# Further Applications

## Summation of Series

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{n} \text{Res}\{ f(z_n) \cot(\pi z_n) \}$$
$$\sum_{n=-\infty}^{\infty} (-)^n f(n) = -\pi \sum_{n} \text{Res}\{ f(z_n) \csc(\pi z_n) \Big\}$$
where $z_n$ are the poles of $f(z)$.

### proof

The major step in the proof is to show that
$$g(z) = \pi \cot(\pi z)$$
has simple poles at $z = n$ with residue 1.
Given the above, the 1st formula is obtained by considering the contour integral which encloses the entire z-plane.

Let $\quad g(z) = \pi \cot(\pi z) = \pi \dfrac{\cos(\pi z)}{\sin(\pi z)}$

$\longrightarrow$ poles of $g(z)$ are simple & at $z = n$.

$$\underset{z=n}{\text{Res}}\, g(z) = \left( \pi \frac{\cos(\pi z)}{\frac{d}{dz}\sin(\pi z)} \right)_{z=n} = 1$$

To evaluate
$$S_N = \sum_{n=-N}^{N} f(n)$$
we use a contour $C_N$ which is a square that is centered at $z = 0$ & intersects the real axis at $x = \pm\left(N + \frac{1}{2}\right)$:

$$\longrightarrow \quad \oint_{C_N} dz\, g(z)\, f(z) = 2\pi i \left\{ \sum_{n=-N}^{N} f(N) + \sum_{k} \text{Res}[\, f(z_k)\, \pi \cot(\pi z_k)\,] \right\}$$

where $k$ runs over all poles of $f$ inside $C_N$.

As $N \longrightarrow \infty$, $\oint_{C_N} dz\, g(z)\, f(z) \longrightarrow 0$ since $|f| \xrightarrow[|z|\to\infty]{} 0$ if $S$ converges. QED.

Proof for the 2nd formula is analogous. All we need is
$$\underset{z=n}{\text{Res}}\, \pi \csc(\pi z) = \left( \pi \frac{1}{\frac{d}{dz}\sin(\pi z)} \right)_{z=n} = \frac{1}{\cos(\pi n)} = (-)^n$$

## Example

$$S = \sum_{n=-\infty}^{\infty} \frac{(-)^n}{(a + n)^2}$$

$$f(z) = \frac{1}{(a + z)^2} \qquad \text{with 2nd order pole at } z = -a$$

$$S = -\pi \operatorname*{Res}_{z=-a} \left\{ \frac{1}{(a + z)^2} \cot(\pi z) \right\}$$

$$= -\pi \left\{ \frac{d}{dz} \csc(\pi z) \right\}_{z=-a}$$

$$= \pi^2 \frac{\cos(\pi a)}{\sin^2(\pi a)}$$

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## Asymptotic Series

### ■ Definition

Let

$$S_N = \sum_{n=0}^{N} \frac{A_n}{z^n} \qquad S = \sum_{n=0}^{\infty} \frac{A_n}{z^n}$$

If

$$f(z) = \phi(z) S$$

$$\lim_{|z| \to \infty} \left\{ z^N \left[ \frac{f(z)}{\phi(z)} - S_N \right] \right\} = 0$$

$\Longrightarrow$ $S$ represents $\frac{f(z)}{\phi(z)}$ **asymptotically**.

**Note:**
Usually, $S$ diverges & there is an optimal $N$ which gives the best approximation.

## ■ Example

**Exponential Function:**

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} d\,t\ \frac{e^{t}}{t}$$

$$E_1(x) = \int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t} = -\mathrm{Ei}(-x)$$

$$= -\int_{x}^{\infty} d\,e^{-t} \cdot \frac{1}{t} = -\left(\frac{e^{-t}}{t}\right)_x^{\infty} - \int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^2}$$

$$= \frac{e^{-x}}{x} - \int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^2}$$

$$= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^3}$$

$$= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\,\frac{e^{-x}}{x^3} - 3!\int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^4}$$

$$= \frac{e^{-x}}{x}\left\{1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-)^n\,\frac{n!}{x^n}\right\} + (-)^{n+1}\,(n+1)!\int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^{n+2}}$$

$$= \frac{e^{-x}}{x}\,S_n + (-)^{n+1}\,(n+1)!\int_{x}^{\infty} d\,t\ \frac{e^{-t}}{t^{n+2}}$$

where $S_n = 1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-)^n\,\frac{n!}{x^n} = \sum_{m=0}^{n} (-)^m\,\frac{m!}{x^m}$

Cauchy Test:

$$\lim_{m\to\infty}\left|\frac{\frac{(m+1)!}{x^{m+1}}}{\frac{m!}{x^m}}\right| = \lim_{m\to\infty}\frac{(m+1)}{|\,x\,|} \longrightarrow \infty$$

$\therefore \quad S = \lim_{n\to\infty} S_n$ diverges.

On the other hand:

$$\lim_{x\to\infty}\left(x^n\left[E_1(x)\,x\,e^x - S_n\right]\right)$$

$$= \lim_{x\to\infty}\left(x^{n+1}\,e^{x}\,(-)^{n+1}\,(n+1)!\int_x^\infty d\,t\;\frac{e^{-t}}{t^{n+2}}\right)$$

$$< \lim_{x\to\infty}\left(x^{-1}\,e^{x}\,(-)^{n+1}\,(n+1)!\int_x^\infty d\,t\;e^{-t}\right)$$

$$= \lim_{x\to\infty}\left\{\,x^{-1}\;(-)^{n+1}\,(n+1)!\right\}$$

$$= 0$$

$\therefore$ $\quad E_1(x) = \dfrac{e^{-x}}{x} S$ is an asymptotic representation.

The error involved in using the $S_n$ is

$$(-)^{n+1}\,(n+1)!\int_x^\infty d\,t\;\frac{e^{-t}}{t^{n+2}}$$

$$< (-)^{n+1}\,(n+1)!\;\frac{1}{x^{n+2}}\int_x^\infty d\,t\;e^{-t}$$

$$= (-)^{n+1}\,(n+1)!\;\frac{1}{x^{n+2}}\,e^{-x}$$

- **Properties**

1. Function represented by asymptotic expansion is not unique.

   eg. $\dfrac{f}{\phi}$ & $\dfrac{f}{\phi} + e^{-z}$ have the same expansion.
2. Phase change in $z$ often produces discontinuities ( **Stokes Phenomena** ).
3. Asymptotic series can be added, multiplied & integrated.
4. Differentiation of asymptotic series is valid only if the derivative function also has an asymptotic series expansion.

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## Method of Steepest Descent
## ( Saddle Point Method )

$$\int_c d\,t\,g(t)\,e^{z\,f\,(t)} = \sum_s g(t_s)\,e^{z\,f\,(t_s)}\;e^{i\,\tau_s}\sqrt{\frac{2\,\pi}{|\;z\,f\;''(t_s)\;|}}$$

where $|\,z\,| >> 1$ ,

$g(t) \approx$const near the saddle points $t_s$,

$e^{z\,f(t)} = 0$ at the end points of $c$.

& $\quad 2\,\tau_s = \pm\,\pi - \text{Arg}\,f\;''(t_s) - \text{Arg}\,z$

with the sign chosen to conform with the original contour.

- **Proof**

Consider a saddle points $t_s$ of $f$
ie. $f\,'(t_s) = 0$.

Let $f(t) = f(t_s) + \frac{1}{2} f\,''(t_s)(t - t_s)^2 + \ldots$

$f\,''(t_s) = |f\,''(t_s)|\, e^{i\phi}$

$t - t_s = T\, e^{i\tau}$

$z = |z|\, e^{i\zeta}$

Keeping only terms up to the 2nd order in $T$:

$$z f(t) = z f(t_s) + \frac{1}{2}\left|z f\,''(t_s)\right| e^{i(\zeta+\phi+2\tau)} T^2$$

$$= z f(t_s) + \frac{1}{2}\left|z f\,''(t_s)\right| T^2 \{\cos(\zeta+\phi+2\tau) + i\sin(\zeta+\phi+2\tau)\}$$

Now, $c$ or $\tau$ is chosen such that along $c$,
1. Re$\{z[f(t) - f(t_s)]\}$ is a maximum
2. Im$\{z[f(t) - f(t_s)]\}$ = const near $t_s$.

This can be accomplished by setting

$$\zeta + \phi + 2\tau = \pm\pi$$

so that $z\{f(t) - f(t_s)\} = \text{Re}\{z[f(t) - f(t_s)]\}$

$$= -\frac{1}{2}\left|z f\,''(t_s)\right| T^2$$

$$< 0$$

on $c$ near $t_s$: $dt = e^{i\tau}\, dT \quad (\tau = \text{const})$

Thus, near $t_s$, the contibution to $I(z)$ is:

$$I_s(z) = e^{z f(t_s)}\, e^{i\tau} \int_{c_s} dT\, e^{-\frac{1}{2}\left|z f''(t_s)\right| T^2}$$

where $c_s$ is the portion ( $\tau$ = const ) of the deformed $c$ which goes through $t_s$.

Since $|z| >> 1$, we can replace $\int_{c_s} dT$ by $2\int_0^\infty dT$.

$$\longrightarrow \quad \int_{c_s} dT\, e^{-\frac{1}{2}\left|z f''(t_0)\right| T^2} \simeq \sqrt{\frac{2\pi}{|z f\,''(t_s)|}}$$

$$I_s(z) = e^{z f(t_s)}\, e^{i\tau} \sqrt{\frac{2\pi}{|z f\,''(t_s)|}}$$

where $2\tau = \pm\pi - \phi - \zeta$

$= \pm\pi - \text{Arg}\, f\,''(t_s) - \text{Arg}\, z$

with the sign chosen to conform with the original contour.

Away from the saddle points, Im$\{z[f(t) - f(t_s)]\}$ varies rapidly so that contributions from different parts of the contour cancels each other. It is therefore a good approximation to write:

$$I(z) = \sum_s I_s(z)$$

$$= \sum_s e^{z f(t_s)}\, e^{i\tau_s} \sqrt{\frac{2\pi}{|z f\,''(t_s)|}}$$

where $2\tau_s = \pm\pi - \phi_s - \zeta$

**Note:**
Substituting $\tau$ into the last expression gives

$$I(z) = \pm e^{z f(t_s)} \sqrt{\frac{-2\pi}{z\ f''(t_s)}}$$

which looks simpler owing to the absence of $\tau$. This is deceptive since the determination of the overall sign always requires the calculation of $\tau$.

It's easy to see that

$$\int_c d\,t\,g(t)\ e^{z f(t)} = \sum_s g(t_s)\, e^{z f(t_s)}\ e^{i \tau_s} \sqrt{\frac{2\pi}{|\ z\ f''(t_s)\ |}}$$

provided $g \simeq$ const near each saddle point.
More precisely, we require $|\Delta g| << |\Delta e^{z f}|$ near each $t_s$.
This is satisfied for any rational function $g$.

Further improvement on the approximation can be achieved by keeping higher order terms in the Taylor series of $f$. The mathematics are more involved. Interested readers can consult:
H.Jeffreys, B.Jeffreys, "Methods of Mathematical Physics", 3rd ed., Chap 17, (56)
P.M.Morse, H.Feshbach, "Method of Theoretical Physics", sec 4.6 (78)

## ■ Example $H_\nu^{(1)}(s)$

$$H_\nu^{(1)}(s) = \frac{1}{\pi i} \int_c d\,z \frac{e^{\frac{s}{2}\left(z - \frac{1}{z}\right)}}{z^{\nu+1}} \quad (\ s \text{ real } \&\ >> 1\ )$$

where $c$ goes from $z = 0$ with initial zero slope via the upper plane to $z = (-\infty, 0)$ asymptotically along the negative x-axis. ( See Arfken, 3rd ed., eg 7.4.1. )

Let $f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right) \qquad g(z) = \frac{1}{\pi\, i\, z^{\nu+1}}$

$\longrightarrow \qquad f' = \frac{1}{2}\left(1 + \frac{1}{z^2}\right)$

The saddle point in the upper plane is $z_0 = i$.

$\longrightarrow \qquad f(z_0) = i \qquad\qquad g(z_0) = -\frac{1}{\pi\ i^\nu}$

$\qquad f'' = -\frac{1}{z^3} \qquad\qquad f''(z_0) = -i = e^{i\frac{3\pi}{2}}$

$\longrightarrow \qquad \phi = \frac{3\pi}{2} \qquad \tau = \pm\ \frac{\pi}{2} - \frac{3\pi}{4} = -\frac{\pi}{4}$ or $-\frac{5\pi}{4}$

To conform with the original $c$, we must have $\tau = -\frac{5\pi}{4}$

$$\longrightarrow \qquad H_\nu^{(1)}(s) = \left(-\frac{1}{\pi\ i^\nu}\right) e^{i s}\, e^{-\frac{5\pi}{4}} \sqrt{\frac{2\pi}{s}}$$

$$= e^{i s - i \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)} \sqrt{\frac{2}{\pi s}}$$

**Example** **Γ(z)**

$$\Gamma(z+1) = z! = \int_0^\infty dx\; x^z e^{-x} \qquad (\,|z| >> 1\,)$$

The original contour is along the positive real axis.
To put the integral into a form to which the saddle point method is applicable, let

$$x = z\,t$$

$$\longrightarrow \quad dx = z\,dt \qquad x^z = z^z t^z \qquad t \in [\,0,\ \infty\,)$$

$$z! = z^{z+1} \int_0^\infty dt\; t^z e^{-zt}$$

$$= z^{z+1} \int_0^\infty dt\; e^{z(\ln t - t)}$$

Let $f(t) = \ln t - t$

$$\longrightarrow \quad f\,' = \frac{1}{t} - 1$$

∴ saddle point at $t_0 = 1$

$$f(1) = -1$$

$$f\,'' = -\frac{1}{t^2} \quad \longrightarrow \quad f\,''(1) = -1 = e^{i\pi} \qquad (\,\phi = \pi\,)$$

$$z = |z|\; e^{i\zeta}$$

$$2\tau = \pm\,\pi - \pi - \zeta = -\zeta \ \text{ or } \ -2\pi - \zeta$$

$$\tau = -\frac{\zeta}{2} \ \text{ or } \ -\pi - \frac{\zeta}{2}$$

We must use $\tau = -\frac{\zeta}{2}$ to conform with the original contour. ( This is obvious for the case of $z$ real, ie. $\zeta = 0$).

$$z! = z^{z+1} e^{-z}\, e^{-i\frac{\zeta}{2}} \sqrt{\frac{2\pi}{|\,z\,|}}$$

$$= z^{z+1} e^{-z} \sqrt{\frac{2\pi}{z}}$$

$$= z^{z+\frac{1}{2}} e^{-z} \sqrt{2\pi}$$