Integral Representations

■ Heaviside Step Function

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$= \frac{1}{2\pi i} \oint_c dk \, \frac{e^{-ikx}}{k}$$

Let C_x be the contour which

1. goes along the real axis from $k = (-\infty, 0)$ to $(-\rho, 0)$, then
2. goes along the upper half circle center at $k = 0$ from $z = (-\rho, 0)$ to $(\rho, 0)$, then
3. goes along the real axis from $k = (\rho, 0)$ to $(\infty, 0)$

For $x > 0$, C is the contour C_x closed by an infinite half circle in the **lower** k plane.

For $x < 0$, C is the contour C_x closed by an infinite half circle in the **upper** k plane.

■ Generating Functions

A family of functions $f_n(x)$ can be defined as the coefficients of a power series expansion of a generating function $g(t, x)$.

$$g(t, x) = \sum_n f_n(x)\, t^n$$

The advantages of doing this is that relations between the member functions can now be investigated systematically. These include:

■ Series expansion of f_n.

Direct expansion of g as a power series & then collect terms proportional to a given power of t often provide an easy way to find the power series of f_n.

■ Recurrence Relations among f_n.

Differentiating the defining eq wrt either t or x & then collect terms proportional to a given power of t gives recurrence relations among f_n.

■ Integral Representation of f_n.

Treating the defining eq as a Laurent series:

$$f_n(x) = \frac{1}{2\pi i} \oint_c dt \, \frac{g(t, x)}{t^{n+1}}$$

where c is any contour that encloses $t = 0$.

Given the integral representaion of f_n, further manipulations of f_n is possible.

These include:

Asymptotic Expansions

Analytic Continuation

Examples

Bessel Function J_n

$$g(t,\ x) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$$J_n(x) = \frac{1}{2\pi i} \oint_c \mathrm{dt}\ \frac{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}}{t^{n+1}} \qquad c = \text{unit circle.}$$

Modified Bessel Function I_n

$$g(t,\ x) = e^{\frac{x}{2}\left(t+\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} I_n(x)\, t^n$$

$$I_n(x) = \frac{1}{2\pi i} \oint_c \mathrm{dt}\ \frac{e^{\frac{x}{2}\left(t+\frac{1}{t}\right)}}{t^{n+1}} \qquad c = \text{unit circle.}$$

Legendre Functions P_n

$$g(t,\ x) = \frac{1}{\sqrt{1-2\,t\,x+t^2}} = \sum_{n=-\infty}^{\infty} P_n(x)\, t^n$$

$$P_n(x) = \frac{1}{2\pi i} \oint_c \mathrm{dt}\ \frac{1}{\sqrt{1-2\,t\,x+t^2}} \cdot \frac{1}{t^{n+1}} \qquad c = \text{unit circle.}$$

Hermite Functions H_n

$$g(t,\ x) = e^{-t^2+2\,t\,x} = \sum_{n=-\infty}^{\infty} \frac{H_n(x)}{n\,!}\, t^n$$

$$H_n(x) = \frac{n!}{2\pi i} \oint_c \mathrm{dt}\ \frac{e^{-t^2+2\,t\,x}}{t^{n+1}} \qquad c = \text{unit circle.}$$

Laguerre Functions L_n

$$g(t,\ x) = \frac{e^{-\frac{x\,t}{1-t}}}{1-t} = \sum_{n=-\infty}^{\infty} L_n(x)\, t^n$$

$$L_n(x) = \frac{1}{2\pi i} \oint_c \mathrm{dt}\ \frac{e^{-\frac{x\,t}{1-t}}}{1-t} \cdot \frac{1}{t^{n+1}} \qquad c = \text{unit circle.}$$

Chebyshev Polynomials T_n

$$g(t,\ x) = \frac{1-t^2}{1-2\,t\,x+t^2} = 2\sum_{n=-\infty}^{\infty} T_n(x)\, t^n$$

$$T_n(x) = \frac{1}{4\pi i} \oint_c \mathrm{dt}\ \frac{1-t^2}{1-2\,t\,x+t^2} \cdot \frac{1}{t^{n+1}} \qquad c = \text{unit circle.}$$