Dispersion Relations

■ Linear Response

■ Definition

The **response** r of a system subject to an external time dependent influence f is defined to be:

$$r(t) = \int_{-\infty}^{\infty} dt' \, g(t, t') \, f(t')$$

When g is independent of f, the response is called **linear**.

■ Impulsive Forces

Consider an influence of the **impluse** type:

$$f_I(t'; t_0) = f_0 \, \delta(t' - t_0) \qquad \longrightarrow \qquad r_I(t; t_0) = f_0 \, g_I(t, t_0)$$

If the system is in equilibrium before the application of f, the response should have the form

$$r_I(t; t_0) = \theta(t - t_0) \, r_I(t - t_0)$$

$$\longrightarrow \qquad g_I(t, t_0) = \theta(t - t_0) \, g(t - t_0)$$

$$g(t - t_0) = r_I(t - t_0) \, / \, f_0 \qquad\qquad r_I(t - t_0) = g(t - t_0) \, f_0$$

■ General Case

Using $f(t') = \int_{-\infty}^{\infty} dt'' \, f(t'') \, \delta(t'' - t')$, we have

$$\begin{aligned}
r(t) &= \int_{-\infty}^{\infty} dt' \, g(t, t') \int_{-\infty}^{\infty} dt'' \, f(t'') \, \delta(t'' - t') \\
&= \int_{-\infty}^{\infty} dt'' \int_{-\infty}^{\infty} dt' \, g(t, t') \, f(t'') \, \delta(t'' - t') \\
&= \int_{-\infty}^{\infty} dt'' \int_{-\infty}^{\infty} dt' \, g(t, t') \, f_I(t''; t') \\
&= \int_{-\infty}^{\infty} dt'' \, r_I(t; t'') \\
&= \int_{-\infty}^{\infty} dt'' \, \theta(t - t'') \, r_I(t - t'') \\
&= \int_{-\infty}^{\infty} dt'' \, \theta(t - t'') \, g(t - t'') \, f(t'')
\end{aligned}$$

$$\longrightarrow \qquad g(t, t') = \theta(t - t') \, g(t - t') \qquad\qquad (\textbf{ causality })$$

$$\therefore \qquad r(t) = \int_{-\infty}^{\infty} dt' \, \theta(t - t') \, g(t - t') \, f(t') = \int_{-\infty}^{t} dt' \, g(t - t') \, f(t')$$

$$\underset{\tau = t - t'}{=} \int_{0}^{\infty} d\tau \, g(\tau) \, f(t - \tau)$$

Dispersion

Fourier Transform $G(\omega)$

Let the fourier transform of a function $h(t)$ be denoted by the capitalized symbol $H(\omega)$:

$$h(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} H(\omega)$$

$$H(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} h(t)$$

Convolution Theorem applied to the linear response

$\longrightarrow$ $\quad R(\omega) = G(\omega) F(\omega)$

where $\quad G(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} \theta(\tau)\, g(\tau) = \int_{0}^{\infty} d\tau\, e^{i\omega\tau} g(\tau)$

$$g(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} G(\omega)$$

Now: $\quad \int_0^{\infty} d\tau\, e^{i\omega\tau} g(\tau) \leq \int_0^{\infty} d\tau \left| e^{i\omega\tau} g(\tau) \right| = \int_0^{\infty} d\tau \left| g(\tau) \right|$

$\therefore$ $\quad G(\omega)$ exists

$\Longrightarrow$ $\quad \int_0^{\infty} d\tau \left| g(\tau) \right|$ exists

$\Longrightarrow$ $\quad \left| g(\tau) \right| \underset{\tau \to \infty}{\longrightarrow} 0$

Analytic Continuation $G(z)$

We now analytic continuate $G(\omega)$ into the entire complex plane $z = \omega + i\,\omega\,'$.

$$G(z) = \int_0^{\infty} d\tau\, e^{i z \tau} g(\tau) = \int_0^{\infty} d\tau\, e^{i\omega\tau - \omega'\tau} g(\tau)$$

$\left| G(z) \right| \underset{|z| \to \infty}{\longrightarrow} 0$ $\qquad$ in the upper half plane.

Using

$$\left| e^{i z \tau} \right|^2 = e^{i z \tau - i z^* \tau} = e^{-2\tau \,\mathrm{Im} z} = e^{-2\tau |z| \sin\theta} \qquad \text{where } \theta = \arg z$$

we have:

$$\left| G(z) \right| \leq \int_0^{\infty} d\tau \left| e^{i z \tau} \right| \left| g(\tau) \right|$$

$$= \int_0^{\infty} d\tau\, e^{-\tau |z| \sin\theta} \left| g(\tau) \right|$$

$$\leq M_G \int_0^\infty \mathrm{d}\tau\, e^{-\tau |z| \sin\theta}$$

$$= \frac{\mathrm{M}_G}{|z| \sin\theta} \qquad \text{if } 0 < \theta < \pi \quad (\text{ upper half } z \text{ - plane })$$

where $M_G = \max |g|$

Hence for $\theta \neq 0, \ \pi$,

$$\left| G(z) \right| \xrightarrow[|z| \to \infty]{} 0$$

For $\theta = 0$ or π, $G(z) = G(\omega)$.

Similar arguments applied to the fact that

$$g(\tau) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-i\omega\tau} G(\omega) \ \text{ exists}$$

gives: $\left| G(\omega) \right| \xrightarrow[|\omega| \to \infty]{} 0$

Hence $\left| G(z) \right| \xrightarrow[|z| \to \infty]{} 0$ in the upper half plane.

$G(z)$ is analytic in the half plane above the real axis