Integral Transform

Definitions

Function Space

function space

A function space is a linear space of functions defined on the same domains & ranges.

Linear Mapping

linear mapping

Let $V(F)$, $W(F)$ be linear spaces over the field F.

A mapping $f : V \longrightarrow W$, $x \mapsto f(x)$ is **linear**.

$\Longleftrightarrow \quad a\,x + b\,y \mapsto f(a\,x + b\,y) = a\,f(x) + b\,f(y) \quad \forall\, x,\ y \in V \ \ \&\ \ a,\ b \in F$

Integral Transform

integral transform def

Let $\mathcal{W}$ be the function space over a linear space $W(\mathbb{K})$,

$\mathcal{V}$ the function space over a linear space $V(\mathbb{K})$.

$\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$

The **integral transform** $\mathcal{T}$ from $\mathcal{V}$ to $\mathcal{W}$ by the **kernel** K is a linear mapping:

$\mathcal{T} : \mathcal{V} \longrightarrow \mathcal{W}$

$f \mapsto F = \mathcal{T}[f]$ where $f \in \mathcal{V}$ & $F \in \mathcal{W}$

such that

$F(\omega) = \mathcal{T}[f(t); \omega] = \int_V \mathrm{dt}\, K(\omega; t)\, f(t)$ where $t \in V$ & $\omega \in W$

$F(\omega)$ is called the **integral transform** of function $f(t)$ by the **kernel** $K(\omega; t)$.

The **inverse transform** $\mathcal{T}^{-1}$ from $\mathcal{W}$ to $\mathcal{V}$ by the **kernel** H is a linear mapping:

$\mathcal{T}^{-1} : \mathcal{W} \longrightarrow \mathcal{V}$

$F \mapsto f = \mathcal{T}^{-1}[F]$ where $f \in \mathcal{V}$ & $F \in \mathcal{W}$

such that

$f(t) = \mathcal{T}^{-1}[F(\omega); t] = \int_W \mathrm{d}\omega\, H(t; \omega)\, F(\omega)$ where $t \in V$ & $\omega \in W$

$f(t)$ is called the **inverse transform** of $F(\omega)$ by the **kernel** $H(t; \omega)$.

Linearity implies:

$\mathcal{T}[a\,f + b\,g] = a\,\mathcal{T}[f] + b\,\mathcal{T}[g] \qquad \forall\, f,\ g \in \mathcal{V}\ \ \&\ \ a,\ b \in \mathbb{K}$

$\mathcal{T}^{-1}[a\,F + b\,G] = a\,\mathcal{T}^{-1}[F] + b\,\mathcal{T}^{-1}[G]$

Fourier (idempotent) Kernel

$V = W$ & $K(\omega; t) = H(t; \omega)$

note: the kernel of the fourier transform is not a fourier kernel.

Convolution (Faltung / Folding)

convolution def

Let $\mathcal{V}$ be the function space over a linear space $V(\mathbb{K})$; $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$

Let $f, g \in \mathcal{V}$

The **convolution** (**faltung**) $f * g$ of f and g in V is defined as

$$(f * g)(t) \equiv C \int_V d\tau \; f(t - \tau)\, g(\tau) \qquad \forall\, t \in V$$

where C is some constant.

Fourier Transform

Kernel $K(\omega, t) = C\, e^{i\omega t}$ $\qquad H(t, \omega) = C'\, e^{-i\omega t}$ $\qquad C C' = \frac{1}{2\pi}$

Definition

fourier transform def

$$F(\omega) = \mathcal{F}[f(t); \omega] = C \int_{-\infty}^{\infty} dt\, e^{i\omega t} f(t)$$

where f is piecewise continuous, differentiable, absolutely integrable ($\int_{-\infty}^{\infty} dt \left| f \right|$ exists).

Fourier Integral Theorem

The proof of this theorem is rather involved.

To begin, we obtain the Riemann-Lebesgue Lemma for finite intervals; then extend it to infinite ones.

This begets the Localization Lemma, 1st for finite, then infinite, intervals.

Finally, we arrive at the Fourier Integral Theorem.

Riemann-Lebesgue Lemma

Riemann-Lebesgue lemma

Let $f(t)$ be piecewise continuous for $0 < a \le t \le b < \infty$

$$\Longrightarrow \quad \int_a^b dt\, f(t) \sin\lambda t \xrightarrow[\lambda\to\infty]{} 0 \qquad \int_a^b dt\, f(t) \cos\lambda t \xrightarrow[\lambda\to\infty]{} 0$$

proof

The proof below assumes f to be continuous in $[a, b]$.

The case for f piece-wise continuous is proved by applying the technique to each continuous segment individuallly.

$$\int_a^b dt\, f(t) \sin\lambda t \xrightarrow[\tau = t - \frac{\pi}{\lambda}]{} - \int_{a-\frac{\pi}{\lambda}}^{b-\frac{\pi}{\lambda}} d\tau\, f\left(\tau + \frac{\pi}{\lambda}\right) \sin\lambda\tau$$

where $\sin(\lambda\tau + \pi) = -\sin\lambda\tau$

$$\therefore \quad 2\int_a^b \mathrm{dt}\, f(t) \sin\lambda t$$

$$= \int_a^b \mathrm{dt}\, f(t) \sin\lambda t - \int_{a-\frac{\pi}{\lambda}}^{b-\frac{\pi}{\lambda}} \mathrm{dt}\, f\left(t+\frac{\pi}{\lambda}\right) \sin\lambda t$$

$$= \int_{b-\frac{\pi}{\lambda}}^{b} \mathrm{dt}\, f(t) \sin\lambda t + \int_a^{b-\frac{\pi}{\lambda}} \mathrm{dt}\, f(t) \sin\lambda t - \int_a^{b-\frac{\pi}{\lambda}} \mathrm{dt}\, f\left(t+\frac{\pi}{\lambda}\right) \sin\lambda t - \int_{a-\frac{\pi}{\lambda}}^{a} \mathrm{dt}\, f\left(t+\frac{\pi}{\lambda}\right) \sin\lambda t$$

$$= \int_{b-\frac{\pi}{\lambda}}^{b} \mathrm{dt}\, f(t) \sin\lambda t + \int_a^{b-\frac{\pi}{\lambda}} \mathrm{dt}\left\{ f(t) - f\left(t+\frac{\pi}{\lambda}\right)\right\} \sin\lambda t - \int_{a-\frac{\pi}{\lambda}}^{a} \mathrm{dt}\, f\left(t+\frac{\pi}{\lambda}\right) \sin\lambda t$$

$$= I_1 + I_2 - I_3$$

where

$$I_1 = \int_{b-\frac{\pi}{\lambda}}^{b} \mathrm{dt}\, f(t) \sin\lambda t$$

$$I_2 = \int_a^{b-\frac{\pi}{\lambda}} \mathrm{dt}\left\{ f(t) - f\left(t+\frac{\pi}{\lambda}\right)\right\} \sin\lambda t$$

$$I_3 = \int_{a-\frac{\pi}{\lambda}}^{a} \mathrm{dt}\, f\left(t+\frac{\pi}{\lambda}\right) \sin\lambda t$$

Since f is piecewise continuous, it is bounded, ie. $\exists\, M > 0 \ni |f| < M$.

Using $\quad \left|\int \mathrm{dt}\, g\right| \le \int \mathrm{dt}\, |g|, \quad |\sin x| \le 1$

we have

$$|I_1| < M \int_{b-\frac{\pi}{\lambda}}^{b} \mathrm{dt} = M\frac{\pi}{\lambda}$$

$$|I_3| < M\frac{\pi}{\lambda}$$

Now, by the mean value theorem,

$$f\left(t+\frac{\pi}{\lambda}\right) = f(t) + f'(s)\frac{\pi}{\lambda} \qquad \text{where } a \le s \le b$$

$$\therefore \quad \left| f(t) - f\left(t+\frac{\pi}{\lambda}\right)\right| \le N\frac{\pi}{\lambda} \qquad \text{where } N = \max |f'|$$

$$|I_2| \le N\frac{\pi}{\lambda}\left(b - \frac{\pi}{\lambda} - a\right)$$

Thus

$$2\left|\int_a^b \mathrm{dt}\, f(t) \sin\lambda t\right| < 2M\frac{\pi}{\lambda} + N\frac{\pi}{\lambda}\left(b - \frac{\pi}{\lambda} - a\right) \underset{\lambda\to\infty}{\longrightarrow} 0$$

Hence: $\quad \displaystyle\int_a^b \mathrm{dt}\, f(t) \sin\lambda t \underset{\lambda\to\infty}{\longrightarrow} 0$

Proof for the cos case proceeds in the same manner.

Writing $e^{i\lambda t} = \cos\lambda t + i \sin\lambda t$, we have

$$\int_a^b \mathrm{dt}\, f(t)\, e^{i\lambda t} \underset{\lambda\to\infty}{\longrightarrow} 0$$

- **Corollary**

Riemann-Lebesgue Corollary

Let

1. $f(t)$ be piecewise continuous for $0 < a \le t < \infty$

2. f is absolutely integrable. ie. $\int_0^\infty dt \left| f \right|$ exists.

$$\Longrightarrow \quad \int_a^\infty dt\ f(t) \sin\lambda t \underset{\lambda\to\infty}{\longrightarrow} 0 \qquad \int_a^\infty dt\ f(t) \cos\lambda t \underset{\lambda\to\infty}{\longrightarrow} 0$$

- **proof**

$$\int_a^\infty dt\ f(t)\sin\lambda t = \int_a^b dt\ f(t)\sin\lambda t + \int_b^\infty dt\ f(t)\sin\lambda t$$

$$\left|\int_b^\infty dt\ f(t)\sin\lambda t\right| \le \int_b^\infty dt \left|f(t)\right| \left|\sin\lambda t\right| < \int_b^\infty dt \left|f(t)\right|$$

f is absolutely integrable $\Longrightarrow$

$$\int_b^\infty dt \left|f(t)\right| \underset{b\to\infty}{\longrightarrow} 0$$

$$\therefore \quad \int_a^\infty dt\ f(t)\sin\lambda t = \lim_{b\to\infty} \lim_{\lambda\to\infty} \int_a^b dt\ f(t)\sin\lambda t = 0$$

- **Localization Lemma**

localization lemma

Let $f'(t)$ be piecewise continuous for $0 < t \le a < \infty$

$$\Longrightarrow \quad \int_0^a dt\ f(t) \frac{\sin\lambda t}{t \longrightarrow \text{Null}} \quad \frac{\pi}{2} f(0_+)$$

- **proof**

The proof below assumes f' to be continuous.

The case for f' piece-wise continuous is proved by applying the technique to each continuous segment individuallly.

Let $\displaystyle \int_0^a dt\ f(t) \frac{\sin\lambda t}{t} = I_1 + I_2$

where $\displaystyle I_1 = f(0_+) \int_0^a dt \frac{\sin\lambda t}{t}$

$$I_2 = \int_0^a dt \{ f(t) - f(0_+) \} \frac{\sin\lambda t}{t}$$

f' is continuous $\longrightarrow$ $\frac{f(t)-f(0_+)}{t}$ is continuous in $[0, a]$

$\longrightarrow$ $\quad I_2 \underset{\lambda\to\infty}{\longrightarrow} 0$

Now:

$$\int_0^a dt \frac{\sin\lambda t}{t} \underset{u=\lambda t}{=} \int_0^{\lambda a} du \frac{\sin u}{u \underset{\lambda\to\infty}{\longrightarrow} \int_0^\infty} du \frac{\sin u}{u} = \frac{\pi}{2}$$

Hence: $\displaystyle \int_0^a dt\ f(t) \frac{\frac{\sin\lambda t}{t} \underset{\lambda\to\infty}{\longrightarrow} \pi}{2} f(0_+)$

Corollary 1

Let

1. $f\,'(t)$ be piecewise continuous for $0 < t < \infty$

2. f is absolutely integrable. ie. $\int_0^\infty \mathrm{dt} \left| f \right|$ exists.

$$\Longrightarrow \quad \int_0^\infty \mathrm{dt}\; f(t)\,\frac{\sin\lambda t}{t} \underset{\lambda\to\infty}{\longrightarrow} \frac{\pi}{2}\, f(0_+)$$

$$\int_0^\infty \mathrm{dt}\; f(t+x)\,\frac{\sin\lambda t}{t} \underset{\lambda\to\infty}{\longrightarrow} \frac{\pi}{2}\, f(x_+)$$

- **proof**

Proof for the 1st part is analogous to that used in the Riemann-Lebesgue corollary.

2nd part is obtained by a simple change of variable.

- **Corollary 2**

Let

1. $f\,'(t)$ be piecewise continuous for $-\infty < t < \infty$

2. f is absolutely integrable. ie. $\int_{-\infty}^\infty \mathrm{dt} \left| f \right|$ exists.

$$\Longrightarrow \quad \int_{-\infty}^\infty \mathrm{dt}\; f(t)\,\frac{\sin\lambda t}{t \longrightarrow \text{Null}}\; \frac{\pi}{2}\,\{\, f(0_+) + f(0_-)\,\}$$

- **proof**

$$\int_{-\infty}^0 \mathrm{dt}\; f(t)\;\frac{\sin\lambda t}{t} \underset{t\to -t}{=} \int_0^\infty \mathrm{dt}\; f(-t)\,\frac{\sin\lambda t}{t} \underset{\lambda\to\infty}{\longrightarrow} \frac{\pi}{2}\, f(0_-)$$

- **Fourier Integral Theorem**

fourier integral thm

Let

1. $f\,'(t)$ be piecewise continuous for $-\infty < t < \infty$

2. f is absolutely integrable. ie. $\int_{-\infty}^\infty \mathrm{dt} \left| f \right|$ exists.

$\Longrightarrow$

$$\frac{1}{2}\,\{\, f(t_+) + f(t_-)\,\} = \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega \int_{-\infty}^\infty \mathrm{d}\tau\;\; f(\tau)\;\cos\omega\,(\tau - t)$$

$$\frac{1}{2}\,\{\, f(t_+) + f(t_-)\,\} = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}\omega \int_{-\infty}^\infty \mathrm{d}\tau\, e^{i\,\omega\,(\tau-t)}\; f(\tau)$$

If f is **continuous** at t.

$$f(t) = \frac{1}{\pi}\int_0^\infty \mathrm{d}\omega \int_{-\infty}^\infty \mathrm{d}\tau\;\; f(\tau)\;\cos\omega\,(\tau - t)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}\omega \int_{-\infty}^\infty \mathrm{d}\tau\, e^{i\,\omega\,(\tau-t)}\; f(\tau)$$

proof

$$\frac{1}{2}\{\, f(t_+) + f(t_-)\,\} = \frac{1}{\pi}\int_{-\infty}^{\infty} d\tau\, f(\tau+t)\ \frac{\sin\lambda t}{t} \qquad (\lambda \longrightarrow \infty)$$

$$\underset{\tau\to\tau-t}{=} \quad \frac{1}{\pi}\int_{-\infty}^{\infty} d\tau\, f(\tau)\ \frac{\sin\lambda(\tau-t)}{\tau-t}$$

Now:

$$\int_0^{\lambda} d\omega\ \ \cos\omega\,(\tau-t) = \frac{\sin\lambda\,(\tau-t)}{\tau-t}$$

$$\therefore \qquad \frac{1}{2}\{\, f(t_+) + f(t_-)\,\} = \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} d\tau\ \ f(\tau)\ \cos\omega\,(\tau-t)$$

Using

- **Inverse Transform**

$$f(t) = \mathcal{F}^{-1}[F(\omega);t] = C\,'\int_{-\infty}^{\infty} d\omega\ e^{-i\,\omega\,t}\,F(\omega) \qquad\qquad \text{CC}\,' = \frac{1}{2\pi}$$

- **proof**

$$\text{FIT} \longrightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\tau\, e^{i\,\omega\,(\tau-t)}\ f(\tau)$$

$$= \ \ C\,'\int_{-\infty}^{\infty} d\omega\ \ e^{-i\,\omega\,t}\, C\int_{-\infty}^{\infty} d\tau\, e^{i\,\omega\,\tau}\ f(\tau)$$

$$= C\,'\int_{-\infty}^{\infty} d\omega\ \ e^{-i\,\omega\,t}\,F(\omega)$$

- **Dirac Delta Function**

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\,\omega\,t}$$

- **proof**

$$\text{FIT} \longrightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\tau\, e^{i\,\omega\,(\tau-t)}\ f(\tau)$$

$$= \ \int_{-\infty}^{\infty} d\tau\ f(\tau)\ \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \ e^{i\,\omega\,(\tau-t)}$$

$$f(t) = \int_{-\infty}^{\infty} d\tau\ f(\tau)\,\delta(\tau-t)$$

$$\longrightarrow \qquad \delta(\tau-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \ e^{i\,\omega\,(\tau-t)}$$

- **Transform of Derivatives**

$$\mathcal{F}\left[\, f^{(n)}(t);\omega\right] = (-i\,\omega\,)^n\,F(\omega)$$

- **proof**

Let $F(\omega) = \mathcal{F}[f(t); \omega] = C \int_{-\infty}^{\infty} \mathrm{dt}\, e^{i\omega t} f(t)$

$F_1(\omega) = \mathcal{F}\left[\frac{d f(t)}{d t}; \omega\right] = C \int_{-\infty}^{\infty} \mathrm{dt}\, e^{i\omega t} \frac{d f(t)}{d t}$

$= C\ f(t)\ e^{i\omega t}\ |_{-\infty}^{\infty} - i\,\omega\, C \int_{-\infty}^{\infty} \mathrm{dt}\, e^{i\omega t} f(t)$

$= -i\,\omega\, F(\omega)$

where $f(t) \xrightarrow[|t| \to \infty]{} 0$ since it is absolutely integrable.

The general case can be proved by induction.

- **Convolution (Faltung) Theorem**

$$\int_{-\infty}^{\infty} \mathrm{d}\tau\ g(\tau)\ f(t-\tau) = \int_{-\infty}^{\infty} \mathrm{d}\omega\ e^{-i\omega t}\, G(\omega)\, F(\omega)$$

where $F,\ G$ are fourier transforms of $f,\ g$, respectively

- **Parseval Relation**

$$\int_{-\infty}^{\infty} \mathrm{dt}\ \mathrm{g}(\mathrm{t})^{*}\, \mathrm{f}(\mathrm{t}) = \frac{\mathrm{C}'}{\mathrm{C}} \int_{-\infty}^{\infty} \mathrm{d}\omega\ \mathrm{G}(\omega)^{*}\, \mathrm{F}(\omega)$$

- **proof**

Using $f(t) = C' \int_{-\infty}^{\infty} \mathrm{d}\omega\ e^{-i\omega t} F(\omega)$

$G(\omega)^{*} = C \int_{-\infty}^{\infty} \mathrm{dt}\, e^{-i\omega t}\, g(t)^{*}$

$$\longrightarrow \int_{-\infty}^{\infty} \mathrm{dt}\ \mathrm{g}(\mathrm{t})^{*}\, \mathrm{f}(\mathrm{t}) = \mathrm{C}' \int_{-\infty}^{\infty} \mathrm{dt}\ \mathrm{g}(\mathrm{t})^{*} \int_{-\infty}^{\infty} \mathrm{d}\omega\ \mathrm{e}^{-\mathrm{i}\omega \mathrm{t}}\, \mathrm{F}(\omega)$$

$$= C' \int_{-\infty}^{\infty} \mathrm{d}\omega\, F(\omega) \int_{-\infty}^{\infty} \mathrm{dt}\ g(t)^{*}\, e^{-i\omega t}$$

$$= \frac{C'}{C} \int_{-\infty}^{\infty} \mathrm{d}\omega\, F(\omega)\, G(\omega)^{*}$$

- **Laplace Transform**

Kernel $K(s, t) = e^{-st}$ $H(t, s) = \frac{1}{2\pi i} e^{st}$

$x \in [0,\ \infty]$ $\mathrm{Re}\, s > s_0 > 0$

- **Definition**

Given a function $f(t)$ of a real variable t.

Its **Laplace transform** $F(s)$ is defined, if the integral exists, as

$$F(s) = \mathfrak{L}[f(t); s] = \int_{0}^{\infty} \mathrm{dt}\, e^{-st} f(t) = \int_{-\infty}^{\infty} \mathrm{dt}\, e^{-st} f(t)\, \theta(t)$$

If $\exists\, s_0,\ t_0 \quad \ni \quad \left| e^{-s_0 t}\ f(t) \right| \le M \qquad \forall\, t > t_0$

$\longrightarrow \quad F(s)$ exists $\forall\, s > s_0$

f is then said to be of **exponential order**.

For $t \longrightarrow 0,\ \ e^{-s t}\ f(t) \longrightarrow f(t)$

$\therefore \quad F(s)$ does not exists if $f \sim t^n \quad \forall\, n \le -1$

- **Inverse Transform**

$$f(t) = \mathfrak{L}^{-1}[F(s); t] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds\, e^{s t}\ F(s)$$

- **proof**

$$\text{FIT} \longrightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\tau\, e^{i\omega(\tau - t)}\ f(\tau)$$

- **Mellin Transform**

Kernel $K(s,\ t) = t^{s-1} \qquad H(t,\ s) = \frac{1}{2\pi i}\ t^{-s}$

$x \in [0,\ \infty] \qquad \text{Re}\, s > s_0 > 0$

- **Hankel Transform**

Kernel $K_n(k,\ r) = r\ J_n(k\, r) \qquad H_n(r,\ k) = k\ J_n(k\, r)$

$k,\ r \in [\,0,\ \infty\,] \qquad n =$ integers

- **References**

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