Fourier Transform

P.M.Morse & H.Feshbach, "Methods of Theoretical Physics", Sec 4.8 (78)

- **Definition**

The Fourier transform $F(k)$ of a function $f(x)$ is defined to be:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{dx}\; e^{i k x}\; f(x)$$

- **Lebesgue Class**

$f \in L^p[a,\, b]$ $\qquad\qquad (p > 0)$

$\Longleftrightarrow$ $\quad f$ is in the Lebesgue class L^p in $[a,\, b]$

$\Longleftrightarrow$ $\quad | f |^p$ is Lebesgue integrable in $[a,\, b]$.

- **Parseval's Formula**

Let $f \in L^2(-\infty,\ \infty)$

$$\int_{-\infty}^{\infty} \mathrm{dk}\, \left| F(k) \right|^2 = \int_{-\infty}^{\infty} \mathrm{dx}\, \left| f(x) \right|^2$$

- **proof**

Let $\quad I = \int_{-\infty}^{\infty} \mathrm{dk}\, \left| F(k) \right|^2\; e^{-\frac{1}{2}\delta^2 k^2}$

Substituting in $F(k)$:

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{dk}\; e^{-\frac{1}{2}\delta^2 k^2} \int_{-\infty}^{\infty} \mathrm{dx}\; e^{i k x}\, f(x) \int_{-\infty}^{\infty} \mathrm{d}\xi\; e^{-i k x}\, f(\xi)^*$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{dx}\; f(x) \int_{-\infty}^{\infty} \mathrm{d}\xi\; f(\xi)^* \int_{-\infty}^{\infty} \mathrm{dk}\; e^{-\frac{1}{2}\delta^2 k^2 + i k (x-\xi)}$$

Using

$$-\frac{1}{2}\delta^2 k^2 + i k (x - \xi) = -\frac{1}{2}\delta^2 \left(k - i\, \frac{x-\xi}{\delta^2} \right)^2 - \frac{(x-\xi)^2}{2\delta^2}$$

& $\quad \int_{-\infty}^{\infty} \mathrm{dk}\; e^{-\frac{1}{2}\delta^2 k^2} = \frac{\sqrt{2\pi}}{\delta}$

we have

$$I = \frac{1}{\sqrt{2\pi}\;\delta} \int_{-\infty}^{\infty} \mathrm{dx}\; f(x) \int_{-\infty}^{\infty} \mathrm{d}\xi\; f(\xi)^*\, e^{-\frac{(x-\xi)^2}{2\delta^2}}$$

$$= \frac{1}{\sqrt{2\pi}\;\delta} \int_{-\infty}^{\infty} \mathrm{dx} \int_{-\infty}^{\infty} \mathrm{d}\xi\; f(x)\, e^{-\frac{(x-\xi)^2}{4\delta^2}}\; f(\xi)^*\, e^{-\frac{(x-\xi)^2}{4\delta^2}}$$

Using Schwarz's inequality:

$$I \le \frac{1}{\sqrt{2\pi}\;\delta} \sqrt{\left(\left\{ \int_{-\infty}^{\infty} \mathrm{dx} \int_{-\infty}^{\infty} \mathrm{d}\xi\; \left| f(x) \right|^2\, e^{-\frac{(x-\xi)^2}{2\delta^2}} \right\} \left\{ \int_{-\infty}^{\infty} \mathrm{dx} \int_{-\infty}^{\infty} \mathrm{d}\xi\; \left| f(\xi) \right|^2\, e^{-\frac{(x-\xi)^2}{2\delta^2}} \right\} \right)}$$

$$= \frac{1}{\sqrt{2\pi}\;\delta} \int_{-\infty}^{\infty} \mathrm{dx} \int_{-\infty}^{\infty} \mathrm{d}\xi\; \left| f(x) \right|^2\, e^{-\frac{(x-\xi)^2}{2\delta^2}}$$

$$= \int_{-\infty}^{\infty} \mathrm{dx}\; \left| f(x) \right|^2$$

Hence, for $\delta \to 0$

$$I = \int_{-\infty}^{\infty} \mathrm{dk} \, \left| F(k) \right|^2 \le \int_{-\infty}^{\infty} \mathrm{dx} \, \left| f(x) \right|^2$$

$\longrightarrow \quad F(k) \in L^2$ if $f(x)$ does.

Using

$$\frac{1}{\sqrt{2\pi}\;\delta} e^{-\frac{(x-\xi)^2}{2\delta^2}} \underset{\delta \to 0}{\longrightarrow} \delta(x-\xi)$$

we have

$$I = \frac{1}{\sqrt{2\pi}\;\delta} \int_{-\infty}^{\infty} \mathrm{dx} \; f(x) \int_{-\infty}^{\infty} \mathrm{d}\xi \; f(\xi)^* \, e^{-\frac{(x-\xi)^2}{2\delta^2}}$$

$$\underset{\delta \to 0}{\longrightarrow} \int_{-\infty}^{\infty} \mathrm{dx} \, \left| f(x) \right|^2 \qquad \text{QED}$$

- **Fourier Integral Theorem**

Let $f \in L^2(-\infty, \infty)$

$$F(k, a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{dx} \; e^{i k x} \, f(x)$$

$$f(x, a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \mathrm{dk} \; e^{-i k x} \, F(k)$$

$\Longrightarrow$ As $a \to \infty$, $F(k, a)$ & $f(x, a)$ converge in the mean to $F(k)$ & $f(x)$, resp.

ie.

$$\lim_{a \to \infty} \int_{-a}^{a} \mathrm{dk} \, \left| F(k, a) - F(k) \right|^2 = 0$$

$$\lim_{a \to \infty} \int_{-a}^{a} \mathrm{dx} \, \left| f(x, a) - f(x) \right|^2 = 0$$