Further Applications

Summation of Series

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_n \text{Res}\{ f(z_n) \cot(\pi z_n) \}$$
$$\sum_{n=-\infty}^{\infty} (-)^n f(n) = -\pi \sum_n \text{Res}\{ f(z_n) \csc(\pi z_n) \}$$

where z_n are the poles of $f(z)$.

proof

The major step in the proof is to show that

$$g(z) = \pi \cot(\pi z)$$

has simple poles at $z = n$ with residue 1.

Given the above, the 1st formula is obtained by considering the contour integral which encloses the entire z-plane.

Let $g(z) = \pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)}$

$\longrightarrow$ poles of $g(z)$ are simple & at $z = n$.

$$\text{Res}_{z=n}\, g(z) = \left(\pi \frac{\cos(\pi z)}{\frac{d}{dz}\sin(\pi z)} \right)_{z=n} = 1$$

To evaluate

$$S_N = \sum_{n=-N}^{N} f(n)$$

we use a contour C_N which is a square that is centered at $z = 0$ & intersects the real axis at $x = \pm \left(N + \frac{1}{2}\right)$:

$$\longrightarrow \quad \oint_{C_N} dz\, g(z)\, f(z) = 2\pi i \left\{ \sum_{n=-N}^{N} f(N) + \sum_k \text{Res}\,[f(z_k)\, \pi \cot(\pi z_k)] \right\}$$

where k runs over all poles of f inside C_N.

As $N \longrightarrow \infty$, $\oint_{C_N} dz\, g(z)\, f(z) \longrightarrow 0$ since $\left| f \right| \xrightarrow[|z| \to \infty]{} 0$ if S converges. QED.

Proof for the 2nd formula is analogous. All we need is

$$\text{Res}_{z=n}\, \pi \csc(\pi z) = \left(\pi \frac{1}{\frac{d}{dz}\sin(\pi z)} \right)_{z=n} = \frac{1}{\cos(\pi n)} = (-)^n$$

Example

$$S = \sum_{n=-\infty}^{\infty} \frac{(-)^n}{(a+n)^2}$$

$f(z) = \frac{1}{(a+z)^2}$ with 2nd order pole at $z = -a$

$$S = -\pi \text{Res}_{z=-a} \left\{ \frac{1}{(a+z)^2} \cot(\pi z) \right\}$$
$$= -\pi \left\{ \frac{d}{dz} \csc(\pi z) \right\}_{z=-a}$$
$$= \pi^2 \frac{\cos(\pi a)}{\sin^2(\pi a)}$$

Asymptotic Series

Definition

Let

$$S_N = \sum_{n=0}^{N} \frac{A_n}{z^n} \qquad\qquad S = \sum_{n=0}^{\infty} \frac{A_n}{z^n}$$

If

$$f(z) = \phi(z)\, S$$

$$\lim_{|z| \to \infty} \left\{ z^N \left[\frac{f(z)}{\phi(z)} - S_N \right] \right\} = 0$$

$\Longrightarrow$ S represents $\frac{f(z)}{\phi(z)}$ **asymptotically**.

Note:

Usually, S diverges & there is an optimal N which gives the best approximation.

Example

Exponential Function:

$$\text{Ei}(x) = \int_{-\infty}^{x} dt\, \frac{e^t}{t}$$

$$E_1(x) = \int_x^{\infty} dt\, \frac{e^{-t}}{t} = -\text{Ei}(-x)$$

$$= -\int_x^{\infty} d\, e^{-t} \cdot \frac{1}{t} = -\left(\frac{e^{-t}}{t}\right)_x^{\infty} - \int_x^{\infty} dt\, \frac{e^{-t}}{t^2}$$

$$= \frac{e^{-x}}{x} - \int_x^{\infty} dt\, \frac{e^{-t}}{t^2}$$

$$= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2 \int_x^{\infty} dt\, \frac{e^{-t}}{t^3}$$

$$= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\, \frac{e^{-x}}{x^3} - 3!\int_x^{\infty} dt\, \frac{e^{-t}}{t^4}$$

$$= \frac{e^{-x}}{x} \left\{1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-)^n \frac{n!}{x^n}\right\} + (-)^{n+1} (n+1)! \int_x^{\infty} dt\, \frac{e^{-t}}{t^{n+2}}$$

$$= \frac{e^{-x}}{x} S_n + (-)^{n+1} (n+1)! \int_x^{\infty} dt\, \frac{e^{-t}}{t^{n+2}}$$

where $S_n = 1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-)^n \frac{n!}{x^n} = \sum_{m=0}^{n} (-)^m \frac{m!}{x^m}$

Cauchy Test:

$$\lim_{m \to \infty} \left| \frac{\frac{(m+1)!}{x^{m+1}}}{\frac{m!}{x^m}} \right| = \lim_{m \to \infty} \frac{(m+1)}{|x|} \longrightarrow \infty$$

$\therefore$ $S = \lim_{n \to \infty} S_n$ diverges.

On the other hand:

$$\lim_{x\to\infty} \left(x^n \left[E_1(x)\, x\, e^x - S_n \right] \right)$$

$$= \lim_{x\to\infty} \left(x^{n+1} e^x (-)^{n+1} (n+1)! \int_x^\infty dt\, \frac{e^{-t}}{t^{n+2}} \right)$$

$$< \lim_{x\to\infty} \left(x^{-1} e^x (-)^{n+1} (n+1)! \int_x^\infty dt\, e^{-t} \right)$$

$$= \lim_{x\to\infty} \left\{ x^{-1} (-)^{n+1} (n+1)! \right\}$$

$$= 0$$

$\therefore$ $E_1(x) = \frac{e^{-x}}{x} S$ is an asymptotic representation.

The error involved in using the S_n is

$$(-)^{n+1} (n+1)! \int_x^\infty dt\, \frac{e^{-t}}{t^{n+2}}$$

$$< (-)^{n+1} (n+1)! \frac{1}{x^{n+2}} \int_x^\infty dt\, e^{-t}$$

$$= (-)^{n+1} (n+1)! \frac{1}{x^{n+2}} e^{-x}$$

■ Properties

1. Function represented by asymptotic expansion is not unique.
   eg. $\frac{f}{\phi}$ & $\frac{f}{\phi} + e^{-z}$ have the same expansion.
2. Phase change in z often produces discontinuities (**Stokes Phenomena**).
3. Asymptotic series can be added, multiplied & integrated.
4. Differentiation of asymptotic series is valid only if the derivative function also has an asymptotic series expansion.

■ Method of Steepest Descent
(Saddle Point Method)

1st Term

■ Formula

$$\int_c dt\, g(t)\, e^{z f(t)} = \sum_s g(t_s)\, e^{z f(t_s)}\, e^{i \tau_s} \sqrt{\frac{2\pi}{|z f''(t_s)|}}$$

$$= \sum_s (\pm)\, g(t_s)\, e^{z f(t_s)} \sqrt{\frac{2\pi}{z f''(t_s)}}$$

where $|z| >> 1$,

$g(t) \approx$const near the saddle points t_s,

$e^{z f(t)} = 0$ at the end points of c.

& $2\tau_s = \pm \pi - \text{Arg } f''(t_s) - \text{Arg } z$

with the sign chosen to conform with the original contour.

- **Proof**

We shall begin with the integral

$I(z) = \int_c \mathrm{dt}\, e^{z\, f(t)}$

Consider a saddle points t_s of f

ie. $f\,'(t_s) = 0.$

Let $f(t) = f(t_s) + \frac{1}{2} f\,''(t_s)\,(t - t_s)^2 + \ldots$

$f\,''(t_s) = \left| f\,''(t_s) \right| e^{i\,\phi}$

$t - t_s = T\, e^{i\,\tau}$

$z = |z|\; e^{i\,\zeta}$

Keeping only terms up to the 2nd order in T:

$z\, f(t) = z\, f(t_s) + \frac{1}{2} \left| z\, f\,''(t_s) \right| e^{i\,(\zeta + \phi + 2\,\tau)}\, T^2$

$= \; z\, \mathrm{f}(t_s) + \frac{1}{2} \left| z\, f\,''(t_s) \right| T^2 \{ \cos(\zeta + \phi + 2\,\tau) + i \sin(\zeta + \phi + 2\,\tau) \}$

Now, c or τ is chosen such that along c,

1. $\mathrm{Re}\{ z\,[f(t) - f(t_s)] \}$ is a maximum
2. $\mathrm{Im}\{ z\,[f(t) - f(t_s)] \} = \mathrm{const}$ near t_s.

This can be accomplished by setting

$\zeta + \phi + 2\,\tau = \pm\pi$

so that $z\{ f(t) - f(t_s) \} = \mathrm{Re}\{ z\,[f(t) - f(t_s)] \}$

$= \; -\frac{1}{2} \left| z\, f\,''(t_s) \right| \; T^2$

< 0

on c near t_s: $\mathrm{dt} = e^{i\,\tau}\, \mathrm{dT}$ $(\tau = \mathrm{const})$

Thus, near t_s, the contibution to $I(z)$ is:

$I_s(z) = e^{z\, f(t_s)}\; e^{i\,\tau} \int_{c_s} \mathrm{dT}\, e^{-\frac{1}{2} \left| z\, f''(t_s) \right| T^2}$

where c_s is the portion $(\tau = \mathrm{const})$ of the deformed c which goes through t_s.

Since $|z| >> 1$, we can replace $\int_{c_s} \mathrm{dT}$ by $2 \int_0^\infty \mathrm{dT}$.

$\longrightarrow \quad \int_{c_s} \mathrm{dT}\, e^{-\frac{1}{2} \left| z\, f''(t_0) \right| T^2} \simeq \sqrt{\frac{2\,\pi}{|z\, f''(t_s)|}}$

$I_s(z) = e^{z\, f(t_s)}\; e^{i\,\tau} \sqrt{\frac{2\,\pi}{|\, z\, f''(t_s)\,|}}$

where $2\,\tau = \pm\,\pi - \phi - \zeta$

$= \; \pm\,\pi - \mathrm{Arg}\, f\,''(t_s) - \mathrm{Arg}\, z$

with the sign chosen to conform with the original contour.

Away from the saddle points, $\mathrm{Im}\{ z\,[f(t) - f(t_s)] \}$ varies rapidly so that contributions from different parts of the contour cancels each other. It is therefore a good approximation to write:

$I(z) = \sum_s I_s(z)$

$= \; \sum_s e^{z\, f(t_s)}\; e^{i\,\tau_s} \sqrt{\frac{2\,\pi}{|z\, f''(t_s)|}}$

where $2\,\tau_s = \pm\,\pi - \phi_s - \zeta$

Substituting τ into the last expression gives

$$I_s(z) = \pm\, e^{z\, f(t_s)} \sqrt{\frac{-2\pi}{z\ f''(t_s)}}$$

with the sign is again chosen to conform with the original contour.
Actually, this result can be directly arrived at following the previous derivation but without introducing all the phase angles.

It's easy to see that

$$\int_c \mathrm{dt}\, \mathrm{g}\,(\mathrm{t})\ \mathrm{e}^{\mathrm{z\,f\,(t)}} = \sum_{\mathrm{s}} \mathrm{g}\,(t_s)\, \mathrm{e}^{\mathrm{z\,f}\,(t_s)}\ \mathrm{e}^{\mathrm{i}\,\tau_s} \sqrt{\frac{2\pi}{|\mathrm{z\,f}''(t_s)|}}$$

$$= \sum_{\mathrm{s}} (\pm)\, \mathrm{g}\,(t_s)\, \mathrm{e}^{\mathrm{z\,f}\,(t_s)} \sqrt{\frac{2\pi}{\mathrm{z\,f}''(t_s)}}$$

provided $g \simeq$ const near each saddle point.
More precisely, we require $|\Delta\, g| << |\Delta\, e^{z\, f}|$ near each t_s.
This is satisfied for any rational function g.

■ Example $H_\nu^{(1)}(s)$

$$H_\nu^{(1)}(s) = \frac{1}{\pi\, i} \int_c \mathrm{dz}\, \frac{e^{\frac{s}{2}\left(z-\frac{1}{z}\right)}}{z^{\nu+1}} \qquad (\ s \text{ real } \&\ >> 1\ )$$

where c goes from $z = 0$ with initial zero slope via the upper plane to $z = (-\infty, 0)$ asymptotically along the negative x-axis. (See Arfken, 3rd ed., eg 7.4.1.)

Let $f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right)$ $\qquad g(z) = \frac{1}{\pi\, i\, z^{\nu+1}}$

$\longrightarrow \quad f\,' = \frac{1}{2}\left(1 + \frac{1}{z^2}\right)$

The saddle point in the upper plane is $z_0 = i$.

$\longrightarrow \quad f(z_0) = i \qquad g(z_0) = -\frac{1}{\pi\ i^\nu}$

$\qquad f\,'' = -\frac{1}{z^3} \qquad f\,''(z_0) = -\,i = e^{i\,\frac{3\pi}{2}}$

$\longrightarrow \quad \phi = \frac{3\pi}{2} \quad \tau = \pm\, \frac{\pi}{2} - \frac{3\pi}{4} = -\frac{\pi}{4}$ or $-\frac{5\pi}{4}$

To conform with the original c, we must have $\tau = -\frac{5\pi}{4}$

$$\longrightarrow \quad H_\nu^{(1)}(s) = \left(-\frac{1}{\pi\ i^\nu}\right) e^{i\,s}\, e^{-\frac{5\pi}{4}} \sqrt{\frac{2\pi}{s}}$$

$$= \mathrm{e}^{i\,s - i\,\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)} \sqrt{\frac{2}{\pi\, s}}$$

If we are to use the other formula, all the phase calculations can be omitted & we arrive directly at:

$$H_\nu^{(1)}(s) = -\left(-\frac{1}{\pi\ i^\nu}\right) e^{i\,s} \sqrt{\frac{2\pi}{s\, i}}$$

where the negative overall sign is used because the upper limit of the original contour is $-\infty$.

■ Example $\Gamma(z)$

$$\Gamma(z+1) = z\,! = \int_0^\infty \mathrm{dx}\ x^z\, e^{-x} \qquad (\ |z| >> 1\ )$$

The original contour is along the positive real axis.
To put the integral into a form to which the saddle point method is applicable, let

$x = z\,t$

$\longrightarrow \quad \mathrm{dx} = z\,\mathrm{dt} \qquad x^z = z^z\, t^z \qquad t \in [0, \infty)$

$$z\,! = z^{z+1} \int_0^\infty \mathrm{dt}\; t^z\, e^{-z\,t}$$

$$= z^{z+1} \int_0^\infty \mathrm{dt}\; e^{z(\ln t - t)}$$

Let $f(t) = \ln t - t$

$\longrightarrow \quad f\,' = \frac{1}{t} - 1$

$\therefore$ saddle point at $t_0 = 1$

$f(1) = -1$

$f\,'' = -\frac{1}{t^2} \quad \longrightarrow \quad f\,''(1) = -1 = e^{i\pi} \qquad (\phi = \pi)$

$z = |z|\, e^{i\zeta}$

$2\,\tau = \pm\,\pi - \pi - \zeta = -\zeta$ or $-2\pi - \zeta$

$\tau = -\frac{\zeta}{2}$ or $-\pi - \frac{\zeta}{2}$

We must use $\tau = -\frac{\zeta}{2}$ to conform with the original contour. (This is obvious for the case of z real, ie. $\zeta = 0$).

$$z\,! = z^{z+1}\, e^{-z}\, e^{-i\frac{\zeta}{2}} \sqrt{\frac{2\pi}{|z|}}$$

$$= z^{z+1}\, e^{-z} \sqrt{\frac{2\pi}{z}}$$

$$= z^{z+\frac{1}{2}}\, e^{-z} \sqrt{2\pi}$$

If we are to use the other formula, all the phase calculations can be omitted & we arrive directly at:

$$z\,! = z^{z+1}\, e^{-z} \sqrt{\frac{2\pi}{z}}$$

Asymptotic Series

Theory

Keeping the higher order terms in the series

$$f(t) = f(t_s) + \frac{1}{2}\, f\,''(t_s)\,(t - t_s)^2 + \ldots$$

gives us an asymptotic series of the integral.
For simplicity, we'll treat only the case

$$I(z) = \int_c \mathrm{dt}\, e^{z\, f(t)}$$

with a single saddle point. Generalization to more complicated situations should be straightforward.

To begin, let

$$f(t) = f(t_s) - w^2$$

where w is real.

$$\longrightarrow \quad I(z) = e^{z\, f(t_s)} \int_c \mathrm{dt}\, e^{-z\, w^2}$$

$$= \quad e^{z\, f(t_s)} \int_c \mathrm{dw}\; \frac{d\,t}{d\,w}\; e^{-z\, w^2}$$

Expanding $\frac{d\,t}{d\,w}$ as a power series:

$$\frac{d\,t}{d\,w} = \sum_{n=0}^{\infty} a_n\, w^n$$

where only even power need be retained.

$$\longrightarrow \quad I(z) = \sum_{n=0}^{\infty} a_n\; e^{z\, f(t_s)} \int_c \mathrm{dw}\; w^n\; e^{-z\, w^2}$$

$$\simeq \pm \sum_{n=0}^{\infty} a_n\; e^{z\, f(t_s)} \int_{-\infty}^{\infty} \mathrm{dw}\; w^n\; e^{-z\, w^2}$$

$$= \quad \pm \sum_{n=0}^{\infty} a_{2\,n}\; e^{z\, f(t_s)}\, \Gamma\!\left(n + \frac{1}{2}\right) \frac{1}{z^{n+\frac{1}{2}}}$$

with the sign chosen to conform with the original contour.

The main job is to find $a_{2\,n}$.

By integration from t_s to t , we see that

$$t - t_s = \sum_{n=0}^{\infty} a_n\, \frac{w^{n+1}}{n+1} \qquad (\; w(t_s) = 0\;)$$

$$= \quad \sum_{n=1}^{\infty} a_{n-1}\, \frac{w^n}{n}$$

On the other hand,

$$w^2 = f(t) - f(t_s) = \sum_{n=2}^{\infty} A_n\, (t - t_s)^n$$

where $\qquad A_n = -\frac{1}{n!}\; f^{(n)}(t_s)$

The calculation of $a_{2\,n}$ is then just the inversion of the power series of w^2.

There are many ways to do so, one of which is by contour integration. (see Morse & Feshbach).

A more elementary way is to substitute the series of $t - t_s$ into that of w^2 & collect coefficients. Thus, putting $T = t - t_s$, we have

$$w^2 = \sum_{n=2}^{\infty} A_n \left\{ \sum_{m=1}^{\infty} a_{m-1}\, \frac{w^m}{m} \right\}^n$$

$$= \quad \sum_{n=2}^{\infty} A_n \left\{ a_0\, w + \frac{a_1}{2}\, w^2 + \frac{a_2}{3}\, w^3 + \ldots \right\}^n$$

$$= \quad \sum_{n=2}^{\infty} A_n\, a_0^{\;n}\, w^n \left\{ 1 + \frac{a_1}{2\,a_0}\, w + \frac{a_2}{3\,a_0}\, w^2 + \ldots \right\}^n$$

$$= \quad A_2\, a_0^{\;2}\, w^2 \left\{ 1 + \frac{a_1}{2\,a_0}\, w + \frac{a_2}{3\,a_0}\, w^2 + \ldots \right\}^2$$

$$+ A_3\, a_0^{\,3}\, w^3 \left\{1 + \frac{a_1}{2\,a_0}\, w + \frac{a_2}{3\,a_0}\, w^2 + \ldots\right\}^3$$

$$+ A_4\, a_0^{\,4}\, w^4 \left\{1 + \frac{a_1}{2\,a_0}\, w + \frac{a_2}{3\,a_0}\, w^2 + \ldots\right\}^4 + \ldots$$

$\longrightarrow$

$w^2:\quad 1 = A_2\, a_0^{\,2}$

$w^3:\quad 0 = A_2\, a_0^{\,2}\, \frac{a_1}{a_0} + A_3\, a_0^{\,3}$

$w^4:\quad 0 = A_2\, a_0^{\,2} \left\{ \left(\frac{a_1}{2\,a_0}\right)^2 + 2\,\frac{a_2}{3\,a_0}\right\} + A_3\, a_0^{\,3} \cdot \frac{3\,a_1}{2\,a_0} + A_4\, a_0^{\,4}$

Hence:

$$a_0 = \frac{1}{\sqrt{A_2}}$$

$$a_1 = -\frac{A_3}{A_2}\, a_0^{\,2} = -\frac{A_3}{A_2^{\,2}}$$

$$0 = A_2\, \frac{a_1^{\,2}}{4} + \frac{2}{3}\, A_2\, a_0\, a_2 + \frac{3}{2}\, A_3\, a_0^{\,2}\, a_1 + A_4\, a_0^{\,4}$$

$$= \frac{1}{4}\,\frac{A_3^{\,2}}{A_2^{\,3}} + \frac{2}{3}\sqrt{A_2}\; a_2 - \frac{3}{2}\,\frac{A_3^{\,2}}{A_2^{\,3}} + \frac{A_4}{A_2^{\,2}}$$

$$\longrightarrow\quad a_2 = \frac{3}{2\sqrt{A_2}} \left\{ \frac{5}{4}\,\frac{A_3^{\,2}}{A_2^{\,3}} - \frac{A_4}{A_2^{\,2}} \right\}$$

It is clear the process is tedious & error prone.
Anyone who intend to further develope the proceedings is strongly advised to use one of the symbolic manipulation programs such as mathematica.

- **Example** **Γ(z)**

$f(t) = \ln t - t$

$\longrightarrow\quad f\,' = \frac{1}{t} - 1$

$\therefore\quad$ saddle point at $t_0 = 1$

$f(1) = -1$

$f\,'' = -\frac{1}{t^2} \quad\longrightarrow\quad f\,''(1) = -1 \qquad A_2 = \frac{1}{2}$

$f\,''' = \frac{2}{t^3} \quad\longrightarrow\quad f\,'''(1) = 2 \qquad A_3 = -\frac{1}{3}$

$f^{(4)} = -\frac{6}{t^4} \quad\longrightarrow\quad f^{(4)}(1) = -6 \qquad A_4 = \frac{1}{4}$

$\therefore\quad a_0 = \sqrt{2}$

$$a_2 = \frac{3}{\sqrt{2}} \left\{ \frac{5}{4} \cdot \frac{8}{9} - 1 \right\} = \frac{1}{3\sqrt{2}}$$

$$z\,! = e^{-z}\, z^{z+1} \left\{ \sqrt{2}\; \Gamma\!\left(\frac{1}{2}\right) \frac{1}{\sqrt{z}} + \frac{1}{3\sqrt{2}}\, \Gamma\!\left(\frac{3}{2}\right) z^{-\frac{3}{2}} + \ldots\right\}$$

$$= e^{-z}\, z^{z+1} \sqrt{2\,\pi} \left\{ z^{-\frac{1}{2}} + \frac{1}{12}\, z^{-\frac{3}{2}} + \ldots\right\}$$

$$= e^{-z}\, z^{z+\frac{1}{2}} \sqrt{2\,\pi} \left\{1 + \frac{1}{12\,z} + \ldots\right\}$$