9. Conformal Mapping

Preservation of Angles

Let C be a smooth curve:

$$z = z(t) \text{ with } \quad a \le t \le b$$

A function f defined on C:

$$w = f(z) \qquad w(t) = f[\, z(t)\,]$$

will describe a curve Γ in the $w-$ plane.

Consider a point $z_0 = z(t_0)$ on C.

Let f be analytic at z_0 with $\left(\dfrac{d f}{d z}\right)_{z=z_0} \neq 0$.

Chain rule: $\dfrac{d w}{d t} = \dfrac{d f}{d z} \cdot \dfrac{d z}{d t}$

At z_0: $\left(\dfrac{d w}{d t}\right)_{t=t_0} = \left(\dfrac{d f}{d z}\right)_{z=z_0} \cdot \left(\dfrac{d z}{d t}\right)_{t=t_0}$

$\longrightarrow \qquad \text{Arg}\left(\dfrac{d w}{d t}\right)_{t=t_0} = \text{Arg}\left(\dfrac{d f}{d z}\right)_{z=z_0} + \text{Arg}\left(\dfrac{d z}{d t}\right)_{t=t_0}$

To simplify the notation a bit, let

$$\Omega_0 = \text{Arg}\left(\frac{d w}{d t}\right)_{t=t_0} \qquad \phi_0 = \text{Arg}\left(\frac{d f}{d z}\right)_{z=z_0} \qquad \zeta_0 = \text{Arg}\left(\frac{d z}{d t}\right)_{t=t_0}$$

$\longrightarrow \qquad \Omega_0 = \phi_0 + \zeta_0$

Since

$\Omega_0 =$ angle of the tangent of Γ at $w_0 \equiv w(t_0)$

$\zeta_0 =$ angle of the tangent of C at $z_0 = z(t_0)$

we see that tangents of Γ is rotated from that of C by ϕ_0.

Consider now another curve C' :

$$z = z'(t') \qquad \text{with} \qquad a' \le t' \le b'$$

The same function f will map this curve into a curve Γ' in the $w-$ plane:

$$w = f(z) \qquad w(t') = f[\, z'(t')\,]$$

Let C & C' intersect at $z = z_0 = z(t_0) = z'(t_0')$

$\longrightarrow \qquad w_0 = w(t_0) = f[\, z(t_0)\,] = f[\, z'(t_0')\,] = w(t_0')$

ie. curves Γ & Γ' also meets since the value of w depends only on that of z.

Analogous to the unprimed case, we have:

$$\Omega_0' = \phi_0 + \zeta_0'$$

where the same ϕ_0 is used here since $\dfrac{d f}{d z}$ is unique if f is analytic.

We thus have

$$\Omega_0 - \Omega_0' = \zeta_0 - \zeta_0'$$

which means the angle between 2 curves at their intersection is preserved under the mapping of an analytic function with non-zero 1st derivative there.

In general, any mapping that preserves angles is called **conformal**.

The reason to insist on $\left(\dfrac{d f}{d z}\right)_{z=z_0} \neq 0$ is because $\phi_0 = \text{Arg}\left(\dfrac{d f}{d z}\right)_{z=z_0}$ is not defined otherwise. On the other hand Ω_0 & ζ_0 involve derivative wrt real variable t so that they're always well defined.

Since f is analytic at $z_0 \longrightarrow$ f is analytic in some $N(z_0)$.

$\left(\frac{d f}{d z}\right)_{z=z_0} \neq 0 \longrightarrow \frac{d f}{d z} \neq 0 \quad \forall \; z \in N(z_0)$

Hence, f is conformal at z_0 $\longrightarrow$ f is conformal in some $N(z_0)$.

Points at which $\frac{d f}{d z} = 0$ are called **critical points**.

Mappings which preserves magnitudes of angles but not their senses are called **isogonal**.

Theorem

Let z_0 be a critical point of $w = f(z)$.

Let $f^{(m)}(z_0) \neq 0$ but $f^{(k)}(z_0) = 0 \quad \forall \; k = 1 \ldots m - 1$

Let the angle between 2 curves be θ at z_0.

$\Longrightarrow$ The angle between the image curves at $f(z_0)$ is $\phi = m\,\theta$.

proof

See ex 80.10

Example 1

$w = e^z$

$\frac{d w}{d z} = e^z = w \neq 0 \qquad \forall \; z$

$\longrightarrow$ w is conformal in the whole z plane.

Let c_1 be the vertical line $x = a, \quad z = a + i\,y$

c_2 be the horizontal line $y = b, \quad z = x + i\,b$

$\therefore \quad c_1 \underset{w}{\longrightarrow} e^a\, e^{i\,y}$ = circle of radius e^a

$c_2 \underset{w}{\longrightarrow} e^x\, e^{i\,b} =$ ray with inclination b.

c_1 & c_2 intersect at $z_0 = a + i\,y = x + i\,b \longrightarrow z_0 = (a, b)$

The inclination of c_1 is $\phi_1 = \frac{\pi}{2}$.

That of c_2 is $\phi_2 = 0$.

The angle from c_1 to c_2 at z_0 is $\phi_2 - \phi_1 = -\frac{\pi}{2}$.

Under the mapping w, the images of c_1 & c_2 meets at $w(z_0) = e^{a+i\,b}$.

The tangent of $w(c_1)$ at z_0 is: $\frac{d w}{d y} = i\,e^a\,e^{i\,b} = e^a\,e^{i\left(b+\frac{\pi}{2}\right)}$

$\longrightarrow$ the inclination is $\theta_1 = b + \frac{\pi}{2}$

The inclination of $w(c_2)$ at z_0 is $\theta_2 = b$.

The angle from $w(c_1)$ to $w(c_2)$ at z_0 is $\theta_2 - \theta_1 = -\frac{\pi}{2}$.

The mapping thus preserves both the magnitude & sense of the angle between c_1 & c_2.

Example 2

Let $\quad f = u + i\,v$

Consider 2 curves c_1 & c_2 in the z plane which are mapped under $w = f(z)$ into level curves in the w plane:

$w(c_1)$: $u = a$ $\qquad$ $w(c_2)$: $v = b$

Let c_1 & c_2 intersect at z_0 where f is analytic & $f\,'(z_0) \neq 0$.

The mapping is therefore conformal at z_0.

Since $w(c_1)$ & $w(c_2)$ are orthogonal at $w(z_0)$, c_1 & c_2 must also be orthogonal at z_0.

Example 3

Let $\quad w = z^*$

The mapping is not analytic ⟶ it's not conformal.

On the other hand, the transformation is simply a reflection about the $x-$ axis.

It's clear geometrically that reflections preserve angles but reverse their senses. The transformation is therefore isogonal.

Example 4

Let $\quad w = 1 + z^2$

⟶ $\quad w\,' = 2\,z$

$\qquad w\,'(0) = 0$

∴ $\quad z = 0$ is a critical point of w

Let $\quad Z = z^2$

⟶ $\quad w = 1 + Z$

∴ $\quad w$ is a composite of Z followed by a translation of 1.

Consider 2 rays c_1 & c_2 starting from $z = 0$ with angles $\theta_1 = \alpha$, $\theta_2 = \beta$, resp.,

ie. $\quad c_1$: $\quad z = r\,e^{i\,\alpha} \qquad 0 \le r < \infty$

$\qquad c_2$: $\quad z = r\,e^{i\,\beta} \qquad 0 \le r < \infty$

Under the mapping w, we have

$\qquad w(c_1) = 1 + r^2\,e^{i\,2\,a} \qquad 0 \le r < \infty$

$\qquad w(c_2) = 1 + r^2\,e^{i\,2\,\beta} \qquad 0 \le r < \infty$

which are rays starting from $z = 1$ with angles $\phi_1 = 2\,\alpha$, $\phi_2 = 2\,\beta$, resp.

The angle $2\,(\alpha - \beta)$ between the image rays are therefore twice that of the original rays.

The mapping is not conformal.

Further Properties

Scale Factor

Let f be conformal at z_0, ie. f is analytic at z_0 & $\left(\frac{df}{dz}\right)_{z=z_0} \neq 0$.

$$\left|\left(\frac{df}{dz}\right)_{z=z_0}\right| = \left|\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right|$$

$$= \lim_{z\to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

$\longrightarrow$ $\left|\frac{df}{dz}\right|$ = **scale factor** of the mapping.

The continuity of all quantities involved also means that the shape of a small region is roughly preserved under a conformal mapping.

Example

$$w = f(z) = z^2$$

Let $z = x + i\,y$

$\longrightarrow$ $w = x^2 - y^2 + i\,2\,x\,y$

$f\,' = 2\,z$

$\therefore$ w is conformal except at $z = 0$.

Consider the half lines

c_1 : $y = x$ $z = (1 + i)\,x$ $(x \geq 0)$

c_2: $x = 1$ $z = 1 + i\,y$ $(y \geq 0)$

which intersect at

$(1 + i)\,x = 1 + i\,y$ or $x = 1,\ y = x = 1.$

ie. $z_0 = 1 + i$

Tangent of c_1 is $\frac{dz}{dx} = 1 + i = \sqrt{2}\,e^{i\,\theta_1}$

with the inclination $\theta_1 = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

Tangent of c_2 is $\frac{dz}{dy} = i = e^{i\,\theta_2}$

with the inclination $\theta_2 = \tan^{-1}\frac{1}{0} = \frac{\pi}{2}$

$\therefore$ $\theta_2 - \theta_1 = \frac{\pi}{4}$

Under the mapping w, c_1 & c_2 becomes

$w(c_1)$: $\quad w = (1+i)^2 x^2 = i\,2\,x^2 \qquad (x \ge 0)$

$\longrightarrow \quad u = 0 \qquad v = 2\,x^2$

which is the vertical half line.

$w(c_2)$: $\quad w = (1+i\,y)^2 = 1 - y^2 + i\,2\,y \quad (y \ge 0)$

$\longrightarrow \quad u = 1 - y^2 \qquad v = 2\,y$

or $\quad u = 1 - \frac{1}{4} v^2 \qquad (\,u \le 1, \quad v \ge 0\,)$

which is the upper half parabola.

The intersect is at $\quad w_0 = w(z_0) = (1+i)^2 = 2\,i.$

Tangent of $w(c_1)$ is $\quad \frac{d\,w}{d\,x} = i\,4\,x = 4\,x\,e^{i\,\theta_1}$

with inclination $\quad \phi_1 = \frac{\pi}{2}$

Tangent of $w(c_2)$ is $\quad \frac{d\,w}{d\,y} = -2\,y + 2\,i$

At z_0 : $\left(\frac{d\,w}{d\,y}\right)_{y=1} = -2 + 2\,i = 2\sqrt{2}\;e^{i\,\phi_2}$

with inclination $\quad \phi_2 = \tan^{-1}\frac{2}{-2} = \frac{3\,\pi}{4} \quad$ (2nd quadrant)

$\therefore \quad \phi_2 - \phi_1 = \frac{\pi}{4} = \theta_2 - \theta_1$ as expected.

So far, we' ve calculated the inclination of the tangent to a curve $w(t)$ of parameter t via its derivative $\frac{d\,w}{d\,t}$. An alternative approach is via derivatives of the curve equation. For example, the curve $w(c_2)$ in the w plane is defined by

$u = 1 - \frac{1}{4} v^2 \qquad (\,u \le 1, \quad v \ge 0\,)$

or $\quad v = 2\sqrt{1-u}$

Its tangent is $\quad \frac{d\,v}{d\,u} = -\frac{1}{\sqrt{1\,-\,u}}$

which at w_0 is $\quad \left(\frac{d\,v}{d\,u}\right)_{u=0} = -\,1$

The inclination at w_0 is therefore $\quad \phi_2 = \tan^{-1}(-1) = \frac{3}{4}\pi$, or $-\frac{\pi}{4}$

Here we run into an arbitrariness which can be resolved only by assigning the sense of travel along the curve. From $u = 1 - y^2$, we see that u decreases as our curve parameter y increases. With the help of the plot in fig 102 , we choose $\phi_2 = \frac{3}{4}\pi$.

■ Local Inverse Theorem

$w = f(z)$ is conformal at z_0

$\Longrightarrow \quad \exists\, f^{-1} \;\ni \quad z_0 = f^{-1}(w_0)$

Note:

Assume the existence of f^{-1}, we have

$$w = f(z) = f\left[f^{-1}(w)\right] \text{ where } z = f^{-1}(w)$$

$$\longrightarrow \quad \frac{d w}{d w} = 1 = \frac{d f(z)}{d z}\cdot\frac{d z}{d w} = \frac{d f(z)}{d z}\cdot\frac{d f^{-1}(w)}{d w}$$

$$\therefore \quad \frac{d f^{-1}(w)}{d w} = \frac{1}{\frac{d f(z)}{d z}} = \left(\frac{d f(z)}{d z}\right)^{-1}$$

provide $\frac{d f(z)}{d z} \neq 0$ or f is conformal.

- **proof**

Writing $z = x + i y$ $w = u + i v$

the transform $w = f(z)$

corresponds to a pair of simultaneous equations

$$u = u(x, y) \qquad v = v(x, y)$$

$$\longrightarrow \quad d u = u_x d x + u_y d y$$

$$d v = v_x d x + v_y d y$$

Now, if the inverse transform f^{-1} exists, we have

$$x = x(u, v) \qquad y = y(u, v)$$

$$\longrightarrow \quad d x = x_u d u + x_v d v$$

$$d y = y_u d u + y_v d v$$

Hence:

$$d x = x_u \left(u_x d x + u_y d y\right) + x_v \left(v_x d x + v_y d y\right)$$

Since $d x$ & $d y$ are independent, their coefficients must vanish.

$$\longrightarrow \quad 1 = x_u u_x + x_v v_x$$

$$0 = x_u u_y + x_v v_y \qquad (1)$$

For a set of given partials of u & v, these eqs admit a solution only if

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0$$

The determinant is well known as the Jacobian in studies of differential eqs.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

To summarize, f^{-1} exists if $J \neq 0$.

Assuming f analytic, the Cauchy-Riemann condition gives

$$J = {u_x}^2 + {v_x}^2 = \left|\frac{\partial f}{\partial x}\right|^2 = \left|\frac{d f}{d z}\right|^2 \neq 0 \qquad \text{QED}$$

Returning to the solution of (1):

$$x_u = \frac{1}{J}\cdot\begin{vmatrix} 1 & v_x \\ 0 & v_y \end{vmatrix} = \frac{v_y}{J}$$

$$x_v = \frac{1}{J}\cdot\begin{vmatrix} u_x & 1 \\ u_y & 0 \end{vmatrix} = -\frac{u_y}{J}$$

By considering $d y$, we can obtain

$$y_u = -\frac{v_x}{J} \qquad y_v = \frac{u_x}{J}$$

Given f analytic, we see that

$$x_u = \frac{u_x}{J} = y_v$$

$$x_v = \frac{v_x}{J} = -y_u$$

$\longrightarrow$ f^{-1} is also analytic.

It should be emphasized here that the above proves only the existence of a **local** inverse. Nothing global is mentioned. In other words, if f is a many to 1 mapping , the f^{-1} discussed so far concerns only a neighborhood in one of the branches of the multi-valued f^{-1}. This point is illustrated in the following example.

■ Example

$$w = f(z) = e^z$$

f is analytic $\forall$ z finite.

$f\,' = e^z \neq 0 \;\; \forall\, z$.

$\longrightarrow$ w is conformal $\forall$ z finite.

The local inverse is

$$z = f^{-1}(w) = \log w \qquad\qquad (w \neq 0)$$

Note:

Since $e^z \neq 0\ \forall\, z$, the range of f doesn't include $w = 0$, likewise the domain of f^{-1} .

For $\log w$, the point $w = 0$ is simgularity (branch point).

Thus, f^{-1} & log are not identical.

The rule that f^{-1} is analytic wherever f is conformal also holds here.

Writing $\quad w = \rho\, e^{i\,\phi}$

$\longrightarrow$ $f^{-1}(w) = \log w = \ln \rho + i\,\phi$

The local inverse is equal to whatever branch of $\log w$ that ϕ falls into.

For example, let $z = 2\,\pi\, i$.

$\longrightarrow$ $w = e^{2\,\pi\, i}$

$$f^{-1}\left(e^{2\,\pi\, i}\right) = 2\,\pi\, i$$

which is the same as

$$\log 1$$

in the branch of , say, $\pi < \phi < 3\,\pi$.

The reciprocal relation of derivatives holds for $w \neq 0$:

$$\frac{d\,f^{-1}}{d\,w} = \frac{1}{w} = \frac{1}{e^z} = \frac{1}{\frac{d\,f}{d\,z}}$$

■ Harmonic Conjugates

■ Theorem

Any harmonic function $u(x,\ y)$ defined on a connected region has a harmonic conjugate.

■ proof

1st, a digression

Consider a curve c in the cartesian $x-y$ plane.

$$c: \quad x = x(t)\; y = y(t) \qquad a \le t \le b$$

The line integral of a $2-D$ vector $V = P\,\hat{x} + Q\,\hat{y}$ along c is defined by

$$\int_c d\,l \cdot V = \int_a^b d\,t \left\{ \frac{d\,x}{d\,t} P + \frac{d\,y}{d\,t} Q \right\}$$

$$= \int_{(x(a),\, y(a))}^{(x(b),\, y(b))} d\,x\,P + d\,y\,Q$$

$$\equiv \int_{x(a)}^{x(b)} d\,x\,P(x,\; y(x)) + \int_{y(a)}^{y(b)} d\,y\,Q(x(y),\; y)$$

The integral will be path independent if the integrand is a total derivative of some function F, ie.

$$\frac{d\,x}{d\,t} P + \frac{d\,y}{d\,t} Q = \frac{d\,F}{d\,t}$$

so that

$$\int_c d\,l \cdot V = \int_a^b d\,t\, \frac{d\,F}{d\,t} = F\,(b) - F(a)$$

depends only on the end points.

Now:

$$\frac{d\,F}{d\,t} = F_x \cdot \frac{d\,x}{d\,t} + F_y \cdot \frac{d\,y}{d\,t}$$

which means

$$P = F_x \qquad Q = F_y$$

or $\quad V = F_x\,\hat{x} + F_y\,\hat{y} = \nabla F$

Assuming the partials of F is continuous

$\longrightarrow \quad F_{x\,y} = F_{y\,x}$

$\therefore \quad P_y = Q_x$

or $\quad \nabla \times V = \left(P_y - Q_x\right)\hat{z} = 0$

which is just the Stokes theorem.

Note also that $\nabla \times V = \nabla \times \nabla F = 0$

Now, if $\quad \nabla \cdot V = 0$

$\longrightarrow \quad 0 = P_x + Q_y = F_{x\,x} + F_{y\,y} = \nabla^2 F$

Back to our problem:

u harmonic $\longrightarrow u_{xx} + u_{yy} = 0$

Setting $\quad P = -u_y \quad Q = u_x$

$\longrightarrow \quad P_y = -u_{yy} \quad Q_x = u_{xx}$

so that $P_y = Q_x$

$$P_x + Q_y = -u_{yx} + u_{xy} = 0$$

Hence $\exists\, v \ni$

$$v(x,\ y) = \int_{(x_0,\, y_0)}^{(x,\, y)} d\,x\left(-u_y\right) + d\ y\, u_x$$

$$= \int_{(x_0,\, y_0)}^{(x,\, y)} d\,s\,[-u_t(s,\ t)] + d\ t\, u_s(s,\ t)$$

& $\quad v_{xx} + v_{yy} = 0 \qquad$ ie. v is harmonic.

Furthermore:

$$v_y = u_x \ \ v_x = -u_y$$

$\longrightarrow \quad v$ is harmonic conjugate to u.

- **Example**

$$u(x,\ y) = x\ y$$

$$u_x = y \qquad u_y = x$$

$$u_{xx} = u_{yy} = 0$$

$$\longrightarrow \quad v(x,\ y) = \int_{(0,\,0)}^{(x,\,y)} d\,x\,(-x) + d\ y\ y$$

$$= \frac{1}{2}\left(-x^2 + y^2\right)$$

$$f(z) = u + i\,v = x\,y + \frac{i}{2}\left(-x^2 + y^2\right)$$

$$= -\frac{i}{2}\,z^2$$

■ Transformation of Harmonics

- **Theorem**

Let f be analytic & define the mapping

$w: \quad D_z \quad \longrightarrow \quad D_w$

$\qquad z \quad \mapsto \quad w = f(z) = u(x,\ y) + i\,v(x,\ y)$

If $h(u,\ v)$ is harmonic in D_w

$\Longrightarrow \quad H(x,\ y) = h[\,u(x,\ y),\ v(x,\ y)\,]$ is harmonic in D_z.

- **proof**

1. Let D_w be simply connected.

$h(u, v)$ is harmonic in D_w

$\longrightarrow$ $\exists\, g(u, v)$ harmonic conjugate to h.

$\therefore$ $\Phi(w) = h(u, v) + i\, g(u, v)$ is analytic in D_w

$F(z) = \Phi[f(z)]$

$= h[u(x, y), v(x, y)] + i\, g[u(x, y), v(x, y)]$

is analytic in D_z.

$\longrightarrow$ $\operatorname{Re} F(z) = h[u(x, y), v(x, y)]$ is harmonic in D_z.

2. D_w not simply connected.

Appy the argument in **1.** to each simply connected sub-domains.

- **Example**

$h(u, v) = e^{-v} \sin u$

is harmonic in $D_w = \{ w = u + i v \mid v > 0 \}$

Let $w = z^2 = x^2 - y^2 + 2 i x y$

$\longrightarrow$ $u = x^2 - y^2$ $\quad v = 2 x y$

$v > 0 \longrightarrow$ x, y are of the same sign. (1st & 3rd quadrant)

For the 1st quadrant, $x, y > 0$

$H(x, y) = e^{-2 x y} \sin\left(x^2 - y^2\right)$

is harmonic.

- **Example**

$h(u, v) = \operatorname{Im} w = v$

$h_u = 0 \qquad h_v = 1$

$h_{uu} = h_{vv} = 0$

$\longrightarrow$ h is harmonic everywhere.

Consider the horizontal strip D_w: $\quad -\frac{\pi}{2} < v < \frac{\pi}{2}$

For $w = \operatorname{Log} z$

D_z that maps into D_w is the right half plane $x > 0$.

Writing $\operatorname{Log} z = \ln \sqrt{x^2 + y^2} + i \tan^{-1} \frac{y}{x}$

$\longrightarrow$ $H(x, y) = \tan^{-1} \frac{y}{x}$

Transformation of Boundary Conditions

Theorem

$$w = f(z) = u(x, y) + i\, v(x, y)$$

Let w be conformal on smooth arc C & $\Gamma = f(C)$

Let $h(u, v)$ satisfies, on Γ, either

$$h = h_0 \qquad \text{or} \qquad \frac{d h}{d n} = 0$$

where $h_0 = \text{const}$ & $d n$ is parallel to the normal to Γ.

$\Longrightarrow$ $H(x, y) = h[\, u(x, y),\ v(x, y)\,]$ satisfies, on C, either

$$H = h_0 \ \text{or} \qquad \frac{d H}{d N} = 0$$

where N is parallel to the normal to C.

proof

1. $h = h_0 \qquad \longrightarrow \qquad H = h_0$

Proof:

This is trivial since by definition

$$H(x, y) = h[\, u(x, y),\ v(x, y)\,]$$

2. $\dfrac{d h}{d n} = 0 \qquad \longrightarrow \qquad \dfrac{d H}{d N} = 0$

Proof:

$$\frac{d h}{d n} = \nabla h \cdot \hat{n} \qquad \frac{d H}{d N} = \nabla H \cdot \hat{N}$$

$$\frac{d h}{d n} = 0 \ \text{ on } \Gamma \quad \longrightarrow \quad \nabla h \cdot \hat{n} = 0 \qquad \text{on } \Gamma$$

$\longrightarrow$ ∇h is orthogonal to the normal to Γ

$\therefore$ it's tangent to Γ

Hence Γ is orthogonal to any level curve $h(u, v) = c$ that intersect it.

The corresponding level curve in the z plane is

$$H(x, y) = h[\, u(x, y),\ v(x, y)\,] = c$$

Under a conformal mapping , this curve should be orthogonal to the image of Γ, ie. C.

Since gradients are orthogonal to level curves, ∇H is tangent to C.

$\longrightarrow$ $\nabla H \cdot \hat{N} = 0$

$\therefore$ $\dfrac{d H}{d N} = \nabla H \cdot \hat{N} = 0$

The above proof assumes $\nabla h \neq 0$.

The case for $\nabla h = 0$ is treated in ex 83.10.

Example

$$h(u, v) = v + 2$$
$$w = i\, z^2 = i\left(x^2 - y^2\right) - 2\, x\, y$$
$$H(x, y) = x^2 - y^2 + 2$$

w is entire.
$w' = 2\, i\, z = 0$ only at $z = 0$
$\longrightarrow$ w is conformal $\forall\, z \neq 0$.

The half line $y = x$, $x > 0$ or $z = (1 + i)\, x$
is mapped into $w = -2\, x^2$ $x > 0$
or $u = -2\, x^2$, $v = 0$
which is the negative $u -$ axis.
Here: $h = 2$, $H = 2$.

The positive $x -$ axis: $y = 0$, $x > 0$ or $z = x$
is mapped into $w = i\, x^2 x > 0$
or $u = 0$, $v = x^2$
which is the positive $v -$axis.
Here: $h = v + 2$, $H = x^2 + 2$
$h_u = 0$, $H_y = 0$