Hilbert Transform

■ Derivation

Let $f(z)$ be analytic & $\left| f \right| \xrightarrow[|z| \to \infty]{} 0$ in the upper half plane.

Let C be the contour shown in fig.

$\longrightarrow \qquad \oint_C dz \, \frac{f(z)}{z-\alpha} = 0$

$$= \int_{S_R} dz \, \frac{f(z)}{z-\alpha} + \int_{-R}^{\alpha-\delta} dx \, \frac{f(x)}{x-\alpha} + \int_{S_\delta} dz \, \frac{f(z)}{z-\alpha} + \int_{\alpha+\delta}^{R} dx \, \frac{f(x)}{x-\alpha}$$

Let $R \longrightarrow \infty$, $\delta \longrightarrow 0$

$\left| f \right| \xrightarrow[|z| \to \infty]{} 0 \qquad \Longrightarrow \qquad \int_{S_R} dz \, \frac{f(z)}{z-\alpha} \xrightarrow[R \longrightarrow \infty]{} 0$

$\int_{-R}^{\alpha-\delta} dx \, \frac{f(x)}{x-\alpha} + \int_{\alpha+\delta}^{R} dx \, \frac{f(x)}{x-\alpha} \xrightarrow[R \longrightarrow \infty]{} \text{PV} \int_{-\infty}^{\infty} dx \, \frac{f(x)}{x-\alpha}$

Using $z - \alpha = \delta \, e^{i\theta}$:

$\int_{S_\delta} dz \, \frac{f(z)}{z-\alpha} = \int_{\pi}^{0} d\theta \; i \, \delta \, e^{i\theta} \, \frac{1}{\delta \, e^{i\theta}} \, f\left(\alpha + \delta \, e^{i\theta}\right) \xrightarrow[\delta \longrightarrow 0]{} \int_{\pi}^{0} d\theta \; i \, f(\alpha) = -i \pi \, f(\alpha)$

$\therefore \qquad \text{PV} \int_{-\infty}^{\infty} dx \, \frac{f(x)}{x-\alpha} = i \pi \, f(\alpha) = \pi \{ \, i \, u(\alpha) - v(\alpha) \, \} = \text{PV} \int_{-\infty}^{\infty} dx \, \frac{u(x) + i \, v(x)}{x-\alpha}$

where $f \equiv u + i \, v$.

Thus

$$f(\alpha) = \frac{1}{\pi i} \, \text{PV} \int_{-\infty}^{\infty} dx \, \frac{f(x)}{x-\alpha}$$

(contrast this with CIF)

$$u(\alpha) = \frac{1}{\pi} \, \text{PV} \int_{-\infty}^{\infty} dx \, \frac{v(x)}{x-\alpha}$$

$$v(\alpha) = -\frac{1}{\pi} \, \text{PV} \int_{-\infty}^{\infty} dx \, \frac{u(x)}{x-\alpha}$$

■ Definition

2 real functions u & v of real variable x forms a **Hilbert transform** pair

$\Longleftrightarrow$

$$u(\alpha) = \frac{1}{\pi} \, \text{PV} \int_{-\infty}^{\infty} dx \, \frac{v(x)}{x-\alpha}$$

$$v(\alpha) = -\frac{1}{\pi} \, \text{PV} \int_{-\infty}^{\infty} dx \, \frac{u(x)}{x-\alpha}$$

- $\text{PV}\int_{-R}^{R} dx \frac{1}{x-\alpha} = \ln\left(\frac{R-\alpha}{R+\alpha}\right)$

- **Proof**

Let $-R < \alpha < R$

$$\text{PV}\int_{-R}^{R} dx \frac{1}{x-\alpha} = \lim_{\delta\to 0}\left\{\int_{-R}^{\alpha-\delta} dx \frac{1}{x-\alpha} + \int_{\alpha+\delta}^{R} dx \frac{1}{x-\alpha}\right\}$$
$$= \lim_{\delta\to 0}\left\{\ln\left(\frac{-\delta}{-R-\alpha}\right) + \ln\left(\frac{R-\alpha}{\delta}\right)\right\}$$
$$= \lim_{\delta\to 0} \ln\left(\frac{\delta}{R+\alpha}\cdot\frac{R-\alpha}{\delta}\right)$$
$$= \ln\left(\frac{R-\alpha}{R+\alpha}\right)$$

- $\text{PV}\int_{-R}^{R} dx \frac{f(x)}{x-\alpha} = f(\alpha)\ln\left(\frac{R-\alpha}{R+\alpha}\right) + \text{PV}\int_{-R}^{R} dx \frac{f(x)-f(\alpha)}{x-\alpha}$

- **Proof**

Using $f(x) = f(\alpha) + f(x) - f(\alpha)$, we have:

$$\text{PV}\int_{-R}^{R} dx \frac{f(x)}{x-\alpha} = \text{PV}\int_{-R}^{R} dx \frac{f(\alpha)}{x-\alpha} + \text{PV}\int_{-R}^{R} dx \frac{f(x)-f(\alpha)}{x-\alpha}$$
$$= f(\alpha)\ln\left(\frac{R-\alpha}{R+\alpha}\right) + \text{PV}\int_{-R}^{R} dx \frac{f(x)-f(\alpha)}{x-\alpha}$$

- **Note**

If $f\,'$ exists at $x = \alpha$,

$\longrightarrow \quad \frac{f(x)-f(\alpha)}{x-\alpha}$ exists at $x = \alpha$

Hence: $\text{PV}\int_{-R}^{R} dx \frac{f(x)-f(\alpha)}{x-\alpha} = \int_{-R}^{R} dx \frac{f(x)-f(\alpha)}{x-\alpha}$

- **Alternate Form of the Hilbert Transform**

$$u(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx \frac{v(x)-v(\alpha)}{x-\alpha}$$
$$v(\alpha) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dx \frac{u(x)-u(\alpha)}{x-\alpha}$$

f is analytic $\longrightarrow$ u, v are differentiable

Using: $\lim_{R\to\infty} \ln\left(\frac{R-\alpha}{R+\alpha}\right) = \ln 1 = 0$

in conjunction with F2 & F3 does the trick.