Integral Transform

Definition of a Laplace Transform

Given a **complex** function $f(t)$ of a **real** variable t.
Its **Laplace transform** $F(s)$ is defined, if the integral exists, as

$$F = \mathfrak{L}[f]$$

One-sided:

$$F(s) = \mathfrak{L}[f(t); s] = \int_0^\infty dt\, e^{-s t}\ f(t) \qquad \text{Re}\, s > 0$$

Two-sided:

$$F(s) = \mathfrak{L}[f(t); s] = \int_{-\infty}^\infty dt\, e^{-s t}\ f(t) \qquad \text{Re}\, s > 0$$

$\mathfrak{L}$ is **linear:** $\mathfrak{L}[a\ f + b\ g] = a\ \mathfrak{L}[f] + b\ \mathfrak{L}[g]$

If $\exists\, s_0,\ t_0 \quad \ni \quad \left| e^{-s_0 t}\ f(t) \right| \leq M \ \ \forall\, t > t_0$

$\longrightarrow \quad F(s)$ exists $\forall\, s > s_0$

f is then said to be of **exponential order**.

For $t \longrightarrow 0$, $e^{-s t}\ f(t) \longrightarrow f(t)$

$\therefore \quad F(s)$ does not exists if $f \sim t^n \quad \forall\, n \leq -1$

Inverse Transform

Bromwich integral:

$$f(t) = \frac{1}{2 \pi i}\, \text{PV} \int_{\gamma - i \infty}^{\gamma + i \infty} ds\, e^{s t}\, F(s) \qquad t > 0, \text{ real}$$

$$= \sum_n \underset{s = s_n}{\text{Res}} \left[\ e^{s t}\, F(s) \right] \qquad \left(F \xrightarrow[|s| \to \infty]{} 0 \right)$$

where the sum is over all poles of F which are assumed to lie on the left side of the vertical line $\text{Re}\, s = \gamma$.

- **proof**

Consider the contour shown in fig 74.
The proof is complete if one can show

$$I = \int_{C_R} \mathrm{ds}\; e^{s\,t}\, F(s) \underset{R\to\infty}{=} 0$$

Now:

$$|\,I\,| \le \max\,|\,F\,| \cdot R \cdot J$$

where
$$J = \left| \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \mathrm{d}\theta\, e^{i\,\theta}\, e^{t\,R\,e^{i\,\theta}} \right|$$

$$= \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \mathrm{d}\theta\, e^{t\,R\cos\theta}$$

$$= \int_{0}^{\pi} \mathrm{d}\phi\, e^{-t\,R\sin\phi} \qquad \phi = \theta - \frac{\pi}{2}$$

$$< \frac{\pi}{R\,t} \qquad \text{(Jordan's inequality)}$$

$$\longrightarrow \quad \left|\,I\,\right| < \max \left|\,F\,\right| \cdot \frac{\pi}{t}$$

$$\therefore \quad I \underset{R\to\infty}{=} 0 \text{ if} \qquad F \underset{|\,s\,|\to\infty}{\longrightarrow} 0$$

- **Example 1**

$$F(s) = \frac{s}{\left(s^2 + a^2\right)^2}$$

Poles of order 2 are at $s = \pm\, i\, a$.

$$\left|\,F\,\right| \frac{\underset{|s|\to\infty}{\longrightarrow}\; 1}{s^3} \to 0$$

$$\underset{s=\pm i\,a}{\mathrm{Res}} \left[\; e^{s\,t}\, F(s)\;\right] = \frac{d}{d\,s}\left(\frac{s\,e^{s\,t}}{(s \pm i\,a)^2}\right)_{s=\pm i\,a}$$

$$= \quad \frac{e^{s\,t}(s\,t+1)}{(s \pm i\,a)^2} - 2\,\frac{s\,e^{s\,t}}{(s \pm i\,a)^3} \quad \Bigg)_{s=\pm i\,a}$$

$$= \quad \mathrm{e}^{\pm i\,a\,t}\left\{\; \frac{\pm i\,a\,t+1}{-4\,a^2} - \frac{1}{-4\,a^2}\;\right\}$$

$$= \quad \mp\, i\;\frac{t}{4\,a}\, e^{\pm i\,a\,t}$$

$$\therefore \quad f(t) = -\,i\;\frac{t}{4\,a}\left(e^{i\,a\,t} - e^{-i\,a\,t}\right) = \frac{t}{2\,a}\,\sin a\,t$$

- **Example 2**

$$F(s) = \frac{\tanh s}{s^2} = \frac{\sinh s}{s^2 \cosh s}$$

$$\left| F \right| \xrightarrow[|s|\to\infty]{} \frac{1}{s^2} \to 0$$

Poles are at $s = 0$, $(2n+1)\frac{\pi}{2} i$.

For $s = 0$:

$$e^{st} F(s) = \frac{1}{s^2}\left(s - \frac{s^3}{3} + \ldots\right)(1 + s\,t + \ldots) = \frac{1}{s} + \ldots$$

$$\therefore \quad \operatorname*{Res}_{s=0}\left[\, e^{st} F(s) \,\right] = 1$$

$$\operatorname*{Res}_{s=(2n+1)\frac{\pi}{2} i}\left[\, e^{st} F(s) \,\right] = \frac{e^{st}}{s^2} = -\frac{4\, e^{(2n+1)\frac{\pi}{2} i t}}{(2n+1)^2 \pi^2}$$

$$f(t) = 1 - \frac{4}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\frac{\pi}{2} i t}}{(2n+1)^2}$$

Now:

$$\sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)\frac{\pi}{2} i t}}{(2n+1)^2} = \sum_{n=0}^{\infty} \frac{e^{(2n+1)\frac{\pi}{2} i t}}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{e^{-(2n-1)\frac{\pi}{2} i t}}{(2n-1)^2}$$

$$= \sum_{n=1}^{\infty} \frac{e^{(2n-1)\frac{\pi}{2} i t}}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{e^{-(2n-1)\frac{\pi}{2} i t}}{(2n-1)^2}$$

$$= 2 \sum_{n=1}^{\infty} \frac{\cos(2n-1)\frac{\pi}{2} t}{(2n-1)^2}$$

$$\longrightarrow \quad f(t) = 1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\frac{\pi}{2} t}{(2n-1)^2}$$

■ Example 3

$$F(s) = \frac{\sinh\left(x\sqrt{s}\right)}{s \sinh \sqrt{s}} \qquad 0 < x < 1$$

Branch point & pole at $s = 0$.

Let branch cut be at $\theta = 0$.

Poles are at $\sqrt{s} = n\pi i$ or $s = -n^2 \pi^2$ $(n \geq 0)$

For $s = 0$

$$\frac{\sinh\left(x\sqrt{s}\right)}{s \sinh \sqrt{s}} = \frac{x\sqrt{s} + \frac{1}{6} x^3 s^{\frac{3}{2}} + \ldots}{s\left(\sqrt{s} + \frac{1}{6} s^{\frac{3}{2}} + \ldots\right)}$$

$$= \frac{x + \frac{1}{6} x^3 s + \ldots}{s\left(1 + \frac{1}{6} s + \ldots\right)}$$

$$\longrightarrow \quad \operatorname*{Res}_{s=0}\left[\, e^{st} F(s) \,\right] = x$$

$$\operatorname*{Res}_{s=-n^2\pi^2}\left[\, e^{st} F(s) \,\right] = e^{st} \frac{\sinh\left(x\sqrt{s}\right)}{s \frac{1}{2\sqrt{s}} \cosh \sqrt{s}} \qquad n \neq 0$$

$$= 2\, \mathrm{e}^{-n^2 \pi^2 t} \frac{\sin(n\pi x)}{n\pi} (-)^n$$

$$\therefore \quad f(t) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \frac{\sin(n\pi x)}{n} (-)^n$$