Further Applications

Summation of Series

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{n} \operatorname{Res}\{ f(z_n) \cot(\pi z_n) \}$$

$$\sum_{n=-\infty}^{\infty} (-)^n f(n) = -\pi \sum_{n} \operatorname{Res}\{ f(z_n) \csc(\pi z_n) \Big\}$$

where z_n are the poles of $f(z)$.

proof

The major step in the proof is to show that

$$g(z) = \pi \cot(\pi z)$$

has simple poles at $z = n$ with residue 1.

Given the above, the 1st formula is obtained by considering the contour integral which encloses the entire z-plane.

Let $\quad g(z) = \pi \cot(\pi z) = \pi \dfrac{\cos(\pi z)}{\sin(\pi z)}$

$\longrightarrow$ poles of $g(z)$ are simple & at $z = n$.

$$\operatorname*{Res}_{z=n} g(z) = \left(\pi \frac{\cos(\pi z)}{\frac{d}{dz}\sin(\pi z)} \right)_{z=n} = 1$$

To evaluate

$$S_N = \sum_{n=-N}^{N} f(n)$$

we use a contour C_N which is a square that is centered at $z = 0$ & intersects the real axis at $x = \pm \left(N + \frac{1}{2}\right)$:

$$\longrightarrow \quad \oint_{C_N} dz\, g(z)\, f(z) = 2\pi i \left\{ \sum_{n=-N}^{N} f(N) + \sum_{k} \operatorname{Res}[\, f(z_k)\, \pi \cot(\pi z_k)\,] \right\}$$

where k runs over all poles of f inside C_N.

As $N \longrightarrow \infty$, $\oint_{C_N} dz\, g(z)\, f(z) \longrightarrow 0$ since $|f| \xrightarrow[|z|\to\infty]{} 0$ if S converges. QED.

Proof for the 2nd formula is analogous. All we need is

$$\operatorname*{Res}_{z=n} \pi \csc(\pi z) = \left(\pi \frac{1}{\frac{d}{dz}\sin(\pi z)} \right)_{z=n} = \frac{1}{\cos(\pi n)} = (-)^n$$

Example

$$S = \sum_{n=-\infty}^{\infty} \frac{(-)^n}{(a + n)^2}$$

$$f(z) = \frac{1}{(a+z)^2} \quad \text{with 2nd order pole at } z = -a$$

$$\begin{aligned} S &= -\pi \operatorname*{Res}_{z=-a} \left\{ \frac{1}{(a+z)^2} \cot(\pi z) \right\} \\ &= -\pi \left\{ \frac{d}{dz} \csc(\pi z) \right\}_{z=-a} \\ &= \pi^2 \frac{\cos(\pi a)}{\sin^2(\pi a)} \end{aligned}$$

Asymptotic Series

Definition

Let

$$S_N = \sum_{n=0}^{N} \frac{A_n}{z^n} \qquad S = \sum_{n=0}^{\infty} \frac{A_n}{z^n}$$

If

$$f(z) = \phi(z)\, S$$

$$\lim_{|z| \to \infty} \left\{ z^N \left[\frac{f(z)}{\phi(z)} - S_N \right] \right\} = 0$$

$\Longrightarrow$ S represents $\frac{f(z)}{\phi(z)}$ **asymptotically**.

Note:
Usually, S diverges & there is an optimal N which gives the best approximation.

■ Example

Exponential Function:

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} d\,t\ \frac{e^{t}}{t}$$

$$E_1(x) = \int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t} = -\mathrm{Ei}(-x)$$

$$= -\int_{x}^{\infty} d\,e^{-t}\cdot\frac{1}{t} = -\left(\frac{e^{-t}}{t}\right)_x^{\infty} - \int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^2}$$

$$= \frac{e^{-x}}{x} - \int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^2}$$

$$= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^3}$$

$$= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\,\frac{e^{-x}}{x^3} - 3!\int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^4}$$

$$= \frac{e^{-x}}{x}\left\{1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-)^n\,\frac{n!}{x^n}\right\} + (-)^{n+1}\,(n+1)!\int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^{n+2}}$$

$$= \frac{e^{-x}}{x}\,S_n + (-)^{n+1}\,(n+1)!\int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^{n+2}}$$

where $S_n = 1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-)^n\,\frac{n!}{x^n} = \sum_{m=0}^{n} (-)^m\,\frac{m!}{x^m}$

Cauchy Test:

$$\lim_{m\to\infty}\left|\frac{\frac{(m+1)!}{x^{m+1}}}{\frac{m!}{x^m}}\right| = \lim_{m\to\infty}\frac{(m+1)}{|\,x\,|} \longrightarrow \infty$$

∴ $S = \lim_{n\to\infty} S_n$ diverges.

On the other hand:

$$\lim_{x\to\infty}\left(x^n\,[\,E_1(x)\,x\,e^x - S_n]\,\right)$$

$$= \lim_{x\to\infty}\left(x^{n+1}\, e^{x}\, (-)^{n+1}\, (n+1)!\int_{x}^{\infty} d\,t\; \frac{e^{-t}}{t^{n+2}} \right)$$

$$< \lim_{x\to\infty}\left(x^{-1}\, e^{x}\, (-)^{n+1}\, (n+1)!\int_{x}^{\infty} d\,t\; e^{-t} \right)$$

$$= \lim_{x\to\infty}\left\{ x^{-1}\; (-)^{n+1}\, (n+1)! \right\}$$

$$= 0$$

$\therefore \quad E_1(x) = \dfrac{e^{-x}}{x} S$ is an asymptotic representation.

The error involved in using the S_n is

$$(-)^{n+1}\, (n+1)!\int_{x}^{\infty} d\,t\; \frac{e^{-t}}{t^{n+2}}$$

$$< (-)^{n+1}\, (n+1)!\, \frac{1}{x^{n+2}}\int_{x}^{\infty} d\,t\; e^{-t}$$

$$= (-)^{n+1}\, (n+1)!\, \frac{1}{x^{n+2}}\, e^{-x}$$

- **Properties**

1. Function represented by asymptotic expansion is not unique.
   eg. $\dfrac{f}{\phi}$ & $\dfrac{f}{\phi} + e^{-z}$ have the same expansion.
2. Phase change in z often produces discontinuities (**Stokes Phenomena**).
3. Asymptotic series can be added, multiplied & integrated.
4. Differentiation of asymptotic series is valid only if the derivative function also has an asymptotic series expansion.

Method of Steepest Descent
(Saddle-Point Method)

- **Concept**

Consider

$$I(z) = \int_{c_0} d\,t\, e^{z\, f(t)} \qquad \text{where} \quad |\, z\,| >> 1$$

& c_0 is such that $e^{z\, f(t)} \longrightarrow 0$ at both ends.

Let $z = |z|\, e^{i\,\zeta}$

$\longrightarrow \quad z\; f = |z|\, e^{i\,\zeta}\; f = |z|\, F$

where $F = e^{i\,\zeta}\; f$

Let $f(t) = u + i\, v$ where $u = \text{Re}\,[\; f(t)\;]$ & $v = \text{Im}\,[\; f(t)\;]$.

$\longrightarrow \quad F = e^{i\,\zeta}(\, u + i\, v\,) = U + i\, V$

where $U = u\cos\zeta - v\sin\zeta \qquad V = u\sin\zeta + v\cos\zeta$

$$\longrightarrow \quad I(z) = \int_{c_0} d\,t\, e^{|z|\,(\,U + i\,V\,)}$$

Within the region where F is analytic, we can deform c_0 at will into any path c with the same end points without changing the value of $I(\alpha)$.

Consider the point t_0 where $f\,'(t_0) = 0$.
f is analytic $\longrightarrow$ $f\,' = u_x + i\,v_x = -i\left(u_y + i\,v_y\right)$
$\therefore$ $u_x(t_0) = u_y(t_0) = v_x(t_0) = v_y(t_0) = 0$
However, $\nabla^2 u = \nabla^2 v = 0$ so that t_0 can only be a saddle point for both u & v.

Let c go through point t_0 in such a way that $U(t_0)$ is a maximum along c & $V(t) =$ const near t_0. The most significant contribution to $I(\alpha)$ is then concentrated near t_0. A Taylor series expansion of f at t_0 then gives an asymptotic expansion of $I(\alpha)$. This is called the **saddle point** or **steepest descent** method.

- **First Approximation**

$$\int_c d\,t\, e^{\alpha\, f(t)} \simeq e^{\alpha\, f(t_0)} \sqrt{\frac{2\,\pi}{z\; e^{i\,\pi}\, f\,''(t_0)}}$$

- **proof**

Let $\quad f(t) = f(t_0) + \frac{1}{2}(t - t_0)^2 f''(t_0) + \ldots$

where $\; f'(t_0) = 0.$

$\longrightarrow \quad f(t) - f(t_0) \simeq \frac{1}{2}(t - t_0)^2 f''(t_0)$

Since, on c, u is a maximum at t_0 and $v(t) = v(t_0)$ near t_0, we have:

$$\alpha\{ f(t) - f(t_0)\} = \alpha\{u(t) - u(t_0)\} \quad \text{is real \& } < 0.$$

$$\simeq \frac{1}{2}\alpha(t - t_0)^2 f''(t_0)$$

$$\equiv -\frac{1}{2}\alpha\,\tau^2$$

where $\quad \tau^2 = -(t - t_0)^2 f''(t_0) \quad$ is real & > 0.

$\quad \tau = \sqrt{-f''(t_0)}\,(t - t_0) \quad$ is real

Let $\quad t - t_0 = r\,e^{i\beta} \quad$ with β =const. for t near t_0.

$\longrightarrow \quad \tau = r\sqrt{e^{i(\pi+2\beta)} f''(t_0)} \qquad$ (β is chosen so that this is real)

$\quad d\tau = \sqrt{e^{i(\pi+2\beta)} f''(t_0)}\; d t \qquad$ near t_0

$$\longrightarrow \quad I(z) = \int_c d t\, e^{\alpha f(t)}$$

$$\simeq \frac{e^{\alpha f(t_0)}}{\sqrt{e^{i(\pi+2\beta)} f''(t_0)}} \int_c d\tau\, e^{-\frac{1}{2}\alpha\tau^2}$$

If the original contour travels in the same direction as c at t_0, we can approximate $\int_c d\tau\, e^{-\frac{1}{2}|z|\tau^2}$ by

$$\int_{-\infty}^{\infty} d\tau\, e^{-\frac{1}{2}|z|\tau^2} = \sqrt{\frac{2\pi}{|z|}}$$

$$\longrightarrow \quad I(z) \simeq e^{z f(t_0)} \sqrt{\frac{2\pi}{|z|\; e^{i(\pi+\theta)} f''(t_0)}}$$

$$= \; \mathrm{e}^{z f(t_0)} \sqrt{\frac{2\pi}{z\; e^{i\pi} f''(t_0)}}$$

- **Example**

Gamma Function:

$$\Gamma(z+1) = \int_0^{\infty} d\tau\; \tau^z e^{-\tau}$$

1st, we put the integrand into the $e^{z f(t)}$ form.

Let $\tau = t\,z \qquad d\tau = z\,d t$

$$\Gamma(z+1) = \int_0^\infty d t\; z\,(t\,z)^z\, e^{-t z}$$

$$= z^{z+1} \int_0^\infty d t\; e^{z \ln t}\; e^{-t z}$$

$$= z^{z+1} \int_0^\infty d t\; e^{z(\ln t - t)}$$

$$= z^{z+1}\, I(z)$$

where $I(z) = \displaystyle\int_0^\infty d t\; e^{z f(t)}, \qquad f(t) = \ln t - t$

$$f\,' = \frac{1}{t} - 1 = 0 \qquad \longrightarrow \qquad t_0 = 1, \qquad f(1) = -1$$

$$f\,'' = -\frac{1}{t^2} \qquad \longrightarrow \qquad f\,''(1) = -1$$

$$\tau = \sqrt{e^{i(\pi+\theta)}\, f\,''(t_0)}\;(t - t_0) = e^{i\theta/2}(t-1)$$

The original contour has $\theta = 0$. This gives the upper limit of τ as $+\infty$.

$$I(z) \simeq e^{z f(t_0)} \sqrt{\frac{2\pi}{z\, e^{i\pi}\, f\,''(t_0)}} = e^{-z}\sqrt{\frac{2\pi}{z}}$$

$$\Gamma(z+1) \underset{z\to\infty}{\longrightarrow} z^{z+1}\, I(z) = \sqrt{2\pi}\; z^{z+\frac{1}{2}}\, e^{-z}$$