Series

Summation of Series

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{n} \text{Res}\{ f(z_n) \cot(\pi z_n) \}$$
$$\sum_{n=-\infty}^{\infty} (-)^n f(n) = -\pi \sum_{n} \text{Res}\{ f(z_n) \csc(\pi z_n) \}$$

where z_n are the poles of $f(z)$.

proof

The major step in the proof is to show that

$$g(z) = \pi \cot(\pi z)$$

has simple poles at $z = n$ with residue 1.

Given the above, the 1st formula is obtained by considering the contour integral which encloses the entire z-plane.

Let $\quad g(z) = \pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)}$

$\longrightarrow$ poles of $g(z)$ are simple & at $z = n$.

$$\underset{z=n}{\text{Res}}\, g(z) = \left(\pi \frac{\cos(\pi z)}{\frac{d}{dz}\sin(\pi z)} \right)_{z=n} = 1$$

To evaluate

$$S_N = \sum_{n=-N}^{N} f(n)$$

we use a contour C_N which is a square that is centered at $z = 0$ & intersects the real axis at $x = \pm \left(N + \frac{1}{2}\right)$:

$$\longrightarrow \quad \oint_{C_N} \text{dz}\, g(z)\, f(z) = 2\pi i \left\{ \sum_{n=-N}^{N} f(N) + \sum_{k} \text{Res}[\, f(z_k)\, \pi \cot(\pi z_k)\,] \right\}$$

where k runs over all poles of f inside C_N.

As $N \longrightarrow \infty$, $\oint_{C_N} \text{dz}\, g(z)\, f(z) \longrightarrow 0$ since $\left| f \right| \xrightarrow[|z| \to \infty]{} 0$ if S converges. QED.

Proof for the 2nd formula is analogous. All we need is

$$\underset{z=n}{\text{Res}}\, \pi \csc(\pi z) = \left(\pi \frac{1}{\frac{d}{dz}\sin(\pi z)} \right)_{z=n} = \frac{1}{\cos(\pi n)} = (-)^n$$

Example

$$S = \sum_{n=-\infty}^{\infty} \frac{(-)^n}{(a+n)^2}$$

$f(z) = \frac{1}{(a+z)^2}$ with 2nd order pole at $z = -a$

$$S = -\pi \underset{z=-a}{\text{Res}} \left\{ \frac{1}{(a+z)^2} \cot(\pi z) \right\}$$
$$= -\pi \left\{ \frac{d}{dz} \csc(\pi z) \right\}_{z=-a}$$
$$= \pi^2 \frac{\cos(\pi a)}{\sin^2(\pi a)}$$

Asymptotic Series

- **Definition**

Let

$$S_N = \sum_{n=0}^{N} \frac{A_n}{z^n} \qquad\qquad S = \sum_{n=0}^{\infty} \frac{A_n}{z^n}$$

If

$$f(z) = \phi(z)\, S$$

$$\lim_{|z| \to \infty} \left\{ z^N \left[\frac{f(z)}{\phi(z)} - S_N \right] \right\} = 0$$

$\Longrightarrow$ S represents $\frac{f(z)}{\phi(z)}$ **asymptotically**.

Note:
Usually, S diverges & there is an optimal N which gives the best approximation.

- **Example**

Exponential Function:

$$\text{Ei}(x) = \int_{-\infty}^{x} dt\, \frac{e^t}{t}$$

$$E_1(x) = \int_x^{\infty} dt\, \frac{e^{-t}}{t} = -\text{Ei}(-x)$$

$$= -\int_x^{\infty} d\, e^{-t} \cdot \frac{1}{t} = -\left(\frac{e^{-t}}{t}\right)_x^{\infty} - \int_x^{\infty} dt\, \frac{e^{-t}}{t^2}$$

$$\frac{=\mathrm{e}^{-x}}{x} - \int_x^{\infty} dt\, \frac{e^{-t}}{t^2}$$

$$\frac{=\mathrm{e}^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\int_x^{\infty} dt\, \frac{e^{-t}}{t^3}$$

$$\frac{=\mathrm{e}^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\,\frac{e^{-x}}{x^3} - 3!\int_x^{\infty} dt\, \frac{e^{-t}}{t^4}$$

$$= \frac{e^{-x}}{x}\left\{1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-)^n \frac{n!}{x^n}\right\} + (-)^{n+1}(n+1)!\int_x^{\infty} dt\, \frac{e^{-t}}{t^{n+2}}$$

$$= \frac{e^{-x}}{x} S_n + (-)^{n+1}(n+1)!\int_x^{\infty} dt\, \frac{e^{-t}}{t^{n+2}}$$

where $S_n = 1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-)^n \frac{n!}{x^n} = \sum_{m=0}^{n} (-)^m \frac{m!}{x^m}$

Cauchy Test:

$$\lim_{m \to \infty} \left| \frac{\frac{(m+1)!}{x^{m+1}}}{\frac{m!}{x^m}} \right| = \lim_{m \to \infty} \frac{(m+1)}{|x|} \longrightarrow \infty$$

$\therefore$ $S = \lim_{n \to \infty} S_n$ diverges.

On the other hand:

$$\lim_{x\to\infty} \left(x^n \left[E_1(x)\, x\, e^x - S_n \right] \right)$$

$$= \lim_{x\to\infty} \left(x^{n+1}\, e^x\, (-)^{n+1}\, (n+1)! \int_x^\infty dt\, \frac{e^{-t}}{t^{n+2}} \right)$$

$$< \lim_{x\to\infty} \left(x^{-1}\, e^x\, (-)^{n+1}\, (n+1)! \int_x^\infty dt\, e^{-t} \right)$$

$$= \lim_{x\to\infty} \left\{ x^{-1}\, (-)^{n+1}\, (n+1)! \right\}$$

$$= 0$$

∴ $E_1(x) = \frac{e^{-x}}{x} S$ is an asymptotic representation.

The error involved in using the S_n is

$$(-)^{n+1}\, (n+1)! \int_x^\infty dt\, \frac{e^{-t}}{t^{n+2}}$$

$$< (-)^{n+1}\, (n+1)!\, \frac{1}{x^{n+2}} \int_x^\infty dt\, e^{-t}$$

$$= (-)^{n+1}\, (n+1)!\, \frac{1}{x^{n+2}}\, e^{-x}$$

- **Properties**

1. Function represented by asymptotic expansion is not unique.
   eg. $\frac{f}{\phi}$ & $\frac{f}{\phi} + e^{-z}$ have the same expansion.
2. Phase change in z often produces discontinuities (**Stokes Phenomena**).
3. Asymptotic series can be added, multiplied & integrated.
4. Differentiation of asymptotic series is valid only if the derivative function also has an asymptotic series expansion.

Method of Steepest Descent (Saddle-Point Method)

- **Concept**

Consider

$$I(z) = \int_c dt\, e^{z f(t)}$$

where c is such that $e^{z f(t)} \longrightarrow 0$ at both ends.

Let $z f(t) = u + i v$ where $u = \text{Re}[\, z f(t)\,]$ & $v = \text{Im}[\, z f(t)\,]$.

$\longrightarrow$ $I(z) = \int_c dt\, e^{u + i v}$

For $|z| \longrightarrow \infty$, we usually have $|u|, |v| \longrightarrow \infty$.

If v varies significantly along c, $e^{z f(t)}$ will oscillate rapidly between positive & negative values of large magnitude. Since subtraction of large numbers leads to large error, very accurate evaluation is required for these portions of the contour.

On the other hand, $\left| e^{z f(t)} \right|$ is large only when u is positive & large. If we choose c so that $v \simeq$ const near the maximum of u along c, the cancellation effect will be minimized. Furthermore, if the maximum is sharp, ie., $\left| e^{z f(t)} \right| \longrightarrow 0$ rapidly away from the maximum, $I(z)$ itself can be adequately approximated by its value near the maximum.

The maximum of f along c is at t_0 where $f\,'(t_0) = 0$.

Near this point we choose c to coincide with the level curve $v(t) = \text{const.} = v(t_0)$.

Hence t_0 is also the maximum of u along c.

Since f is analytic, t_0 can at most be a saddle point, ie., f and hence u, is the minimum along another level curve $c\,'$ that passes through t_0.

Now, from ex 22.14:

level curves of u, v = const. are orthogonal to each other wherever $f\,' \neq 0$.

$\longrightarrow$ u changes most rapidly along level curve v = const.

Thus, t_0 is a maximum on c implies that c gives the steepest descent from the saddle point.

Note also that level curves of u, v = const. need no longer be orthogonal at t_0 since $f\,'(t_0) = 0$. In fact, the curve $c\,'$ gives the steepest ascent from t_0.

- **First Approximation**

Let $\quad f(t) = f(t_0) + \frac{1}{2}(t - t_0)^2 f\,''(t_0) + \ldots$ where $f\,'(t_0) = 0$

$\longrightarrow \quad f(t) - f(t_0) \simeq \frac{1}{2}(t - t_0)^2 f\,''(t_0)$

Let $\quad z = |z| e^{i\theta}$

$$\longrightarrow \quad z\{ f(t) - f(t_0) \} \simeq |z| e^{i\theta} \frac{1}{2}(t - t_0)^2 f\,''(t_0)$$

$$\equiv -\frac{1}{2}|z|\tau^2$$

where

$$\tau^2 = -e^{i\theta}(t - t_0)^2 f\,''(t_0)$$

$$\tau = \sqrt{e^{i(\pi+\theta)} f\,''(t_0)}\,(t - t_0)$$

$$\mathrm{d}\tau = \sqrt{e^{i(\pi+\theta)} f\,''(t_0)}\,\mathrm{dt}$$

$$\longrightarrow \quad I(z) = \int_c \mathrm{dt}\, e^{z f(t)}$$

$$\simeq \frac{e^{z f(t_0)}}{\sqrt{e^{i(\pi+\theta)} f\,''(t_0)}} \int_c \mathrm{d}\tau\, e^{-\frac{1}{2}|z|\tau^2}$$

Approximate $\int_c \mathrm{d}\tau\, e^{-\frac{1}{2}|z|\tau^2}$ by

$$\int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{-\frac{1}{2}|z|\tau^2} = \sqrt{\frac{2\pi}{|z|}}$$

$$\longrightarrow \quad I(z) \simeq e^{z f(t_0)} \sqrt{\frac{2\pi}{|z|\, e^{i(\pi+\theta)} f\,''(t_0)}}$$

$$= \mathrm{e}^{z f(t_0)} \sqrt{\frac{2\pi}{z\, e^{i\pi} f\,''(t_0)}}$$