8. Mapping by Elementary Functions

■ Introduction

The main purpose of this chapter is to build up a basic library of conformal mappings which is crucial in the solution of $2-D$ Laplace equations with complicated boundary conditions.
A collection of some mappings of interest can be found in App 2 of Churchill.
The program **F(z).exe** can be used if more details are desired.

The complete mapping of a complex function cannot be shown in a single graph since it involves a space of 4 dimensions. The approach taken here is to list the mappings of a few representative domains. Since in applications, one is required to find a mapping which reduces a complicated boundary to a simple one, some familarity with a collection of such mappings is needed to get one started. Fortunately, simple composites of elementary functions already provide a respectable arsenal adequate for most situations of interest.

Some domains of common interest are

1. Fixed points: $z = f(z)$
2. Lines. Of particular interest are the horizontal & vertical ones.
3. Circles.
4. Rectangles, half-planes, quadrants, strips.
5. Disks, annular region.
6. Sectors of disks or annular region.

■ $w = a z + b$

Alias: **Linear Transformations**

Fixed points at $z = \dfrac{b}{1-a}$ $\quad (a \neq 1)$

■ $w = a z$

Let $\quad a = |a|\, e^{i\phi_0}, \quad z = |z|\, e^{i\phi}$

$\longrightarrow \quad w = |a|\,|z|\, e^{i(\phi+\phi_0)}$

Interpretation: expand ($|a| > 1$), contract ($|a| < 1$), & rotate.

■ $w = z + b$

Interpretation: translation

■ $w = 1/z$

Fixed points at $z^2 = 1$ or $z = \pm 1$

■ 1st Interpretation

$$w^* = \frac{1}{z^*} = \frac{z}{|z|^2}$$

$\longrightarrow \quad |w^*| = \dfrac{1}{|z|} \quad \arg w^* = \arg z$

Interpretation: move points inside the unit circle to the outside radially & vice versa

$w = \frac{1}{z} = \frac{z^*}{|z|^2}$

$\longrightarrow \quad |w| = \frac{1}{|z|} \qquad \arg w = \arg z^* = -\arg z$

Interpretation:

1st: move points inside the unit circle to the outside radially & vice versa

then: reflects about the real axis.

$w = u + i\,v = \frac{1}{z} = \frac{z^*}{|z|^2} = \frac{x - i\,y}{x^2 + y^2}$

$\longrightarrow \quad u = \frac{x}{x^2 + y^2} = \frac{x}{|z|^2} \qquad v = -\frac{y}{x^2 + y^2} = -\frac{y}{|z|^2}$

$z = \frac{1}{w} = \frac{u - i\,v}{u^2 + v^2}$

$\longrightarrow \quad x = \frac{u}{u^2 + v^2} = \frac{u}{|w|^2} \qquad y = -\frac{v}{u^2 + v^2} = -\frac{v}{|w|^2}$

Interpretation: transforms lines into circles & vice versa.

The inside of unit disk $\longleftrightarrow$ region outside.

- **2nd Interpretation**

A circle in the $(x,\ y)$ plane is described by:

$$(x - a)^2 + (y - b)^2 = r^2$$

or $\quad x^2 + y^2 - 2\,a\,x - 2\,b\,y + a^2 + b^2 - r^2 = 0$

A line by:

$$y = m\,x + b$$

Therefore, the eq

$$A\left(x^2 + y^2\right) + B\,x + C\,y + D = 0$$

describes a circle, when $A \neq 0$, with

$$a = -\frac{B}{2\,A} \qquad b = -\frac{C}{2\,A}$$

$$r^2 = a^2 + b^2 - \frac{D}{A} = \frac{1}{4\,A^2}\left(B^2 + C^2 - 4\,A\,D\right) > 0$$

or a line, when $A = 0$, with:

$$m = -\frac{B}{C} \qquad b = -\frac{D}{C} \qquad (C \neq 0)$$

$$m\,x + b = B\,x + D = 0 \quad (C = 0)$$

In the case of a circle, the distance from its center $(a,\ b)$ to the origin is $r_0 = \sqrt{a^2 + b^2}$.

Its radius r is $\quad r^2 = r_0{}^2 - \frac{D}{A}$.

$\therefore$ The circle passes through the origin when $D = 0$.

Using

$x = u\left(x^2+y^2\right) y = v\left(x^2+y^2\right)$

$\longrightarrow \quad \left(u^2+v^2\right)\left(x^2+y^2\right) = 1$

The eq $A\left(x^2 + y^2\right) + B x + C y + D = 0$

becomes $\quad \left(x^2 + y^2\right)(A + B u + C v) + D = 0$

$\longrightarrow \quad A + B u + C v + D\left(x^2 + y^2\right) = 0$

Hence, lines & circles are mapped into lines & circles.

- **Summary**

1. A circle not passing through the origin:

$$A\left(x^2 + y^2\right) + B x + C y + D = 0$$

is mapped into a circle not passing through the origin:

$$A + B u + C v + D\left(x^2 + y^2\right) = 0$$

2. A circle passing through the origin:

$$A\left(x^2 + y^2\right) + B x + C y = 0$$

is mapped into a line not passing through the origin:

$$A + B u + C v = 0$$

3. A line not passing through the origin:

$$B x + C y + D = 0$$

is mapped into a circle passing through the origin:

$$B u + C v + D\left(x^2 + y^2\right) = 0$$

4. A line passing through the origin:

$$B x + C y = 0$$

is mapped into a line passing through the origin:

$$B u + C v = 0$$

- **Example 1**

Vertical line $\quad x = c$

The quickest way is to apply the formulae developed above. However, direct derivation for each special case is more appealing to those who detest memorization.

$$z = c + i y$$

$$\longrightarrow \quad w = \frac{1}{z} = \frac{c - i y}{c^2 + y^2} = u + i v$$

$$u = \frac{c}{c^2 + y^2} \qquad v = -\frac{y}{c^2 + y^2}$$

$$\longrightarrow \quad u^2 + v^2 = \frac{1}{c^2 + y^2} = \frac{u}{c}$$

or $\quad \left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$

which is a circle of radius $\frac{1}{2c}$, centered at $\left(\frac{1}{2c}, 0\right)$ & passes through the origin.

Note also that

1. The circle & line are on the same side of the y –axis.
2. The size of the circle is inversely proportional to the distance of the line from the y –axis.

Example 2

Horizontal line $y = c$

$$z = x + i\,c$$

$$\longrightarrow \quad w = \frac{1}{z} = \frac{x - i\,c}{x^2 + c^2} = u + i\,v$$

$$u = \frac{x}{x^2 + c^2} \qquad v = -\frac{c}{x^2 + c^2}$$

$$\longrightarrow \quad u^2 + v^2 = \frac{1}{x^2 + c^2} = -\frac{v}{c}$$

or $$u^2 + \left(v + \frac{1}{2\,c}\right)^2 = \left(\frac{1}{2\,c}\right)^2$$

which a circle of radius $\frac{1}{2\,c}$, centered at $\left(0, -\frac{1}{2\,c}\right)$ & passes through the origin.

Note also that

1. The circle & line are on the opposite side of the x –axis.
2. The size of the circle is inversely proportional to the distance of the line from the x –axis.

- **Example 3**

Half plane $x \geq c \quad (c > 0)$

$$z = x + i\,y$$

$$\longrightarrow \quad w = \frac{1}{z} = \frac{x - i\,y}{x^2 + y^2} = u + i\,v$$

$$u = \frac{x}{x^2 + y^2} \qquad v = -\frac{y}{x^2 + y^2}$$

$$\longrightarrow \quad u^2 + v^2 = \frac{1}{x^2 + y^2} = \frac{u}{x} = -\frac{v}{y}$$

$$x = \frac{u}{u^2 + v^2} \geq c > 0$$

$$\longrightarrow \quad u^2 + v^2 - \frac{u}{c} \leq 0$$

or $$\left(u - \frac{1}{2\,c}\right)^2 + v^2 \leq \left(\frac{1}{2\,c}\right)^2$$

which denotes the region inside a circle of radius $\frac{1}{2\,c}$, centered at $\left(\frac{1}{2\,c}, 0\right)$ & passes through the origin.

- $w = \dfrac{a\,z + b}{c\,z + d} \quad (a\,d - b\,c \neq 0)$

Fixed points at

$$c\,z^2 + (d - a)\,z - b = 0$$

$$\longrightarrow \quad z = \frac{1}{2\,c}\left\{a - d \pm \sqrt{(d - a)^2 - 4\,c\,b}\right\}$$

Mobius Transformation

$\longrightarrow \quad c\,z\,w - a\,z + d\,w - b = 0$

$\quad \equiv A\,z\,w + B\,z + C\,w + D$

The last form is called the **Mobius** or **bilinear** transformation.

Since $A = c,\ \ B = -a,\ \ C = d,\ \ D = -b,$

$\longrightarrow \quad a\,d - b\,c = -B\,C + D\,A \neq 0$

Properties in the finite plane

For $c = 0$:

$$w = \frac{a}{d}\,z + \frac{b}{d}$$ is a linear transform with $d \neq 0$.

For $d = 0$:

$$w = \frac{a}{c} + \frac{b}{c}\cdot\frac{1}{z}$$ is a combination of translation & inversion with $c \neq 0$.

In general, assuming $c \neq 0$:

$$w = \frac{a\,z + b}{c\,z + d} = \frac{a}{c} + \left(b - \frac{a\,d}{c}\right)\cdot\frac{1}{c\,z + d}$$

Let $\quad Z = c\,z + d \qquad W = \frac{1}{Z} = \frac{1}{c\,z + d}$

$$\longrightarrow \quad w = \frac{a}{c} + \left(b - \frac{a\,d}{c}\right)W$$

Provided $\quad a\,d - b\,c \neq 0$,

it is a composition of linear transformations & inversion.

$\therefore \quad$ lines & circles $\longleftrightarrow$ lines & circles

The inverse of the transform

$$w = \frac{a\,z + b}{c\,z + d}$$

is:

$$z = \frac{-d\,w + b}{c\,w - a} \qquad a\,d - b\,c \neq 0$$

For $c \neq 0$, the transform is $1 - 1$ provided

1. $c\,z + d \neq 0$ or $z \neq -\frac{d}{c}$.
2. $c\,w - a \neq 0$ or $w \neq \frac{a}{c}$.

For $c = 0$, we have:

$$w = \frac{a}{d}\,z + \frac{b}{d}$$

$$z = \frac{d}{a}\,w - \frac{b}{a}$$

The transform is $1 - 1$ provided $a,\ d \neq 0$.

- **Extended Plane**

For the extended complex plane which includes the point ∞ , the above pair of transform is everywhere $1-1$. To be more precise, let

$$T(z) = \frac{a\,z + b}{c\,z + d}$$

$$T^{-1}(w) = \frac{-d\,w + b}{c\,w - a}$$

Points at which these transforms fail to exist in the finite complex plane now becomes well defined.

Thus, for $c \neq 0$,

1. $T\left(-\frac{d}{c}\right) = \infty$
2. $T^{-1}\left(\frac{a}{c}\right) = \infty$

In addition, we need to consider the transforms for the extended point ∞.

$$T(\infty) = \frac{a}{c}$$

$$T^{-1}(\infty) = -\frac{d}{c}$$

For $c = 0$, the undefined points of the finite plane transform are simply ∞ so that

$T(\infty) = \infty$ ($d = 0$)

$T^{-1}(\infty) = \infty$ ($a = 0$)

- **Example 1**

Find transformation which maps

$-1,\ 0,\ 1$ into $-i,\ 1,\ i$

Let the transformation be $w(z) = \frac{a\,z + b}{c\,z + d}$

The mapping of the 3 points give:

$$-i = \frac{-a + b}{-c + d} \qquad 1 = \frac{b}{d} \qquad i = \frac{a + b}{c + d}$$

$\longrightarrow$ $b = d$

$$-i = \frac{-a + b}{-c + b} \qquad i = \frac{a + b}{c + b}$$

$$\longrightarrow \quad \frac{a - b}{-c + b} = \frac{a + b}{c + b}$$

$$2\,a\,c - 2\,b^2 = 0 \text{ or } \qquad c = \frac{b^2}{a}$$

$$\longrightarrow \quad i = \frac{a + b}{\frac{b^2}{a} + b} = \frac{a}{b} \qquad \text{or} \qquad b = -i\,a$$

$\therefore$ $d = -i\,a$ $\quad c = -a$

$$\longrightarrow \quad w = \frac{z - i}{-z - i} = \frac{i - z}{i + z}$$

- **Example 2**

Find transformation which maps
$-1,\ 0,\ 1$ into $1,\ \infty,\ i$

Let the transformation be $w(z) = \dfrac{a z + b}{c z + d}$

The mapping of the 3 points give:

$$1 = \frac{-a + b}{-c + d} \qquad \infty = \frac{b}{d} \quad i = \frac{a + b}{c + d}$$

$\longrightarrow \quad d = 0$

$$1 = \frac{-a + b}{-c} \qquad i = \frac{a + b}{c}$$

$\longrightarrow \quad a - b = -i\,(a + b) \quad$ or $\quad b = \frac{1+i}{1-i}\, a = i\, a$

$\therefore \quad c = (1 - i)\, a$

$$\longrightarrow \quad w = \frac{z + i}{(1 - i)\, z} = \frac{(z + i)(1 + i)}{2 z} = \frac{(1 + i)\, z + i - 1}{2 z}$$

- **Implicit Form**

Since 3 points defines a linear fractional transformation (see eg 1 & 2), we can write

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_2)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_2 - z_1)}$$

which is called the **implicit form** of the linear fractional transformation.

One can prove the above assertion by demonstration (see Churchill sec 71).
However, this leaves the form itself a product of pure inspiration. The following derivation aims to rectify this shortcoming.

- **proof**

It's well-known that a linear transformation

$$w = A z + B$$

can be written as

$$\frac{w - w_1}{w_2 - w_1} = \frac{z - z_1}{z_2 - z_1} \qquad (1)$$

showing that 2 points determine the transformation uniquely.

We now try to obtain a similar form for the bilinear (Mobius) transformation:

$$A z w + B z + C w + D = 0$$

The presence of the ($z w$) term then suggests the form

$$\frac{w - w_1}{w_2 - w_1} \cdot \frac{w_3 - w_2}{w - w_2} = \frac{z - z_1}{z_2 - z_1} \cdot \frac{z_3 - z_2}{z - z_2} \qquad (2)$$

where we've multiply (1) with the reciprocal of another linear transform.

$$\frac{w - w_2}{w_3 - w_2} = \frac{z - z_2}{z_3 - z_2}$$

The use of the reciprocal is to create ($z w$) terms.

Writing (2) as

$$(w - w_1)(z - z_2)(z_2 - z_1)(w_3 - w_2)$$
$$= (z - z_1)(w - w_2)(z_3 - z_2)(w_2 - w_1) \qquad (3)$$

we see that it reduces to the bilinear form after expanding the 1st two factors on each side & collecting terms.
What remains is to show that (3) is satisfied by points (w_i, z_i) $\forall\, i = 1, 2, 3$.
For $i = 1, 2$, both sides = 0.
For $i = 3$, we have

$$(w_3 - w_1)(z_2 - z_1) = (z_3 - z_1)(w_2 - w_1)$$

Incorporating this condition into (3) then gives

$$(w - w_1)(z - z_2)(z_2 - z_1)(w_3 - w_2)$$
$$= (z - z_1)(w - w_2)(z_3 - z_2)(w_3 - w_1)\frac{(z_2 - z_1)}{z_3 - z_1}$$
$$\longrightarrow \qquad \frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)} = \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$$

which the correct transform satisfying $w_i = w(z_i)$ for $i = 1, 2, 3$.
To bring it into complete agreement with eq(1) in sec71 of Churchill, we make the interchange of index $2 \leftrightarrow 3$, giving

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

The last form is easier to memorize since the subscripts appear in the order 1, 2, 3 & 3, 2, 1 in the numerator & denominator, resp.
The implicit form is particularly suited for cases where the transorm of 3 points are known, as will be illustrated in the following examples.

- **Example 1**

Find transformation which maps
$-1,\ 0,\ 1$ into $-i,\ 1,\ i$

Using

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

we have

$$\frac{(w + i)(1 - i)}{(w - i)(1 + i)} = \frac{(z + 1)(-1)}{(z - 1)(1)}$$
$$\longrightarrow \quad \frac{w + i}{w - i}\, i = \frac{z + 1}{z - 1}$$
$$w = \frac{(z - 1) - i(z + 1)}{i(z - 1) - (z + 1)}$$
$$= \frac{(1 - i)z - (1 + i)}{(i - 1)z - (1 + i)}$$
$$= \frac{-iz - 1}{iz - 1}$$
$$= \frac{-z + i}{z + i}$$

- **Example 2**

Find transformation which maps
$-1,\ 0,\ 1$ into $1,\ \infty,\ i$

Using

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

we have

$$\frac{(w-1)(\infty)}{(w-i)(\infty)}=\frac{(z+1)(-1)}{(z-1)(1)}$$

$$\longrightarrow \quad \frac{w-1}{w-i}=-\frac{z+1}{z-1}$$

$$w=\frac{i(z+1)+z-1}{z-1+z+1}$$

$$=\frac{(1+i)z+i-1}{2z}$$

- **Mappings of Upper Half Plane**

$$w=e^{i\phi}\cdot\frac{z-z_0}{z-z_0{}^*} \qquad \operatorname{Im} z_0>0$$

is the mapping which maps the upper half plane into the unit circle:

$$\left.\begin{array}{l}\operatorname{Im} z>0\\ \operatorname{Im} z=0\end{array}\right\} \longrightarrow \left\{\begin{array}{l}|w|<1\\ |w|=1\end{array}\right.$$

- **proof**

Assuming the transform to be bilinear:

$$w=\frac{az+b}{cz+d} \qquad (ad-bc\neq 0)$$

The task is to determine a, b, c, d.

For convenience, we'll adopt the convention $|f|\equiv F$ for all complex number f. Thus:

$$\begin{array}{ll} z=Z e^{i\zeta} & 0\leq\zeta\leq\pi \\ w=W e^{i\Omega} & 0\leq W\leq 1 \\ a=A e^{i\alpha} & b=B e^{i\beta} \\ c=C e^{i\gamma} & d=D e^{i\delta} \end{array}$$

Consider 1st points on the real axis:

$\operatorname{Im} z=0$, $z=\pm Z$:

$$1=\left|\frac{\pm Za+b}{\pm Zc+d}\right|$$

For the origin $z=0$:

$$1=\frac{B}{D}\longrightarrow \quad D=B\neq 0$$

For $Z\to\infty$:

$$1=\frac{A}{C}\longrightarrow \quad C=A\neq 0$$

For the rest of the upper plane:

$$\text{Im}\, z > 0$$

$$w = \frac{a\,z + b}{c\,z + d} \underset{z\to\infty}{\longrightarrow} \frac{a}{c}$$

$$\therefore \quad W \underset{z\to\infty}{\longrightarrow} \frac{A}{C} = 1$$

This is to be expected from the real axis case. It also emphasizes the fact that all 'points' with $|z| = \infty$ is considered as a single point ∞.

Using the fact that $D = B \neq 0$, $C = A \neq 0$ we can write

$$w = \frac{a}{c} \cdot \frac{z + \frac{b}{a}}{z + \frac{d}{c}}$$

$$= e^{i(\alpha-\gamma)} \cdot \frac{z + \frac{B}{A} e^{i(\beta-\alpha)}}{z + \frac{B}{A} e^{i(\delta-\gamma)}}$$

For points on the real axis:

$$1 = \left| \frac{\pm Z + \frac{B}{A} e^{i(\beta-\alpha)}}{\pm Z + \frac{B}{A} e^{i(\delta-\gamma)}} \right|$$

Using $|a-b|^2 = A^2 + B^2 - 2\,A\,B \cos(\alpha - \beta)$,

$\longrightarrow \quad \cos(\beta - \alpha) = \cos(\delta - \gamma)$

We therefore have

either $\quad \beta - \alpha = \delta - \gamma \quad \longrightarrow \quad w = e^{i(\alpha-\gamma)}$

or $\quad \beta - \alpha = -(\delta - \gamma) \quad \longrightarrow \quad w = e^{i(\alpha-\gamma)} \cdot \dfrac{z + \frac{B}{A} e^{i(\beta-\alpha)}}{z + \frac{B}{A} e^{-i(\beta-\alpha)}}$

The 1st choice is clearly too restrictive.

The 2nd option, which we shall adopt, can be beautified with the following notations:

$$\alpha - \gamma = \phi \qquad \frac{B}{A} e^{i(\beta-\alpha)} = -z_0 = -Z_0\, e^{i\zeta_0}$$

so that

$$w = e^{i\phi} \cdot \frac{z - z_0}{z - z_0^*}$$

with 3 free parameters ϕ, Z_0, & ζ_0.

To ensure the upper plane is mapped inside the circle, we need

$$1 > \left| \frac{z - z_0}{z - z_0^*} \right|$$

$\longrightarrow \quad \cos(\zeta + \zeta_0) < \cos(\zeta - \zeta_0) \quad$ for $\quad 0 < \zeta < \pi$

$\quad -\sin\zeta \sin\zeta_0 < \sin\zeta \sin\zeta_0$

$\quad -\sin\zeta_0 < \sin\zeta_0$

$\longrightarrow \quad \pi > \zeta_0 > 0 \quad$ ie. $\quad \text{Im}\, z_0 > 0$

Note: This condition can be obtained by inspection from fig 79 since z_0^* is always further away from z than z_0 does whenever z is in the upper plane.

See example 2 for the case $\text{Im}\, z_0 = 0$.

- **Example 1**

$$w = \frac{i - z}{i + z}$$

$$w = -\frac{z - i}{z + i}$$
$$= e^{i\pi} \cdot \frac{z - i}{z + i}$$

$\longrightarrow \quad z_0 = i \qquad \text{Im}\, z_0 = 1 > 0$

$\therefore$ w is of the form that maps the upper plane into a circle.

- **Example 2**

$$w = \frac{z - 1}{z + 1}$$

$z_0 = 1 \longrightarrow \qquad \text{Im}\, z_0 = 0$

The range of the mapping is no longer confined as witnessed by

$w(-1) = \infty$

Using $z = x + i\,y$, $\quad y > 0$,

$$\longrightarrow \quad w = \frac{x - 1 + i\,y}{x + 1 + i\,y}$$
$$= \frac{(x - 1 + i\,y)(x + 1 - i\,y)}{(x + 1)^2 + y^2}$$
$$= \frac{x^2 - 1 + y^2}{(x + 1)^2 + y^2} + i\,\frac{2\,y}{(x + 1)^2 + y^2}$$

$\therefore \quad \text{Im}\, w > 0 \qquad \text{for } y > 0.$

The upper half $z -$ plane is mapped onto the upper half $w -$ plane with

$w(\infty) = -1.$

The real axis $y = 0$ is mapped onto the real axis of the $w -$ plane.

$$w(x) = \frac{x^2 - 1}{(x + 1)^2}$$

For points near -1, let $x = -1 + \epsilon$

$$\longrightarrow \quad w(x) = \frac{(-1 + \epsilon)^2 - 1}{\epsilon^2} = \frac{-2 + \epsilon}{\epsilon}$$
$$= 1 - \frac{2}{\epsilon} \underset{\epsilon \to 0_{\mp}}{\longrightarrow} \pm\infty$$

which shows the entire real w axis is indeed covered.

- **$w = e^z$**

Let $\quad z = x + i\,y$

$w = e^z = e^x\, e^{i\,y} = \rho\, e^{i\,\phi}$

$\longrightarrow \quad \rho = e^x \qquad \phi = y + 2\,n\,\pi$

The mapping is in general many to 1.

The **vertical line** $x = c$, or $z = c + i\,y$ is mapped into

$w = e^c\, e^{i\,y} \qquad \text{with} \qquad |\,w\,| = e^c$

which is a **circle** of radius e^c.

Moving upward along the line (y increases) results in moving counterclockwise on the circle (ϕ increases).

Any y interval of length $2\,\pi$ will cover the whole circle once.

This mapping is many to 1.

The **horizontal line** $y = c$, or $z = x + i\,c$ is mapped into

$$w = e^x\, e^{i\,c} \qquad \text{with} \qquad |\,w\,| = e^x$$

which is a **ray** (line extending from the origin to infinity) in the direction $\phi = c$.

A **rectangular region**

$$a \le x \le b \qquad c \le y \le d$$

is mapped into

$$w = \rho\, e^{i\,\phi}$$

with limits $e^a \le \rho \le e^b \qquad c \le \phi \le d$

An **infinite strip** $a \le y \le b$ is mapped into

$$w = \rho\, e^{i\,\phi} \qquad \text{with} \qquad a \le \phi \le b$$

which is a fan shape region of infinite extend.

- ***w* = ln *z***

The mapping is $1 - 1$ only within a single branch of $\ln z$.

Let $\quad z = r\, e^{i\,\theta} \qquad \alpha \le \theta \le \alpha + 2\,\pi$

$\longrightarrow \quad w = \ln z = \ln r + i\,\theta$

which is simply the inverse of the exponential mapping.

- **Example**

$$w = \text{Log}\ \frac{z - 1}{z + 1}$$

Let $\quad Z = \dfrac{z - 1}{z + 1}$

$\longrightarrow \quad w = \text{Log}\ Z$

Since Z maps the upper half $z -$ plane into the upper half $Z -$ plane,
& Log maps the upper half $Z -$ plane into the infinite strip $0 < \text{Im}\, w < \pi$,
the composite maps the upper half $z -$ plane into the infinite strip.

- ***w* = sin *z***

Let

$$z = x + i\,y \qquad w = u + i\,v$$

$\longrightarrow \quad w = \sin z = \sin x\, \cosh y + i\cos x\, \sinh y$

$$u = \sin x\, \cosh y \qquad v = \cos x\, \sinh y$$

The mapping is many to 1 with period $2\,\pi$ in the x direction.

Fixed point at

$$x = \sin x\, \cosh y \qquad y = \cos x\, \sinh y$$

$\longrightarrow \quad x = 0, \quad y = 0. \quad \text{or} \quad z = 0$

- **Ranges**

From the definitions

$$\sinh y = \frac{1}{2}\,(e^y - e^{-y}) \quad \cosh y = \frac{1}{2}\,(e^y + e^{-y})$$

One sees that for $-\infty \le y \le \infty$

$\sinh(-y) = -\sinh y \qquad \cosh(-y) = \cosh y$

$-\infty \le \sinh y \le \infty \qquad 1 \le \cosh y \le \infty$

For a **finite range** $a \le y \le b$

we have $\sinh a \le \sinh y \le \sinh b.$

$\cosh c \le \cosh y \le \cosh d$

where

$$c = \begin{cases} \min\{\,|a|,\ |b|\,\} \\ 0 \end{cases} \text{ if } a,\ b \text{ are of the } \begin{cases} \text{same} \\ \text{opposite} \end{cases} \text{ sign.}$$

$$d = \max\{\,|a|,\ |b|\,\}$$

The situation for $\sin x$ & $\cos x$ is even more complicated owing to their periodicity. We'll dispense with an enumeration of the results for every possible combinations in the perhaps unjustifiable hope that an average student should be able to work it out for his/her-self.

- **Vertical Line**

The **vertical line** $x = c$ or $z = c + i\,y$
is mapped into

$$w = \sin c\ \cosh y + i \cos c \sinh y$$

$$u = \sin c\ \cosh y \qquad v = \cos c\ \sinh y$$

The bounds for u & v must be carefully calculated according to the rules given in sec Ranges.

For the case

$$0 \le y \le b, \qquad c \ne n\,\frac{\pi}{2}$$

we have

$$1 \le \frac{u}{\sin c} \le \cosh b \qquad 0 \le \frac{v}{\cos c} \le \sinh b$$

Using $\cosh^2 y - \sinh^2 y = 1$

$$\longrightarrow \quad \frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$$

which is a **hyperbola** with foci points at

$$w = \pm\sqrt{\sin^2 c + \cos^2 c} = \pm 1$$

The signs of $\sin c$ & $\cos c$ determine into which portion of the hyperbola is mapped.

If $c = n\,\frac{\pi}{2}$, we have u or $v = 0$ for $n =$ even or odd, which corresponds to the imaginary or real axis, resp. Further more, the $n =$ odd case uses only portions of real axis which satisfies $1 \le \frac{u}{\sin\left(n\,\frac{\pi}{2}\right)} \le \infty$.

- **Vertical Strip**

For the **semi-infinite vertical strip**

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2} \qquad y \ge 0$$

we have

$$-1 \le \sin x \le 1 \qquad 0 \le \cos x \le 1$$

$$0 \le \sinh y \le \infty \quad 1 \le \cosh y \le \infty$$

$$\longrightarrow \quad -\infty \le u \le \infty \qquad 0 \le v \le \infty$$

which is the **upper half w − plane**.

- **Horizontal Line**

The **horizontal line** $y = c$ or $z = x + i\,c$ is mapped into

$$w = \sin x \cosh c + i \cos x \sinh c$$

$$u = \sin x \cosh c \qquad v = \cos x \sinh c$$

$$\longrightarrow \quad -1 \le \frac{u}{\cosh c} \le 1 \qquad -1 \le \frac{v}{\sinh c} \le 1$$

Using $\cos^2 y + \sin^2 y = 1$

$$\longrightarrow \quad \frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$$

which is an **ellipse** with foci points at

$$w = \pm\sqrt{\cosh^2 c - \sinh^2 c} = \pm 1$$

- **Rectangular region**

Region $-\frac{\pi}{2} \le x \le \frac{\pi}{2} \qquad 0 \le y \le b$

$$\longrightarrow \quad -1 \le \sin x \le 1 \qquad 0 \le \cos x \le 1$$

$$1 \le \cosh y \le \cosh b \qquad 0 \le \sinh y \le \sinh b$$

$$u = \sin x \cosh y \qquad v = \cos x \sinh y$$

$$\longrightarrow \quad -\cosh b \le u \le \cosh b \qquad 0 \le v \le \sinh b$$

For the vertical boundaries:

1. $x = \frac{\pi}{2} \qquad 0 \le y \le b$

   is mapped into a segment of the real axis with

   $$u = \cosh y \in [1, \cosh b] \qquad v = 0$$

2. $x = -\frac{\pi}{2}\, 0 \le y \le b$

   is mapped into a segment of the real axis with

   $$u = -\cosh y \in [-\cosh b, -1] \quad v = 0$$

For the horizontal boundaries:

1. $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $y = b$

   is mapped into a segment of the upper ellipse with

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1 \quad v > 0$$

2. $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $y = 0$

   is mapped into a segment of the real axis with

$$u = \sin x \in [-1, 1] \qquad v = 0$$

- **$w = \cos z$**

$$w = \cos z = \sin\left(z + \frac{\pi}{2}\right) = \sin Z$$

where $Z = z + \frac{\pi}{2}$.

The mapping is therefore a composition of a translation & sin transformation.

- **$w = \sinh z$**

$$w = \sinh z = i \sin(i z) = i \sin Z$$

where $Z = i z$

The mapping is therefore a composition of a rotation, a sin transform, & then a rotation.

- **$w = \sqrt{z}$**

Let $z = r e^{i\theta} = r e^{i(\theta + 2\pi n)}$

$\longrightarrow$ $w = \sqrt{z} = \sqrt{r}\ e^{i\frac{(\theta+2\pi n)}{2}}$

$= \sqrt{r}\ e^{i\frac{\theta}{2} + i\pi n}$ $\qquad n = 0,\ 1$

$n = 0$ is the principal branch when $-\pi < \theta < \pi$.

Alternatively, the same result can be obtained as follows:

$$w = e^{\frac{1}{2}\ln z} = e^{\frac{1}{2}\{\ln r + i(\theta + 2\pi n)\}}$$

$$= \sqrt{r}\ e^{i\frac{\theta}{2} + i\pi n}$$

A fan shaped region

$$0 \le r \le R \qquad a \le \theta \le b$$

is mapped into another fan shaped region

$$0 \le \rho \le \sqrt{R} \qquad \frac{a}{2} \le \phi \le \frac{b}{2}$$

where $w = \rho e^{i\phi}$.

- **$w = z^{\frac{1}{n}}$**

Generalization of the above treatment to this case is obvious.

- **Example**

$$w = \sqrt{\sin z} = \sqrt{Z} \qquad Z = \sin z$$

with $\quad | Z | > 0 \qquad 0 < \text{Arg}\, Z < \frac{\pi}{2}$

This composite mapping is most easily illustrated graphically. (see fig 91).

- $\boldsymbol{w = \sqrt{P_n(z)}}$

Here P_n is a polynomial of order n.

ie. $\quad P_n(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{j=1}^{n} \left(z - z_j\right)$

where z_j are its roots.

$\longrightarrow \quad w = \sqrt{a_n} \prod_{j=1}^{n} \left(z - z_j\right)^{\frac{1}{2}}$

We have therefore $n + 1$ branch points at $z = z_j$ & ∞ if all roots are distinct.

In general, there are $m + 1$ branch points where m is the number of distinct roots of odd order. (roots of even order are not branch points)

Branch cuts are obtained by joining pairs of branch points. Since ∞ represents the entire region outside the finite plane, the number of choices of cuts is infinite & one just pick the most convenient ones.

The examples in Churchill were already discussed in Chap 3 & 6 so we won't repeat here.

- **Example**

$$w = f(z) = \sqrt{z^2 - 1}$$

$$w = \sqrt{(z-1)(z+1)}$$
$$= e^{\frac{1}{2}\{\log(z-1) + \log(z+1)\}}$$

Branch points are at $z = \pm 1$.

Caution: $\quad z = \infty$ is not a branch point since

$$w \underset{z\to\infty}{\longrightarrow} \sqrt{z^2} = z \qquad \text{is single valued.}$$

The branch cut is therefore the line segment joining points $z = \pm 1$.

It's also legitimate to use 2 lines each extending from a branch point to ∞ in an arbitrary direction. Since ∞ is a single point, these 2 lines are actually 1. The situation is best visualized in the Reimann sphere.

With the line between $z = \pm 1$ as the branch cut, we have:

$$\sqrt{z-1} = \sqrt{r_-}\, e^{i \frac{\phi_-}{2} + i n_- \pi} \qquad -\pi < \phi_- < \pi$$

$$\sqrt{z+1} = \sqrt{r_+}\, e^{i \frac{\phi_+}{2} + i n_+ \pi} \qquad 0 < \phi_+ < 2\pi$$

where any 2 consecutive n 's are adequate to denote the 2 branches.

$\longrightarrow \qquad w = \sqrt{r_+ r_-}\, e^{i\theta}$

where $\theta = \frac{1}{2}(\phi_+ + \phi_-) + (n_+ + n_-)\pi$

If ϕ_+ & ϕ_- remains in a specific branch, we see that

$$-\frac{\pi}{2} + (n_+ + n_-)\pi < \theta < \frac{3\pi}{2} + (n_+ + n_-)\pi$$

with a range of 2π.

The minimum of θ occurs at $\phi_- = -\pi$ & $\phi_+ = 0$, which is a point just beneath the branch cut.

The maximum of θ occurs at $\phi_- = \pi$ & $\phi_+ = 2\pi$, which is a point just above the branch cut.

This confirms the correctness of the branch cut assignment.