■ $\nabla^2 \psi + k^2 \psi = 0$

■ Notation

Let $\nabla^2 \psi + k^2 \psi = 0$

$k^2 = 0 \quad \Longrightarrow \quad$ Laplace eq

$k^2 > 0 \quad \Longrightarrow \quad$ Helmholtz eq

$k^2 < 0 \quad \Longrightarrow \quad$ Diffusion eq

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \qquad \textbf{Cylindrical Coord}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2 \sin^2 \theta} \left\{ \sin\theta \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{\partial^2}{\partial \phi^2} \right\} \qquad \textbf{Spherical Coord}$$

■ Cylindrical Coord

Let $\psi(\rho, \phi, z) = P(\rho)\, \Phi(\phi)\, Z(z)$

$$\longrightarrow \quad \frac{d^2 Z}{d z^2} = l^2 Z$$

$$\frac{d^2 \Phi}{d \phi^2} = -m^2 \Phi$$

$$\left\{ \frac{d^2}{d \rho^2} + \frac{1}{\rho} \frac{d}{d \rho} - \frac{m^2}{\rho^2} + l^2 + k^2 \right\} P = 0$$

■ Spherical Coord

Let $\psi(r, \theta, \phi) = R(r)\, Y(\theta, \phi) = R(r)\, \Theta(\theta)\, \Phi(\phi)$

$$\longrightarrow \quad \frac{d^2 \Phi}{d \phi^2} = -m^2 \Phi$$

$$\left\{ \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) - \frac{m^2}{\sin^2\theta} + l(l+1) \right\} \Theta = 0$$

$$\left\{ \frac{d^2}{d r^2} + \frac{2}{r} \frac{d}{d r} - \frac{l(l+1)}{r^2} + k^2 \right\} R = 0$$

■ Bessel Functions

■ Bessel's Eq

$$x^2 Z_\nu'' + x Z_\nu' + \left(x^2 - \nu^2 \right) Z_\nu = 0$$

■ Helmholtz Eq in Cylindrical Coord

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d} \rho^2} + \frac{1}{\rho} \frac{\mathrm{d}}{\mathrm{d} \rho} - \frac{\mathrm{m}^2}{\rho^2} + \mathrm{l}^2 + \mathrm{k}^2 \right\} \mathrm{Z_m} (\mathrm{n}\, \rho) = 0 \qquad n^2 = l^2 + k^2$$

■ Generating Function (ν = n)

$$g(x, t) = e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \qquad 0 < |t| < \infty$$

Integral Representation

Integral Order

Treating the above as a Laurent series in complex t.

$$J_n(x) = \frac{1}{2\pi i} \oint_c \mathrm{dt} \, \frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{n+1}}$$

where c encloses the essential singularity $t = 0$.

Let c be the unit circle: $t = e^{i\theta}$

$$\mathrm{dt} = i\, e^{i\theta}\, \mathrm{d}\theta \qquad\qquad t - \frac{1}{t} = 2\, i \sin\theta$$

$$\longrightarrow \qquad J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \; e^{-i n\theta + i x \sin\theta}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \{ \cos[-n\theta + x\sin\theta] + i \sin[-n\theta + x\sin\theta] \}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \; \cos[-n\theta + x\sin\theta] \text{ since } \sin[-n\theta + x\sin\theta] \text{ is odd.}$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \mathrm{d}\theta \; \cos[-n\theta + x\sin\theta]$$

Non-Integral Order

$$\mathrm{J}_\nu(\mathrm{x}) = \frac{1}{2\pi \mathrm{i}} \oint_c \mathrm{dt} \, \frac{\mathrm{e}^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{\mathrm{t}^{\nu+1}}$$

where c must now encloses both the branch point $t = 0$ & the branch cut.

It can be shown that (Arfken , 3rd ed., Ex 11.1.17.)

$$J_\nu(x) = \frac{1}{\pi} \int_0^{\pi} \mathrm{d}\theta \; \cos[-\nu\theta + x\sin\theta] - \frac{\sin\nu\pi}{\pi} \int_0^{\infty} \mathrm{d}\theta \; e^{-\nu\theta + x\sinh\theta}$$

Series Expansion

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{x}{2}\right)^l t^l \cdot \sum_{m=0}^{\infty} \frac{(-)^m}{m!} \left(\frac{x}{2}\right)^m t^{-m} \qquad\qquad 0 < |t| < \infty$$

$$= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-)^m}{l!\, m!} \left(\frac{x}{2}\right)^{l+m} t^{l-m}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{(-)^m}{(n+m)!\, m!} \left(\frac{x}{2}\right)^{n+2m} t^n \qquad\qquad \text{where } n = l - m$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$$\longrightarrow \qquad J_n(x) = \sum_{m=0}^{\infty} \frac{(-)^m}{(n+m)!\, m!} \left(\frac{x}{2}\right)^{n+2m}$$

Generalization to **non-integral order:**

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-)^m}{(\nu+m)!\, m!} \left(\frac{x}{2}\right)^{\nu+2m}$$

where $(\nu + m)! \equiv \Gamma(\nu + m + 1)!$

- $\boldsymbol{J_{-n}(x) = (-)^n J_n(x)}$

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-)^m}{(-n+m)!\, m!} \left(\frac{x}{2}\right)^{-n+2m}$$

$$= \sum_{m=n}^{\infty} \frac{(-)^m}{(-n+m)!\, m!} \left(\frac{x}{2}\right)^{-n+2m} \qquad \text{where} \quad (-n)! = \infty$$

$$= \sum_{l=0}^{\infty} \frac{(-)^{l+n}}{l!\,(l+n)!} \left(\frac{x}{2}\right)^{n+2l} \qquad \text{where} \quad l = -n + m$$

$$= (-)^n J_n(x)$$