11. Schwarz-Christoffel Transformation

Real Axis $\longrightarrow$ Polygon

- **Single Segment**

A conformal mapping $w = f(z)$ with $\arg \left( \frac{df}{dz} \right)_{z=z_0} = \phi = \text{const}$ maps a segment of the real axis into a line segment with inclination $\phi$.

- **proof**

Consider a mapping $w = f(z)$ which is conformal at $z_0$. Let $C: z = z(s)$ be a smooth curve passing through $z_0 = z(s_0)$ with

$$\arg \left( \frac{dz}{ds} \right)_{s=s_0} = t_0$$

$\Gamma: w = f(z(s)) = w(s)$ be the image of $C$ & passes through $w_0 = w(s_0)$ with

$$\arg \left( \frac{dw}{ds} \right)_{s=s_0} = \tau_0$$

From

$$\frac{dw}{ds} = \frac{df}{dz} \cdot \frac{dz}{ds}$$

$$\rightarrow \quad \tau_0 = t_0 + \phi_0$$

where $\phi_0 = \arg \left( \frac{df}{dz} \right)_{z=z_0}$

Let $C$ be a segment of the real axis, i.e., $z = x$.

Since $x$ is real, we can use it as the curve parameter, namely, $s = x$.

$$\rightarrow \quad \frac{dz}{dx} = 1 \quad \Rightarrow \quad t = \arg \left( \frac{dz}{dx} \right) = 0 \forall z \in C$$

$$\therefore \quad \tau = \frac{dw}{dx} = \phi = \left( \frac{df}{dz} \right)_{z=z_0} \quad \forall w \in \Gamma$$

Hence:

$$\phi = \text{constant} \quad \Rightarrow \quad \tau = \text{constant.}$$

$\Gamma$ is a line segment with inclination $\phi$.

- **Construction of $f$**

Let $\frac{df}{dz} = g(z)$

The task is to find $g \ni \arg g(z) = \text{const}$ when $z$ is in some interval on the real axis. The choices are obviously numerous.

The simplest example is:

$$g(z) = z - x_0 = Ze^{i\phi} \quad \text{where} \quad Z = |z - z_0|$$

$$\rightarrow \quad \arg g(z) = \phi = \begin{cases} 0 & z = x > x_0 \\ \pi & z = x < x_0 \end{cases}$$
Another is:

\[ g(z) = (z - x_0)^p = Z^p e^{i p \theta} \]

where \( Z = |z - z_0| \)

\[ \arg g(z) = p \phi = \begin{cases} 0 & z = x > x_0 \\ p \pi & z = x < x_0 \end{cases} \]

\( g \) is multi-valued with a branch point at \( x_0 \) if \( p \neq \text{integer} \).

For \( p > 0 \), \( g \) is entire & \( g \to \infty \) as \( z \to \infty \).

For \( p < 0 \), \( g \) has a pole at \( x_0 \) (of order \( p \) if \( p = \text{integer} \)) & \( g \to 0 \) as \( z \to \infty \).

A more general example is:

\[ g(z) = h(z) (1 + i \alpha) \]

where \( h(z) = h(z^*) \) & \( \alpha = \text{real const.} \)

\[ \arg g(z) = \tan^{-1} \alpha \quad \forall x \]

**Multiple Segments**

Let the real axis be divided into \( n \) segments by \( n - 1 \) points \( x_j \), \( j = 1, 2, \ldots n - 1 \).

For convenience, we set \( x_0 = -\infty, x_n = \infty, \) & \( x_j < x_{j+1} \).

The task is to map these segments into an \( n \)-sided polygon with vertices at

\[ w_j = f(x_j) \]

where \( j = 0, 1, \ldots, n \) & \( w_0 = w_n \).

Consider \( w = f(z) \) with

\[ \frac{df}{dz} = A \prod_{j=1}^{n-1} (z - x_j)^{-p_j} \]

Since

\[ \arg \frac{df}{dz} = \arg A - \sum_{j=1}^{n-1} p_j \arg (z - x_j) \]

\& \( \arg (z - x_j) = \begin{cases} 0 & x > x_j \\ \pi & x < x_j \end{cases} \)

we see that for \( x_k < x < x_{k+1} \), \( 0 \leq k \leq n - 1 \):

\[ \arg \left( \frac{df}{dz} \right)_{z=x} = \arg A - \sum_{j=k+1}^{n-1} p_j \pi \]

with the understanding that for \( k = n - 1 \), \( \arg \left( \frac{df}{dz} \right)_{z=x} = \arg A \).

Let us trace \( \Gamma \) as \( z \) moves along the \( x \)-axis in the positive direction.

Beginning at \( w_0 = f([-\infty, 0]) \), assumed to be somewhere in the finite \( w \)-plane, one moves in the direction
\[ \tau_0 = \text{arg } A - \sum_{j=1}^{n-1} p_j \pi \]

until \( z \) reaches \( x_1 \) & \( w, w_1 \). As \( z \) moves from \( x_1 \) towards \( x_2 \), \( \Gamma \) moves from \( w_1 \) towards \( w_2 \) in a new direction

\[ \tau_1 = \text{arg } A - \sum_{j=2}^{n-1} p_j \pi = \tau_0 + p_1 \pi \]

Hence, \( \Delta \tau_1 = \tau_1 - \tau_0 = p_1 \pi \)

which can be positive or negative depending on the sign of \( p_1 \).

Since all changes of direction can be denoted in an interval \((-\pi, \pi)\), we can set

\[-1 < p_1 < 1\]

without lost of generality.

Similarly, as \( z \) moves from \( x_k \) towards \( x_{k+1} \), \( \Gamma \) moves from \( w_k \) towards \( w_{k+1} \) in the direction

\[ \tau_k = \text{arg } A - \sum_{j=k+1}^{n-1} p_j \pi = \tau_{k-1} + p_k \pi \]

so that
\[ \Delta \tau_k = \tau_k - \tau_{k-1} = p_k \pi \]

with \(-1 < p_k < 1\) & \( 1 \leq k \leq n-1 \). ( \( \tau_{n-1} = \text{arg } A \))

The total change of direction as one moves from \( w_0 \) to \( w_n \) is therefore

\[ \Delta \tau = \tau_n - \tau_0 = \sum_{k=1}^{n} \Delta \tau_k = \pi \sum_{k=1}^{n-1} p_k + \Delta \tau_n \]

where \( \Delta \tau_n = \tau_n - \tau_{n-1} \).

In order to form a closed \( \Gamma \), i.e., \( w_0 = w_n \), we must have

\[ \Delta \tau = 2 \pi m \]

We are interested only in \( \Gamma \) being a convex polygon. This means all \( \Delta \tau_k \)'s or \( p_k \)'s must be of the same sign, which we take to be positive so that \( \Gamma \) is traversed in the clockwise direction as \( x \) increases.

Next, if \( \Gamma \) is to avoid crossing itself, we need \( \Delta \tau \leq 2 \pi \), or \( m = 0 \) or 1.

The case \( m = 0 \) is excluded since it is incompatible with \( w_0 = w_n \).

Thus \( 2 \pi = \pi \sum_{k=1}^{n-1} p_k + \Delta \tau_n \)

Writing \( \Delta \tau_n = p_n \pi \)

\[ \rightarrow \quad p_n = 2 - \sum_{k=1}^{n-1} p_k \]

We have a \( n \)-sided convex polygon.

For the case \( p_n = 0 \) or \( \tau_n = \tau_{n-1} \).

\[ \rightarrow \quad \sum_{k=1}^{n-1} p_k = 2 \]

Geometrically, this means \( w_0 = w_n \) is on the line from \( w_{n-1} \) to \( w_1 \).

The polygon has only \( n - 1 \) sides.

This is the device we use to map the real axis onto a polygon whose vertices are all images of points on the \textbf{finite} real axis.
Schwarz-Christoffel Transformation

We've seen that the transformation \( w = f(z) \) with
\[
\frac{df}{dz} = A \prod_{j=1}^{n-1} (z - x_j)^{p_j}
\]
with \( p_n = 2 - \sum_{j=1}^{n-1} p_j, \quad 0 < p_j < 1 \)

maps the real axis onto a convex polygon of the positive sense.

Thus, \( \frac{df}{dz} \neq 0 \) for finite \( z \).

If all the branch cuts are oriented toward the lower plane, the mapping will be analytic & hence conformal in the finite upper half plane \( y \geq 0 \) except for the branch points \( z = x_j \).

Let us denote the region of analyticity for \( f \) by \( R \).

\[
\rightarrow f(z) = f(z_0) + \int_{z_0}^{z} ds \left( \frac{df(s)}{ds} \right) \forall z, \quad z_0 \in R
\]
\[
= B + A \int_{z_0}^{z} ds \prod_{j=1}^{n-1} (s - x_j)^{-p_j} \quad B = f(z_0)
\]

\( w = f(z) \) is known as the **Schwarz-Christoffel Transformation** (SCT).

Some properties of the SCT will be studied in some detail in the following:

Existence

An implicit assumption is that the integral in the SCT exist.

Since \( \left| \frac{df}{dz} \right| \rightarrow A \left| \frac{1}{z} \right|^{\Sigma p_j} \)

a necessary condition is therefore:
\[
\sum_{j=1}^{n-1} p_j = 2 - p_n > 1
\]

which is automatically satisfied since all \( p_j \), including \( p_n \), obeys criterion \( p_j < 1 \).
- **f** is continuous at \( z = x_f \)

Near each \( x_k \), we can write

\[
\frac{df}{dz} = (z - x_k)^{-p_k} \phi(z)
\]

where

\[
\phi(z) = A \prod_{j \neq k} (z - x_j)^{-p_j}
\]

is analytic at \( x_k \).

\[
\phi(z) = \sum_{m=0}^{\infty} \frac{\phi^{(m)}(x_k)}{m!} (z - x_k)^m
\]

\[
= \phi(x_k) + (z - x_k) \psi(z)
\]

where

\[
\psi(z) = \sum_{m=1}^{\infty} \frac{\phi^{(m)}(x_k)}{m!} (z - x_k)^{m-1}
\]

is analytic at \( x_k \).

Hence

\[
\frac{df}{dz} = (z - x_k)^{-p_k} \phi(x_k) + (z - x_k)^{1-p_k} \psi(z)
\]

\[
f(z) = f(z_0) + \int_{z_0}^{z} \left( (s - x_k)^{-p_k} \phi(x_k) + (s - x_k)^{1-p_k} \psi(z) \right) ds
\]

where for our purpose here, \( z, \ z_0 \approx x_k \).

Now: \( |p_k| < 1 \quad \implies \quad 1 - p_k > 0 \)

\[
\int_{z_0}^{z} ds (s - x_k)^{1-p_k} \psi(z)
\]

is analytic & therefore continuous at \( x_k \) as a function of \( z \).

Next

\[
\int_{z_0}^{z} ds (s - x_k)^{-p_k} = - \frac{1}{p_k} \left( (z - x_k)^{1-p_k} - (z_0 - x_k)^{1-p_k} \right)
\]

is analytic & therefore continuous at \( x_k \) as a function of \( z \).

Thus, \( f(z) \) is the sum of 2 continuous functions so that it’s continuous at \( x_k \) too.

- **Maps region** \( y > 0 \) **onto interior of Polygon**

Consider the mapping of a point \( z \) in the upper half plane.

If \( x_k < \text{Re} \ z < x_{k+1} \)

let \( z_1 = x \) be a point on the real axis with \( x_k < x < x_{k+1} \).

The line \( c \) going from \( z_1 \) to \( z \) make an angle \( \theta \) with the real axis with \( 0 < \theta < \pi \).

The image \( y \) of \( c \) goes from point \( f(x) \) on one leg of the polygon to a point \( w = f(z) \).

Since the mapping is conformal, the tangent of \( y \) at \( f(x) \) makes the same angle \( \theta \) with the image of the real axis segment which is a side of the polygon. \( w \) is therefore on the left of the polygon leg. Since the polygon is in the positive sense, \( w \) is inside it.
Interpretation

Parameters in the SCT are: \( j = 1 \ldots n \)

- \( x_j : \) \( w(x_j) = w_j = \) position of vertex \( j \).
- \( p_j : \) \( p_j \pi = \) exterior angle at vertex \( w_j \).
- \( A : \) entire polygon is rotated by an angle \( \alpha = \arg A \) & scaled by a factor \( |A| \).
- \( f(z_0) : \) position of polygon is shifted by \( f(z_0) \).
- \( z_0 : \) arbitrary constant; has no effect on polygon.

The positions of the \( n \) vertices describes the polygon completely.

This takes \( 2n \) real parameters.

If we don’t care about its exact position (2 real parameters), its orientation (1 real parameter), or its size (1 real parameter), we need only \( 2n - 4 \) parameters.

Hence, there’re only \( 2n - 4 \) amount the \( 2n \) \( x_j \) & \( p_j \) parameters that are necessary.

Now, there’s always the contraint \( p_n = 2 - \sum_{k=1}^{n-1} p_k \).

If we’re mapping the whole real axis, \( x_n = \infty \) is also fixed. Hence, there’re only \( 2n - 2 \) \( x_j \) & \( p_j \) parameters available.

Thus, 2 of them can be chosen arbitrarily if we only wish to produce a polygon similar to the chosen one. However, the \( p_j \)'s are needed to satisfy the angles at each vertex so that only 2 of the \( x_j \)'s can be arbitrary.

On the other hand, if only a finite portion of the real axis is mapped, \( x_n \) is a free parameter so that we have 3 in total for the same purpose.

Triangles & Rectangles

Triangles

To map 3 points \( x_1 \), \( x_2 \) & \( x_3 \) to the vertices \( w_1 \), \( w_2 \) & \( w_3 \) of a triangle, we need

\[
w = f(z) = A \int_{z_0}^{z} ds (s - x_1)^{-p_1} (s - x_2)^{-p_2} (s - x_3)^{-p_3} + B
\]

where \( p_1 + p_2 + p_3 = 2 \)
The exterior angle \( \phi_j \) at vertex \( w_j \) is \( \phi_j = p_j \pi \).
The corresponding interior angle \( \theta_j \) is \( \theta_j = \pi - \phi_j = (1 - p_j) \pi \)

\[
\rightarrow \quad p_j = 1 - \frac{\theta_j}{\pi}
\]

If \( x_3 = \infty \), we have

\[
w = f(z) = A \int_{z_0}^{z} ds (s - x_1)^{-p_1} (s - x_2)^{-p_2} + B
\]
Equilateral Triangle

\[ p_1 = p_2 = p_3 = p \quad \rightarrow \quad p = \frac{2}{3} \]

Setting \( x_1 = -1, \quad x_2 = 1, \quad x_3 = \infty \)

\[ \rightarrow \quad w = f(z) = A \int_{z_0}^{z} d\,s (s + 1)^{-2/3} (s - 1)^{-2/3} + B \]

Let's try \( A = 1, \quad B = 0, \quad z_0 = 1 \) (see interpretation)

\[ \rightarrow \quad w = f(z) = \int_{1}^{z} d\,s (s + 1)^{-2/3} (s - 1)^{-2/3} \]

which means \( w_2 = f(x_2) = f(1) = 0. \)

Next:

\[ w_1 = f(x_1) = f(-1) \]

\[ = \int_{1}^{-1} d\,s (s + 1)^{-2/3} (s - 1)^{-2/3} \]

We can choose the path to follow the \( x \)-axis so that

\[ w_1 = \int_{1}^{-1} d\,x (x + 1)^{-2/3} (x - 1)^{-2/3} \]

\[ = -e^{-2\pi i / 3} \int_{1}^{0} d\,x (1 + x)^{-2/3} (1 - x)^{-2/3} \]

\[ = 2 e^{\pi i / 3} \int_{0}^{1} d\,x (1 - x^3)^{-2/3} \]

Setting \( x = \sqrt[3]{t} \quad \rightarrow \quad d\,x = \frac{1}{2\sqrt[3]{t^2}} d\,t \)

\[ \therefore \quad w_1 = e^{\pi i / 3} \int_{0}^{1} d\,t \, t^{-1/2} (1 - t)^{-2/3} = e^{\pi i / 3} B\left(\frac{1}{2}, \frac{1}{3}\right) = e^{\pi i / 3} b \]

where \( b = B\left(\frac{1}{2}, \frac{1}{3}\right). \)

The 3rd vertex is:

\[ w_3 = f(x_3) = f(\infty) \]
The 3rd vertex is:

\[ w_3 = f(H_x L) = f(H_\theta L) = x_1 = \frac{x_1 - 1}{d_x H_x} + \frac{1}{L} - \frac{2}{3} H_x - \frac{1}{L} - \frac{2}{3} \]

We can again choose the path to follow the x-axis so that

\[ w_3 = \int_1^\infty dx \left( x^2 - 1 \right)^{-2/3} \]

Since \( f(z) = f(re^{i\theta}) \) is independent of \( \theta \) as \( r \to \infty \), we have \( f(\infty) = f(-\infty) \).

\[ w_3 = \int_1^\infty dx \left( x^2 - 1 \right)^{-2/3} \]

Now:

\[ \int_{-1}^\infty dx \left( x^2 - 1 \right)^{-2/3} = -\int_{-1}^1 dx \left( x^2 - 1 \right)^{-2/3} \]

\[ = -e^{-4\pi i/3} \int_1^\infty dx (x^2 - 1)^{-2/3} = e^{-\pi i/3} w_3 \]

\[ \int_{-1}^1 dx \left( x^2 - 1 \right)^{-2/3} = w_1 = e^{\pi i/3} b \]

\[ \to w_3 = \frac{e^{-\pi i/3}}{1 - e^{-\pi i/3}} b = \frac{1}{2} \left( 1 + \sqrt{3} i \right) \]

To summarize, we have:

\[ w_1 = e^{\pi i/3} b, \quad w_2 = 0, \quad w_3 = b \]

which forms an equilateral triangle of side \( b \).

Rectangles

Fluid Flow