{n=1}^{\infty}\frac{e^{(2\,n-1)\,\frac{\pi}{2}\,i\,t}}{(2\,n-1)^2} + \sum_{n=1}^{\infty}\frac{e^{-(2\,n-1)\,\frac{\pi}{2}\,i\,t}}{(2\,n-1)^2}$$

$$= 2\sum_{n=1}^{\infty}\frac{\cos\,(\,2\,n-1)\,\frac{\pi}{2}\,t}{(2\,n-1)^2}$$

$$\longrightarrow \qquad f(t) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\,(\,2\,n-1)\,\frac{\pi}{2}\,t}{(2\,n-1)^2}$$

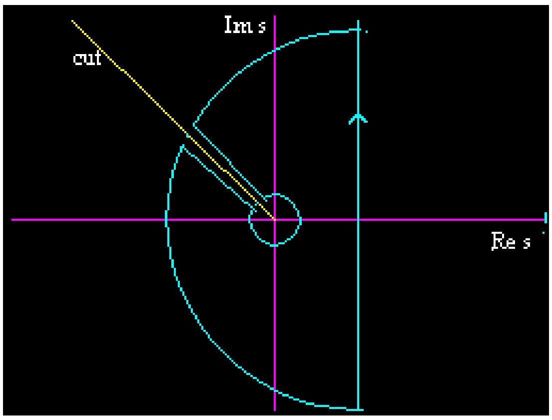
- **Example 3**

$$F(s) = \frac{\sinh\left(x\,\sqrt{s}\,\right)}{s\,\sinh\,\sqrt{s}} \qquad\qquad 0 < x < 1$$

Branch point & pole at $s = 0$.

Simple poles are at $\quad \sqrt{s} = n\,\pi\,i \quad$ or $\quad s = -\,n^2\,\pi^2 \quad (\,n \geq 0\,)$

The branch cut must not be the negative real axis to avoid the poles. Also, we need $\gamma > 0$ to push all singularities to the left of the $\text{Re}\,s = \gamma$ line. The branch cut must accordingly points to the left (see fig)



Now:

$$2\left|\sinh\sqrt{s}\right| = \left|e^{\sqrt{s}} - e^{-\sqrt{s}}\right|$$

$$\leq \left|e^{\sqrt{s}}\right| + \left|e^{-\sqrt{s}}\right|$$

Let $\quad s = R\,e^{i\theta}$

$$\longrightarrow \quad \sqrt{s} = \sqrt{R}\;e^{i\frac{\theta}{2}} = \sqrt{R}\left\{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right\}$$

$$2\sinh\sqrt{s} = e^{\sqrt{R}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)} - e^{-\sqrt{R}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}$$

$$\xrightarrow[R\to\infty]{} e^{\sqrt{R}\left|\cos\frac{\theta}{2}\right|}\cdot e^{\pm i\sqrt{R}\sin\frac{\theta}{2}}$$

where the ± sign is the same as that in

$$\cos\frac{\theta}{2} = \pm\left|\cos\frac{\theta}{2}\right|.$$

$$\therefore \quad \left|2\sinh\sqrt{s}\right| \xrightarrow[R\to\infty]{} e^{\sqrt{R}\left|\cos\frac{\theta}{2}\right|}$$

$$\longrightarrow \quad \left|\frac{\sinh x\sqrt{s}}{s\sinh\sqrt{s}}\right| \xrightarrow[R\to\infty]{} \frac{1}{R}\cdot e^{\sqrt{R}\left|\cos\frac{\theta}{2}\right|(x-1)}$$

$$\longrightarrow 0 \qquad \text{for} \qquad 0 < x < 1$$

Let the branch be $\alpha < \theta < \alpha + 2\pi$ where

$$\frac{\pi}{2} < \alpha < \frac{3}{2}\pi \qquad \& \qquad \alpha \neq \pi$$

The segments of the contour on both sides of the branch cut give:

$$\int_R^\rho \frac{dr}{r}\cdot\frac{\sinh\left(x\sqrt{r}\;e^{i\left(\frac{\alpha}{2}+\pi\right)}\right)}{\sinh\left(\sqrt{r}\;e^{i\left(\frac{\alpha}{2}+\pi\right)}\right)} - \int_R^\rho \frac{dr}{r}\cdot\frac{\sinh\left(x\sqrt{r}\;e^{i\frac{\alpha}{2}}\right)}{\sinh\left(\sqrt{r}\;e^{i\frac{\alpha}{2}}\right)} = 0$$

where we have used $e^{i\pi} = -1$ & $\sinh(-z) = -\sinh z$.

$f(t)$ is therefore once again simply the sum of the residues of all poles.

For $s = 0$

$$\frac{\sinh\left(x\sqrt{s}\right)}{s\sinh\sqrt{s}} = \frac{x\sqrt{s} + \frac{1}{6}x^3 s^{\frac{3}{2}} + \ldots}{s\left(\sqrt{s} + \frac{1}{6}s^{\frac{3}{2}} + \ldots\right)}$$

$$= \frac{x + \frac{1}{6}x^3 s + \ldots}{s\left(1 + \frac{1}{6}s + \ldots\right)}$$

$$\longrightarrow \quad \operatorname*{Res}_{s=0}\left[e^{st}F(s)\right] = x$$

$$\operatorname*{Res}_{s=-n^2\pi^2}\left[e^{st}F(s)\right] = e^{st}\frac{\sinh\left(x\sqrt{s}\right)}{s\,\frac{1}{2\sqrt{s}}\cosh\sqrt{s}} \qquad n \neq 0$$

$$= 2\,e^{-n^2\pi^2 t}\,\frac{\sin(n\pi x)}{n\pi}(-)^n$$

$$\therefore \quad f(t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\,\frac{\sin(n\pi x)}{n}(-)^n$$