4. Differential Forms

A. The algebra and integral calculus of forms

B. The differential calculus of forms and its applications

4.1. Definition of Volume – the Geometrical Role of Differential Forms

Any pair of infinitesimal vectors in Euclidean space defines an infinitesimal **area** equal to the area of the parallelogram that uses these vectors as sides. Obviously, the area bounded by 2 parallel vectors is zero. Addition of areas is simply addition of the corresponding vectors. [see Fig.4.3] Thus, we define the area as an antisymmetric bilinear mapping

$$area : V \times V \to \mathbb{R} \qquad \text{by} \qquad (x, y) \mapsto area(x, y)$$

such that

$$area(\bar{a},\bar{b}) = -area(\bar{b},\bar{a}) \qquad \text{so that} \qquad area(\bar{a},\bar{a}) = 0$$

$$area(\bar{a},\bar{b}) + area(\bar{a},\bar{c}) = area(\bar{a},\bar{b}+\bar{c})$$

In other words, *area*(,) is an antisymmetric $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor. In terms of Cartesian components, we have

$$area(\bar{V},\bar{W}) = \begin{vmatrix} V^x & V^y \\ W^x & W^y \end{vmatrix}$$

4.2. Notation and Definitions for Antisymmetric Tensors

A $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor $\tilde{\omega}$ is said to be **antisymmetric** if

$$\tilde{\omega}\left(\bar{U},\bar{V}\right) = -\tilde{\omega}\left(\bar{V},\bar{U}\right) \qquad \text{for all vectors } \bar{U},\bar{V} \tag{4.1}$$

Similarly, a $\begin{pmatrix} 0 \\ p \end{pmatrix}$ tensor, where $p \ge 3$, is **completely antisymmetric** if it changes sign when any pair of its arguments is interchanged. Any tensor can be anti-symmetrized as a signed sum of all permutations of its arguments. For example,

$$\tilde{\omega}_A\left(\bar{U},\bar{V}\right) = \frac{1}{2!}\left[\tilde{\omega}\left(\bar{U},\bar{V}\right) - \tilde{\omega}\left(\bar{V},\bar{U}\right)\right] \tag{4.2}$$

$$\begin{aligned} \tilde{p}_A\left(\bar{U},\bar{V},\bar{W}\right) = \frac{1}{3!}\Big[& \tilde{\omega}\left(\bar{U},\bar{V},\bar{W}\right) + \tilde{\omega}\left(\bar{V},\bar{W},\bar{U}\right) + \tilde{\omega}\left(\bar{W},\bar{U},\bar{V}\right) \\ & -\tilde{\omega}\left(\bar{V},\bar{U},\bar{W}\right) - \tilde{\omega}\left(\bar{W},\bar{V},\bar{U}\right) - \tilde{\omega}\left(\bar{U},\bar{W},\bar{V}\right)\Big] \end{aligned} \tag{4.3}$$

where the subscript A stands for **antisymmetric part** and the factors 2! and 3! are added for normalization. In index notations, these become

$$\left(\tilde{\omega}_A\right)_{ij} = \frac{1}{2!}\left(\omega_{ij} - \omega_{ji}\right) \equiv \omega_{[ij]} \tag{4.4}$$

$$\left(\tilde{p}_A\right)_{ijk} = \frac{1}{3!}\left(p_{ijk} + p_{jki} + p_{kij} - p_{jik} - p_{kji} - p_{ikj}\right) \equiv p_{[ijk]} \tag{4.5}$$

where we have used $\left[ij\cdots k\right]$ to denote the normalized totally antisymmetric sum. Hereafter, we shall use the overhead ~ to denote a totally antisymmetric $\begin{pmatrix} 0 \\ p \end{pmatrix}$ tensor.

Note that in an n-D vector space, a completely antisymmetry $\begin{pmatrix} 0 \\ p \end{pmatrix}$ tensor has at most

$$C_p^n = \frac{n!}{p!\left(n-p\right)!} \tag{4.6}$$

independent components.

4.3. Differential Forms

A ***p*-form** is defined as a completely antisymmetric tensor of type $\begin{pmatrix} 0 \\ p \end{pmatrix}$. Here, *p* is the **degree** of the form. Obviously, antisymmetry is irrelevant for $p = 0,1$. It is left as exercise to show that the set of all forms of a given *p* at point $x \in M$ is a vector space $\Omega_x^p(M)$ of dimension $C_p^n = \dfrac{n!}{p!(n-p)!}$, where *n* is the dimension of the manifold.

To facilitate the building up of *p*-forms of high degrees, we introduce the **wedge**, or **exterior**, **product** $\wedge$. To begin, we define

$$\tilde{p} \wedge \tilde{q} \equiv \tilde{p} \otimes \tilde{q} - \tilde{q} \otimes \tilde{p} \qquad (4.8)$$

if $\tilde{p}$ and $\tilde{q}$ are 1-forms. Obviously, $\tilde{p} \wedge \tilde{q}$ is a 2-form with $\tilde{p} \wedge \tilde{p} = 0$. Given a vector basis $\{\bar{e}_i\}$ and its dual 1-form basis $\{\tilde{e}^i\}$, it is straightforward to show that the set of C_2^n independent forms in $\{\tilde{e}^i \wedge \tilde{e}^j\}$ is a basis for 2-forms. In particular,

$$\tilde{\alpha} = \sum_{i<j=1}^{n} \alpha_{ij} \tilde{e}^i \wedge \tilde{e}^j = \frac{1}{2!} \alpha_{ij} \tilde{e}^i \wedge \tilde{e}^j \qquad (4.9)$$

where

$$\alpha_{ij} \equiv \tilde{\alpha}(\bar{e}_i, \bar{e}_j) = \frac{1}{2!} \alpha_{kl} \left(\tilde{e}^k \otimes \tilde{e}^l - \tilde{e}^l \otimes \tilde{e}^k \right)(\bar{e}_i, \bar{e}_j)$$

$$= \frac{1}{2!} \alpha_{kl} \left[\tilde{e}^k(\bar{e}_i) \tilde{e}^l(\bar{e}_j) - \tilde{e}^l(\bar{e}_i) \tilde{e}^k(\bar{e}_j) \right] = \frac{1}{2!} \alpha_{kl} \left(\delta_i^k \, \delta_j^l - \delta_i^l \, \delta_j^k \right)$$

$$= \frac{1}{2!} \left(\alpha_{ij} - \alpha_{ji} \right) = -\alpha_{ji} = \alpha_{[ij]}$$

are the components of the 2-form $\tilde{\alpha}$.

In order to handle *p*-forms with *p* > 2, we demand the wedge product to be **associative**, i.e.,

$$\tilde{p} \wedge (\tilde{q} \wedge \tilde{r}) = (\tilde{p} \wedge \tilde{q}) \wedge \tilde{r} = \tilde{p} \wedge \tilde{q} \wedge \tilde{r} \qquad (4.10)$$

where $\tilde{p}, \tilde{q}, \tilde{r}$ are forms of arbitrary degrees. The set of all forms of arbitrary degrees, together with an antisymmetric multiplication $\wedge$, is called a **Grassmann algebra**. Its dimension is $\sum_{p=0}^{n} C_p^n = (1+1)^n = 2^n$, where *n* is the dimension of the manifold.

Exercise 4.8

Show that

$$(\tilde{p} \wedge \tilde{q})_{i\cdots jk\cdots l} = C_p^{p+q} p_{[i\cdots j} q_{k\cdots l]} \qquad (4.11)$$

For $\tilde{p}$ a 1-form and $\tilde{q}$ a 2-form, we have

$$(\tilde{p} \wedge \tilde{q})_{ijk} = p_i q_{jk} + p_j q_{ki} + p_k q_{ij} \quad = 3 p_{[i} q_{jk]}$$

4.4. Manipulating Differential Forms

4.4.1. Commutation

Consider a p-form $\tilde{p}$ and a q-form $\tilde{q}$. Given a 1-form basis $\{\tilde{e}^i\}$, we can write

$$\tilde{p} = \frac{1}{p!} p_{i_1\cdots i_p}\, \tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_p} \qquad \text{and} \qquad \tilde{q} = \frac{1}{q!} q_{i_1\cdots i_q}\, \tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_q}$$

so that

$$\begin{aligned}
\tilde{p} \wedge \tilde{q} &= \frac{1}{p!q!} p_{i_1\cdots i_p} q_{i_{p+1}\cdots i_{p+q}} \left(\tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_p}\right) \wedge \left(\tilde{e}^{i_{p+1}} \wedge \cdots \wedge \tilde{e}^{i_{p+q}}\right) \\
&= (-)^p \frac{1}{p!q!} p_{i_1\cdots i_p} q_{i_{p+1}\cdots i_{p+q}}\, \tilde{e}^{i_{p+1}} \wedge \left(\tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_p}\right) \wedge \left(\tilde{e}^{i_{p+2}} \wedge \cdots \wedge \tilde{e}^{i_{p+q}}\right) \\
&= (-)^{2p} \frac{1}{p!q!} p_{i_1\cdots i_p} q_{i_{p+1}\cdots i_{p+q}}\, \tilde{e}^{i_{p+1}} \wedge \tilde{e}^{i_{p+2}} \wedge \left(\tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_p}\right) \wedge \left(\tilde{e}^{i_{p+3}} \wedge \cdots \wedge \tilde{e}^{i_{p+q}}\right) \\
&\ \vdots \\
&= (-)^{pq} \frac{1}{p!q!} p_{i_1\cdots i_p} q_{i_{p+1}\cdots i_{p+q}} \left(\tilde{e}^{i_{p+1}} \wedge \cdots \wedge \tilde{e}^{i_{p+q}}\right) \wedge \left(\tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_p}\right) \\
&= (-)^{pq}\, \tilde{q} \wedge \tilde{p} \qquad (4.12)
\end{aligned}$$

4.4.2. Contraction

The **contraction** of a *p*-form $\tilde{\alpha}$ with a 1-vector $\bar{\xi}$ results in a (*p*–1)-form. To simplify the notations, we denote this by

$$\tilde{\alpha}\left(\bar{\xi}\right) \equiv \tilde{\alpha}\left(\bar{\xi},\cdots\right) \qquad (4.13a)$$

where … denotes *p*–1 empty slots. In terms of components, we have

$$\left[\tilde{\alpha}\left(\bar{\xi}\right)\right]_{j\cdots k} = \alpha_{i\,j\cdots k}\xi^{i} \qquad (4.13)$$

Note that in (4.13a), putting $\bar{\xi}$ in some other slot than the first can effect at most a sign change. To see this, consider the 2-form $\tilde{\alpha} = \tilde{p}\wedge\tilde{q}$, where $\tilde{p},\tilde{q}$ are 1-forms:

$$\left(\tilde{p}\wedge\tilde{q}\right)\left(\bar{\xi}\right) = \left(\tilde{p}\otimes\tilde{q} - \tilde{q}\otimes\tilde{p}\right)\left(\bar{\xi}\right) \quad = \tilde{p}\left(\bar{\xi}\right)\tilde{q} - \tilde{q}\left(\bar{\xi}\right)\tilde{p}$$

Thus, even though $\bar{\xi}$ is nominally at the 1^{st} slot, the antisymmetry of the wedge product ensures that it is contracted with both $\tilde{p}$ and $\tilde{q}$. Thus, putting it in the 2^{nd} slot only results in a sign change. Obviously, the same also applies to any *p*-form that can be written as a wedge product of *p* 1-forms. In particular,

$$\left(\tilde{e}^{i_1}\wedge\cdots\wedge\tilde{e}^{i_p}\right)\left(\bar{\xi}\right) = \left(\tilde{e}^{i_1}\otimes\tilde{e}^{i_2}\otimes\cdots\otimes\tilde{e}^{i_p} - \tilde{e}^{i_2}\otimes\tilde{e}^{i_1}\otimes\cdots\otimes\tilde{e}^{i_p} + \cdots\right)\left(\bar{\xi}\right)$$

$$= \tilde{e}^{i_1}\left(\bar{\xi}\right)\tilde{e}^{i_2}\otimes\cdots\otimes\tilde{e}^{i_p} - \tilde{e}^{i_2}\left(\bar{\xi}\right)\tilde{e}^{i_1}\otimes\cdots\otimes\tilde{e}^{i_p} + \cdots$$

$$= \xi^{i_1}\,\tilde{e}^{i_2}\otimes\cdots\otimes\tilde{e}^{i_p} - \xi^{i_2}\,\tilde{e}^{i_1}\otimes\cdots\otimes\tilde{e}^{i_p} + \cdots \qquad [\,p!\text{ terms}]$$

$$= p!\,\xi^{[i_1}\,\tilde{e}^{i_2}\otimes\cdots\otimes\tilde{e}^{i_p]}$$

Since a general *p*-form can be written as

$$\tilde{\alpha} = \frac{1}{p!}\alpha_{i_1\cdots i_p}\,\tilde{e}^{i_1}\wedge\cdots\wedge\tilde{e}^{i_p}$$

we have

$$\tilde{\alpha}\left(\bar{\xi}\right) = \alpha_{i_1 i_2\cdots i_p}\,\xi^{[i_1}\,\tilde{e}^{i_2}\otimes\cdots\otimes\tilde{e}^{i_p]}$$

Using $\alpha_{i_1\cdots i_p} = \alpha_{[i_1\cdots i_p]}$, we can write

$$\tilde{\alpha}\left(\bar{\xi}\right) = \xi^{i_1}\alpha_{[i_1 i_2\cdots i_p]}\tilde{e}^{i_2}\otimes\cdots\otimes\tilde{e}^{i_p} \quad = \xi^{i_1}\alpha_{i_1 i_2\cdots i_p}\,\tilde{e}^{[i_2}\otimes\cdots\otimes\tilde{e}^{i_p]}$$

$$= \frac{1}{\left(p-1\right)!}\,\xi^{i_1}\alpha_{i_1 i_2\cdots i_p}\,\tilde{e}^{i_2}\wedge\cdots\wedge\tilde{e}^{i_p} \qquad (4.15)$$

= (*p*–1)-form with components $\left[\tilde{\alpha}\left(\bar{\xi}\right)\right]_{i_2\cdots i_p} = \xi^{i_1}\alpha_{i_1 i_2\cdots i_p}$

Finally, we mention that if $\tilde{\alpha}$ is an arbitrary form, we have

$$\left(\tilde{p}\wedge\tilde{\alpha}\right)\left(\bar{\xi}\right) = \tilde{p}\left(\bar{\xi}\right)\wedge\tilde{\alpha} + \left(-\right)^{p}\,\tilde{p}\wedge\tilde{\alpha}\left(\bar{\xi}\right) \qquad (4.16)$$

4.5. Restriction of Forms

Since a *p*-form $\tilde{\alpha}$ is a $\begin{pmatrix} 0 \\ p \end{pmatrix}$ tensor, its domain is the product space $\otimes^p V = \underbrace{V \otimes \cdots \otimes V}_{p\ factors}$, where *V* is the entire vector space. The **restriction** $\tilde{\alpha}\big|_W$ of $\tilde{\alpha}$ to a subspace *W* of *V* is the same *p*-form $\tilde{\alpha}$ with domain restricted to $\otimes^p W$. Hence,

$$\tilde{\alpha}\big|_W \left(\bar{X}, \cdots, \bar{Y}\right) = \tilde{\alpha}\left(\bar{X}, \cdots, \bar{Y}\right) \qquad \forall\ \bar{X}, \cdots, \bar{Y} \in W$$

Note that if $\dim W < p$, then $\tilde{\alpha}\big|_W \equiv 0$ since all *p*-forms vanish on a vector space with dimension less than *p*. If $\dim W = p$, then $\tilde{\alpha}\big|_W$ has only one component.

The restricting of a form is also called **sectioning**. This comes from treating *W* as a (hyper) plane passing through, thus sectioning, the series of planes representing the form. A form $\tilde{\alpha}$ is said to be **annulled** by *W* if $\tilde{\alpha}\big|_W = 0$.

4.6. Fields of Forms

A **field** of *p*-forms on a manifold *M* is a rule (with appropriate differentiability conditions) that gives a *p*-form at each point of *M*. The properties of *p*-forms discussed so far with respect to a vector space *V* are applicable to the tangent space T_P for each point *P* in *M*. The only point needs mentioning is that, since a submanifold *S* of *M* picks out a subspace V_P of T_P for all points $P \in S$, we can define the **restriction** of a *p*-form field $\tilde{\alpha}$ to *S* as the field formed by restricting $\tilde{\alpha}$ at *P* to V_P.

4.7. Handedness and Orientability

In an n-D manifold M, the space $\Omega_P^n(M) = \wedge^n {}^*T_P = \underbrace{{}^*T_P \wedge \cdots \wedge {}^*T_P}_{n\ factors}$ of all n-forms at each point in M is 1-D. Consider now an n-form field $\tilde{\omega}$. If $\{\overline{e}_i\}$ is a vector basis at point P, then $\tilde{\omega}(\overline{e}_1, \cdots, \overline{e}_n) = 0$ iff $\tilde{\omega} = 0$ at P. Hence, $\tilde{\omega}$ separates the set of all vector bases at P into 2 classes: those with $\tilde{\omega}(\overline{e}_1, \cdots, \overline{e}_n) > 0$ are called **right-handed** and those with $\tilde{\omega}(\overline{e}_1, \cdots, \overline{e}_n) < 0$, **left-handed**. This classification is actually independent of the exact value of $\tilde{\omega}$ used. For, the 1-D nature of $\Lambda^n {}^*T_P$ means that any non-zero n-form $\tilde{\omega}'$ can be written as $\tilde{\omega}' = f\,\tilde{\omega}$, where f is a non-zero number. Hence, any pair of bases that belong to the same class under $\tilde{\omega}$ will remain so under $\tilde{\omega}'$. A manifold M is called **orientable** if it is possible to define handedness continuously over it, i.e., there exists a continuous basis $\{\overline{e}_i(P)\}$ with the same handedness everywhere on M. Obviously, this is equivalent to the existence of an n-form field that is continuous and non-zero everywhere. For example, the Euclidean space is orientable but the Mobius band is not.

4.8. Volumes and Integration on Oriented Manifolds

4.8.1. Integration of a Function

In an n-D manifold, a set of n linearly independent infinitesimal vectors define an n-D parallelepiped whose volume, which is a number, can be obtained by the contraction of these vectors with an n-form.

Let $\tilde{\omega}$ be an n-form on an region U of an n-D manifold M with coordinates $\{x^1,\cdots,x^n\}$. Since the n-form space is 1-D, we can write

$$\tilde{\omega} = f\ \tilde{d}x^1 \wedge\cdots\wedge \tilde{d}x^n$$

where $f = f\left(x^1,\cdots,x^n\right)$ is some function on M. To integrate over U, we divide it into cells spanned by n-tuples of infinitesimal vectors $\left\{\Delta x^1 \frac{\partial}{\partial x^1},\cdots,\Delta x^n \frac{\partial}{\partial x^n}\right\}$. The volume of one cell can be written as

$$\Delta x^1\cdots\Delta x^n = \tilde{d}x^1 \wedge\cdots\wedge \tilde{d}x^n \left(\Delta x^1 \frac{\partial}{\partial x^1},\cdots,\Delta x^n \frac{\partial}{\partial x^n}\right)$$

so that the integration of f over this cell is

$$\begin{aligned}\int_{cell} f\left(x^1,\cdots,x^n\right) d^n x &\simeq f\,\Delta x^1\cdots\Delta x^n \\ &= f\ \tilde{d}x^1 \wedge\cdots\wedge \tilde{d}x^n \left(\Delta x^1 \frac{\partial}{\partial x^1},\cdots,\Delta x^n \frac{\partial}{\partial x^n}\right) \\ &= \tilde{\omega}\left(\Delta x^1 \frac{\partial}{\partial x^1},\cdots,\Delta x^n \frac{\partial}{\partial x^n}\right) \\ &= \tilde{\omega}(cell) \qquad (4.17)\end{aligned}$$

Adding up the contributions of all cells gives

$$\begin{aligned}\int_U f\ d^n x = \sum_{cell} \tilde{\omega}(cell) &\equiv \int_U \tilde{\omega} \qquad (4.18) \\ &= \int_U f\ \tilde{d}x^1 \wedge\cdots\wedge \tilde{d}x^n\end{aligned}$$

4.8.2. Change of Variables

The integral $\int \tilde{\omega}$ is independent of coordinates up to an overall sign. For example, on a 2-D manifold with coordinates (λ, μ), eq(4.18) becomes

$$\int f(\lambda,\mu)\, d\lambda\, d\mu = \int \tilde{\omega} \quad = \int f(\lambda,\mu)\, \tilde{d}\lambda \wedge \tilde{d}\mu$$

Under a change of coordinates $(\lambda,\mu) \to (x,y)$, we have

$$\tilde{d}\lambda = \frac{\partial \lambda}{\partial x}\tilde{d}x + \frac{\partial \lambda}{\partial y}\tilde{d}y$$

$$\tilde{d}\mu = \frac{\partial \mu}{\partial x}\tilde{d}x + \frac{\partial \mu}{\partial y}\tilde{d}y$$

so that

$$\begin{aligned}
\tilde{d}\lambda \wedge \tilde{d}\mu &= \left(\frac{\partial \lambda}{\partial x}\tilde{d}x + \frac{\partial \lambda}{\partial y}\tilde{d}y\right) \wedge \left(\frac{\partial \mu}{\partial x}\tilde{d}x + \frac{\partial \mu}{\partial y}\tilde{d}y\right) \\
&= \left(\frac{\partial \lambda}{\partial x}\frac{\partial \mu}{\partial y} - \frac{\partial \lambda}{\partial y}\frac{\partial \mu}{\partial x}\right)\tilde{d}x \wedge \tilde{d}y \qquad (4.19) \\
&= \frac{\partial(\lambda,\mu)}{\partial(x,y)}\tilde{d}x \wedge \tilde{d}y
\end{aligned}$$

where $\frac{\partial(\lambda,\mu)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} \\ \frac{\partial \mu}{\partial x} & \frac{\partial \mu}{\partial y} \end{vmatrix} = J$ is the **Jacobian** of the transformation $(\lambda,\mu) \to (x,y)$. Now, in the ordinary Riemann integral of a function of multi-variables, the same coordinate transformation is related by

$$d\lambda d\mu = |J| dxdy \qquad (4.19a)$$

where the absolute value of *J* is used. To reconcile this difference, we first point out that if we write the transformation as $(\lambda,\mu) \to (y,x)$, (4.19) and (4.19a) become

$$\tilde{d}\lambda \wedge \tilde{d}\mu = \frac{\partial(\lambda,\mu)}{\partial(y,x)}\tilde{d}y \wedge \tilde{d}x \quad = -J\ \tilde{d}y \wedge \tilde{d}x \quad = J\ \tilde{d}x \wedge \tilde{d}y$$

$$d\lambda d\mu = |J| dydx \quad = |J| dxdy$$

Thus, the values of both expressions remain unchanged, as they must be, but for different reasons. In particular, the infinitesimal area defined in terms of the exterior product (4.19) has an orientation, which is ignored in (4.19a) but handled eventually in terms of the integration limits of the Riemann integral.

4.8.3. Orientability

Consider again the integral

$$\int \tilde{\omega} = \sum_{cell} \tilde{\omega}(cell) \quad = \int f \; d^n x$$

where

$$\tilde{\omega} = f \; \tilde{d}x^1 \wedge \cdots \wedge \tilde{d}x^n$$

$$cell = \left(\Delta x^1 \frac{\partial}{\partial x^1}, \cdots, \Delta x^n \frac{\partial}{\partial x^n} \right) \quad = \left(\Delta x^1 \; \overline{e}_1, \cdots, \Delta x^n \; \overline{e}_n \right)$$

If we switch to another basis $\left(\overline{e}_1', \cdots, \overline{e}_n' \right)$ which differs from $\left(\overline{e}_1, \cdots, \overline{e}_n \right)$ only in handedness, we have

$$cell' = \left(\Delta x^{1'} \; \overline{e}_1', \cdots, \Delta x^{n'} \; \overline{e}_n' \right)$$

and

$$\tilde{\omega}(cell') = f \; \tilde{d}x^1 \wedge \cdots \wedge \tilde{d}x^n \left(\Delta x^{1'} \; \overline{e}_1', \cdots, \Delta x^{n'} \; \overline{e}_n' \right) \quad = -\tilde{\omega}(cell)$$

Assuming that the entire region of integration is **orientable**, i.e., it can be described by a set of bases of the same handedness, we have

$$\sum_{cell} \tilde{\omega}(cell) = -\sum_{cell'} \tilde{\omega}(cell')$$

Therefore, in relating $\int \tilde{\omega}$ to the Riemann integral $\int f \; d^n x$, there is an unavoidable ambiguity in sign. By convention, a right-handed basis is always assumed in the definition (4.18).

4.8.4. Integration on Submanifold

By definition, the form $\tilde{\omega}$ in the integration $\int \tilde{\omega}$ is always of the maximum degree: n-form on an n-D manifold M, or p-form over a p-D submanifold S. In every case, $\int \tilde{\omega}$ exists only if its domain is (**internally**) orientable. The relation, if any, between the orientabilities of M and S is therefore of interest. Let M be orientable and P a point of S. A right-handed n-form $\tilde{\omega}$ at P can be reduced to a p-form by contracting with $n-p$ linearly independent '**normal vectors**' $\left\{\overline{n}_1, \cdots, \overline{n}_{n-p}\right\}$ at P that are not tangent to S. The resultant p-form $\tilde{\omega}\left(\overline{n}_1, \cdots, \overline{n}_{n-p}\right)$ then represents a right-handed restriction of $\tilde{\omega}$ to S. Obviously, this restriction depends on the exact choice of the normal vectors, including the order in which they are labelled. A given choice of $\left\{\overline{n}_1, \cdots, \overline{n}_{n-p}\right\}$ is called an **external orientation** for S at P. S is said to be **externally orientable** if it is possible to define an external orientation continuously over it.

If some open region of M containing S is orientable, then either S is both internally and externally orientable, or it is neither. If no such region of M is orientable, S may be one but not both. For example, consider a Mobius strip as a 2-D submanifold of R^3 (figure 4.5) and a curve in the strip as a 1-D submanifold of the strip (figure 4.6). Set up a right-handed triad of vectors at any point P of the strip, two lying in the strip and one out of it. Carry them continuously once around the strip, keeping the two always tangent to it. The outward pointing one always returns pointing to the opposite side: the Mobius band is not externally orientable in R^3. Similarly, set up two vectors in the strip, one tangent to the curve C_1 and the other not. Transport these continuously around and the outward pointing one returns pointing to the other side of the curve in the strip. Since we know that the curve is internally orientable (this is a property independent of any space it is embedded in) it cannot be externally orientable in a larger nonorientable manifold. By contrast, the curve C_2 is both internally and externally orientable in the strip because it does not 'feel' the nonorientability of the strip: it has a neighborhood in the strip which is orientable.

4.9. *N*-vectors, Duals, and the Symbol $\varepsilon_{ij\ldots k}$

4.9.1. Dual Maps

At each point P of an n-D manifold M, there are 4 vector spaces of the same dimension, i.e., the spaces of p-forms, (n-p)-forms, p-vectors, and (n-p)-vectors. There are various 1-1 mappings between them:

$$\begin{array}{ccc} p-vector & \overset{\tilde{\omega}}{\longleftrightarrow} & (n-p)-forms \\ g\updownarrow & & \updownarrow g \\ p-forms & \overset{\tilde{\omega}}{\longleftrightarrow} & (n-p)-vectors \end{array}$$

where g is the metric tensor and $\tilde{\omega}$ the volume n-form. The $\tilde{\omega}$ map is also called a **dual map**. The **dual** *$\mathbf{T}$ of a q-vector $\mathbf{T}$ with components $T^{i\cdots k} = T^{[i\cdots k]}$ (q indices) is defined as

$$*\mathbf{T} = \tilde{\omega}(\mathbf{T}) \qquad (4.21)$$

In terms of components, we start with

$$\tilde{\omega} = \frac{1}{n!}\omega_{i\cdots j}\,\tilde{e}^i \wedge\cdots\wedge \tilde{e}^j \qquad (n \text{ indices})$$

$$\mathbf{T} = \frac{1}{q!}T^{k\cdots l}\,\bar{e}_k \wedge\cdots\wedge \bar{e}_l \qquad (q \text{ indices})$$

$$= T^{k\cdots l}\,\bar{e}_k \otimes\cdots\otimes \bar{e}_l \quad = T^{[k\cdots l]}\,\bar{e}_k \otimes\cdots\otimes \bar{e}_l$$

Thus,

$$\tilde{\omega}(\mathbf{T}) = \frac{1}{n!}\omega_{i\cdots j}T^{k\cdots l}\,\tilde{e}^i \wedge\cdots\wedge \tilde{e}^j\left(\bar{e}_k \otimes\cdots\otimes \bar{e}_l\right)$$

Since the contraction involves only the 1st q products of 1-forms, we have

$$\omega_{i\cdots j}T^{k\cdots l}\,\tilde{e}^i \otimes\cdots\otimes \tilde{e}^j\left(\bar{e}_k \otimes\cdots\otimes \bar{e}_l\right) = \omega_{k\cdots l m\cdots j}T^{k\cdots l}\,\tilde{e}^m \otimes\cdots\otimes \tilde{e}^j$$

Since there are C^n_q ways to pick q 1-forms out of n 1-forms without regard to order, we have,

$$\tilde{\omega}(\mathbf{T}) = \frac{1}{n!}C^n_q\,\omega_{k\cdots l m\cdots j}T^{k\cdots l}\,\tilde{e}^m \wedge\cdots\wedge \tilde{e}^j \qquad (\, m\ldots j \text{ contain } (n-q) \text{ indices}\,)$$

$$= \frac{1}{q!(n-q)!}\,\omega_{k\cdots l m\cdots j}T^{k\cdots l}\,\tilde{e}^m \wedge\cdots\wedge \tilde{e}^j \qquad (n\text{-}q)\text{-form}$$

so that

$$(*\mathbf{T})_{m\cdots j} = \left[\tilde{\omega}(\mathbf{T})\right]_{m\cdots j} = \frac{1}{q!}\omega_{k\cdots l m\cdots j}T^{k\cdots l} \qquad (n\text{-}q \text{ free indices}) \qquad (4.20)$$

$$= (*\mathbf{T})_{[m\cdots j]}$$

4.9.2. Cross Products

As an example, consider Euclidean space E^3 as our vector space. By definition, the Cartesian components of a vector and its associated 1-form are equal. Consider then 2 vectors $\overline{U}$, $\overline{V}$ and their associated 1-forms $\tilde{U}$, $\tilde{V}$. The 2-form $\tilde{U} \wedge \tilde{V}$ has $C_2^3 = 3$ components, namely, $U_2V_3 - U_3V_2$, $U_3V_1 - U_1V_3$, and $U_1V_2 - U_2V_1$.

By (4.20), it can be considered as the dual of a vector $\overline{W}$ so that [Exercise 4.10]

$$*\overline{W} = \tilde{U} \wedge \tilde{V} \tag{4.22}$$

In fact, since the components of $\overline{W}$ are the same as those of $\tilde{U} \wedge \tilde{V}$, we see that $\overline{W} = \overline{U} \times \overline{V}$. Thus, the cross product exists only in E^3, where the dimensions of the 2-forms and 1-vectors are the same. Also, $\overline{W}$ is an "axial" vector since the volume 3-form $\tilde{\omega}$ and hence * changes sign, but $\tilde{U} \wedge \tilde{V}$ doesn't, when the handedness of the coordinate system is changed.

4.9.3. Dual of Forms

Analogous to (4.20), we define the dual $\mathbf{S} = *\tilde{B}$ of a p-form $\tilde{B}$ as the (n-p)-vector with components

$$S^{i\cdots k} = \frac{1}{p!}\omega^{l\cdots m\, i\cdots k} B_{l\cdots m} \qquad (4.26)$$

where $\omega^{i\cdots k}$ is the inverse of the volume n-form $\tilde{\omega}$, i.e.,

$$\omega^{12\cdots n} = \frac{1}{\omega_{12\cdots n}} \qquad (4.24)$$

$$\omega^{i\cdots k}\omega_{i\cdots k} = n! \qquad (4.23)$$

4.9.4. Properties of Duals

Taking a function *f* as a 0-vector, its dual is an *n*-form $*f = f\tilde{\omega}$. The dual of $*f$ is the original 0-vector:

$$*(*f) = *(f\tilde{\omega}) = \frac{1}{n!}\omega^{l\cdots m}(f\omega_{l\cdots m}) = f$$

i.e., $**f = f$. Indeed, for any *p*-form $\tilde{B}$ or *p*-vector **T**, we have

$$**\tilde{B} = (-)^{p(n-p)}\tilde{B} \qquad (4.27a)$$

$$**\mathbf{T} = (-)^{p(n-p)}\mathbf{T} \qquad (4.27b)$$

The proof of this is as follows. Let **S** be the (*n*-*p*)-vector dual to the *p*-form $\tilde{B}$:

$$\mathbf{S} = *\tilde{B} \qquad \text{with} \qquad S^{i\cdots k} = \frac{1}{p!}\omega^{l\cdots m\, i\cdots k}B_{l\cdots m}$$

The dual of **S** is

$$(*\mathbf{S})_{j\cdots l} = \frac{1}{(n-p)!}\omega_{i\cdots k\ j\cdots l}\, S^{i\cdots k} \qquad [(n\text{-}p) \text{ indices in } i\ldots k \text{ and } p \text{ in } j\ldots l\,]$$

$$= \frac{1}{p!(n-p)!}\omega_{i\cdots k\ j\cdots l}\ \omega^{r\cdots s\, i\cdots k}B_{r\cdots s}$$

$$= \frac{(-)^{p(n-p)}}{p!(n-p)!}\omega_{i\cdots k\ j\cdots l}\ \omega^{i\cdots k\ r\cdots s}B_{r\cdots s}$$

where the sign factor arises from interchanging the *p* indices *r*…*s* with the (*n*-*p*) indices *i*…*k* in $\omega^{r\cdots s\, i\cdots k}$ to get $\omega^{i\cdots k\ r\cdots s}$. Now, for a given set of indices *i*…*k*, only *r*…*s* that are permutations of *j*…*l* contributes. Furthermore, in permuting the indices *r*…*s*, there is no sign changes in the product $\omega^{i\cdots k\ r\cdots s}B_{r\cdots s}$ (no summation in repeated indices) since ω and *B* change sign together. Using $\omega_{i\cdots k\ j\cdots l}\ \omega^{i\cdots k\ j\cdots l} = 1$ (no summation), we see that for a given *i*…*k*, we have

$$\omega_{i\cdots k\ j\cdots l}\ \omega^{i\cdots k\ r\cdots s}B_{r\cdots s} = p!\,B_{j\cdots l} \qquad (\text{ summation in } r\ldots s)$$

Since there are $(n-p)!$ permutations in *i*…*k*, we have

$$(*\mathbf{S})_{j\cdots l} = (-)^{p(n-p)}B_{j\cdots l}$$

which proves (4.27a).

Combining the metric maps with the dual maps, we obtain maps between p- and $(n-p)$- forms or vectors. These maps are also called * but caution must be exercised when the metric is indefinite (as in relativity). [see Exercise 5.13].

4.9.5. Levi-Civita Symbols

For an *n*-D space, the completely antisymmetric **Levi-Civita symbols** are defined as

$$\varepsilon_{ij\cdots k} = \varepsilon^{ij\cdots k} = \begin{cases} +1 \\ -1 \\ 0 \end{cases} \quad \text{if} \quad j\cdots k = \begin{cases} \textit{even permutation of } 12\cdots n \\ \textit{odd permutation of } 12\cdots n \\ \textit{otherwise} \end{cases} \tag{4.28}$$

irregardless of the coordinate system used. For example, on a 3-D manifold, the form $\tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3$ has components ε_{ijk} in the coordinate system $\left(x^1, x^2, x^3\right)$.

In another coordinate system, its components become $h\varepsilon_{ijk}$, where *h* is some function.

Another example is that the volume *n*-form can be writtens as

$$\omega_{ij\cdots k} = f\,\varepsilon_{ij\cdots k} \tag{4.29}$$

while

$$\omega^{ij\cdots k} = \frac{1}{f}\varepsilon^{ij\cdots k} \tag{4.30}$$

4.10. Tensor Densities

Given a set of coordinates $\left\{x^i \,;\, i=1,\cdots,n\right\}$, we can define a volume *n*-form by

$$\tilde{\varepsilon} = \frac{1}{n!}\varepsilon_{ij\cdots k}\, \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge\cdots\wedge \tilde{d}x^n$$

Any *n*-form $\tilde{\omega}$ can be written as

$$\tilde{\omega} = \mathfrak{w}\,\tilde{\varepsilon} \qquad \Rightarrow \qquad \omega_{ij\cdots k} = \mathfrak{w}\,\varepsilon_{ij\cdots k}$$

where $\mathfrak{w}$ is called a **scalar density**. Under a coordinate transformation to $\left\{x'^i\right\}$, a straightforward generalization of (4.19) gives

$$\tilde{d}x^1 \wedge \tilde{d}x^2 \wedge\cdots\wedge \tilde{d}x^n = J\; \tilde{d}x'^1 \wedge \tilde{d}x'^2 \wedge\cdots\wedge \tilde{d}x'^n$$

where J is the Jacobian of the transformation. Since the Levi-Civita symbols take on the same values in every coordinate system, the *n*-form $\tilde{\omega}$ in the $\left\{x'^i\right\}$ system becomes

$$\tilde{\omega} = \mathfrak{w}'\,\tilde{\varepsilon}' \quad = \mathfrak{w}'\; \tilde{d}x'^1 \wedge \tilde{d}x'^2 \wedge\cdots\wedge \tilde{d}x'^n \quad = \mathfrak{w}'\,\frac{1}{J}\,\tilde{d}x^1 \wedge \tilde{d}x^2 \wedge\cdots\wedge \tilde{d}x^n$$

so that

$$\mathfrak{w}' = \mathfrak{w}\, J$$

In general, a quantity $\mathfrak{T}$ that transforms like a tensor but with an extra factor J^n, i.e.,

$$\mathfrak{T}^{c'd'\cdots}_{a'b'\cdots} = J^n\, \Lambda^a_{\ a'}\Lambda^b_{\ b'}\cdots\Lambda^{c'}_{\ c}\Lambda^{d'}_{\ d}\cdots\mathfrak{T}^{cd\cdots}_{ab\cdots} \qquad (4.31')$$

is called a **tensor density of weight *n***. Thus, a volume *n*-form is a scalar density of weight 1. [Note that Lawrie called $\mathfrak{T}$ a density of weight $-n$].

4.11. Generalized Kronecker Deltas

Consider a volume n-form $\tilde{\omega} = f\tilde{\varepsilon}$ with components $\omega_{ij\cdots k} = f\varepsilon_{ij\cdots k}$ and its dual $*\tilde{\omega}$ with components $\omega^{ij\cdots k} = \frac{1}{f}\varepsilon^{ij\cdots k}$. It is straightforward to show, by induction, that

$$\omega_{ij\cdots k}\omega^{lm\cdots r} = \varepsilon_{ij\cdots k}\varepsilon^{lm\cdots r} = \delta^l_i \delta^m_j \cdots \delta^r_k - \delta^m_i \delta^l_j \cdots \delta^r_k + \cdots$$

$$= n!\, \delta^l_{[i}\, \delta^m_j \cdots \delta^r_{k]} \qquad (4.33)$$

Defining the ***p*-delta** by

$$\delta^{i\cdots j}_{k\cdots l} = p!\, \delta^i_{[k} \cdots \delta^j_{l]} \qquad (4.34)$$

where the sets $(i\cdots j)$ and $(k\cdots l)$ each contains p indices, we cane write (4.33) as

$$\varepsilon_{ij\cdots k}\varepsilon^{lm\cdots r} = \delta^{lm\cdots r}_{ij\cdots k} \qquad (4.35)$$

Consider now the contraction of a (p+1)-delta:

$$\delta^{i\,jk\cdots l}_{i\,mr\cdots s} = (p+1)!\, \delta^i_{[i}\, \delta^j_m \delta^k_r \cdots \delta^l_{s]}$$

Using [cf. (4.5)],

$$(p+1)![imr\cdots s] = p!\{i[mr\cdots s] - m[ir\cdots s] - r[mi\cdots s] - \cdots - s[mr\cdots i]\}$$

we have

$$\delta^{i\,jk\cdots l}_{i\,mr\cdots s} = p!\{\delta^i_i\, \delta^j_{[m}\delta^k_r \cdots \delta^l_{s]} - \delta^i_m\, \delta^j_{[i}\delta^k_r \cdots \delta^l_{s]} - \delta^i_r\, \delta^j_{[m}\delta^k_i \cdots \delta^l_{s]} - \cdots - \delta^i_s\, \delta^j_{[m}\delta^k_r \cdots \delta^l_{i]}\}$$

$$= p!\{n\, \delta^j_{[m}\delta^k_r \cdots \delta^l_{s]} - \delta^j_{[m}\delta^k_r \cdots \delta^l_{s]} - \delta^j_{[m}\delta^k_r \cdots \delta^l_{s]} - \cdots - \delta^j_{[m}\delta^k_r \cdots \delta^l_{s]}\}$$

$$= p!\,(n-p)\,\delta^j_{[m}\delta^k_r \cdots \delta^l_{s]}$$

$$= (n-p)\,\delta^{jk\cdots l}_{mr\cdots s} \qquad (4.36)$$

which is a p-delta in an n-D space.

Example

Consider the triple cross product $\bar{W}\times(\bar{U}\times\bar{V})$ in E^3. In Cartesian coordinates, we have $U_i = U^i$ and

$$\left(\bar{U} \times \bar{V}\right)_i = \varepsilon_{ijk} U^j V^k \qquad \left(\bar{U} \times \bar{V}\right)^i = \varepsilon^{ijk} U_j V_k$$

so that

$$\left[\bar{W} \times \left(\bar{U} \times \bar{V}\right)\right]_i = \varepsilon_{ijk} W^j \varepsilon^{klm} U_l V_m \quad = \varepsilon_{kij} \varepsilon^{klm} W^j U_l V_m$$

$$= (3-2)! \delta_{ij}^{lm} W^j U_l V_m \quad = \left(\delta_i^l \delta_j^m - \delta_j^l \delta_i^m\right) W^j U_l V_m$$

$$= W^m U_i V_m - W^l U_l V_i \quad = \left(\bar{W} \cdot \bar{V}\right) U_i - \left(\bar{W} \cdot \bar{U}\right) V_i$$

or, in the more familiar vector notations

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{w} \cdot \mathbf{v}) \mathbf{u} - (\mathbf{w} \cdot \mathbf{u}) \mathbf{v}$$

4.12. Determinants and $\varepsilon_{ij\ldots k}$

Consider a 2×2 matrix $A=\{A^{ij}\}$.

$$\varepsilon_{ij}A^{1i}A^{2j}=\varepsilon_{12}A^{11}A^{22}+\varepsilon_{21}A^{12}A^{21} \quad =A^{11}A^{22}-A^{12}A^{21} \quad =\det(A) \tag{4.38}$$

It is left as an exercise [see Exercise 4.12] to show that, for a $n\times n$ matrix A,

$$\det(A)=\varepsilon_{ij\cdots k}A^{1i}A^{2j}\cdots A^{nk} \tag{4.39}$$

$$=\frac{1}{n!}\varepsilon_{ab\cdots c}\varepsilon_{ij\cdots k}A^{ai}A^{bj}\cdots A^{ck} \tag{4.39a}$$

Also [see Exercise 4.13], if $\{\tilde{e}^i\}$ is a set of orthonormal basis 1-forms and $\{x^{k'}\}$ is an arbitrary coordinate system, then the **preferred volume *n*-form**

$$\tilde{\omega}=\tilde{e}^1\wedge\tilde{e}^2\wedge\cdots\wedge\tilde{e}^n$$

can be written as

$$\tilde{\omega}=\sqrt{|g|}\,\tilde{d}x^{1'}\wedge\tilde{d}x^{2'}\wedge\cdots\wedge\tilde{d}x^{n'} \tag{4.40}$$

where $g=\det(g_{i'j'})$ with $\{g_{i'j'}\}$ being the matrix tensor expressed in the $\{x^{k'}\}$ coordinates.

Example

In E^3 with Cartesian coordinates, the volume of a parallelepiped of sides $\bar{a},\bar{b}$ and $\bar{c}$ is equal to the determinant of a matrix A with these vectors as rows, i.e.,

$$A^{1j}=a^j \qquad A^{2j}=b^j \qquad A^{3j}=c^j$$

By (4.39), we have

$$volume=\det(A) \quad =\varepsilon_{ijk}a^ib^jc^k \quad =a^i\varepsilon_{ijk}b^jc^k \quad =a^i\left(\bar{b}\times\bar{c}\right)_i \quad =\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$$

4.13. Metric Volume Elements

In an n-D manifold with an orthonormal 1-form basis $\{\tilde{e}^i\}$, the preferred volume n-form $\tilde{\omega} = \tilde{e}^1 \wedge \cdots \wedge \tilde{e}^n$ is unique up to a sign, i.e., given another orthonormal basis $\{\tilde{e}^{i'}\}$ and corresponding volume n-form $\tilde{\sigma} = \tilde{e}^{1'} \wedge \cdots \wedge \tilde{e}^{n'}$, we have $\tilde{\omega} = \pm\tilde{\sigma}$. The proof is as follows. By definition, the components of these n-forms are $\omega_{ij\cdots k} = \varepsilon_{ij\cdots k}$ and $\sigma_{i'j'\cdots k'} = \varepsilon_{i'j'\cdots k'}$. With respect to the basis $\{\tilde{e}^{i'}\}$, the components of $\tilde{\omega}$ are $\omega_{i'j'\cdots k'} = J\,\varepsilon_{i'j'\cdots k'}$, where J is the Jacobian of the transformation $\{\tilde{e}^i\} \to \{\tilde{e}^{i'}\}$. The proof is completed if we can show that $J = \pm 1$.

To begin, applying the transformation matrix $\Lambda^i_{\ j'}$ to the metric tensor gives

$$g_{i'j'} = \Lambda^i_{\ i'}\Lambda^j_{\ j'}\,g_{ij}$$

which can be written in matrix language as

$$\mathbf{g}' = \quad ^{T} \qquad \text{where} \qquad \mathbf{\Lambda}_{ij'} = \Lambda^i_{\ j'}$$

The determinant of this is

$$g' = \det(\mathbf{g}') = \det(\ \ ^{T})\, g\, \det(\ \ ) \ = g\left[\det(\mathbf{\Lambda})\right]^2$$

Since everything are real, we see that g and g' must have the same sign and

$$\det(\mathbf{\Lambda}) = \pm\sqrt{\frac{g'}{g}}$$

Assuming $\{g_{ij}\}$ is written with respect to an orthonormal basis, we have [see §2.29] $g_{ij} = \pm\delta_{ij}$ so that $g = \pm 1$ and $\det(\mathbf{\Lambda}) = \pm\sqrt{|g'|}$. [Note that the three ± signs in the last sentence are not correlated]. If the bases for $\{g_{ij}\}$ and $\{g_{i'j'}\}$ are both orthonormal, we have

$$\left[\det(\mathbf{\Lambda})\right]^2 = 1 \qquad \Rightarrow \qquad J = \det(\mathbf{\Lambda}) = \pm 1$$

which completes the proof.

Since the volume n-form $\tilde{\omega}$ is unique only up to a $\pm$ sign, so is its inverse $\omega^{ij\cdots k}$. By definition,

$$\omega_{ij\cdots k}\,\omega^{ij\cdots k} = n! \qquad \text{with} \qquad \omega^{12\cdots n} = \frac{1}{\omega_{12\cdots n}}$$

On the other hand, given a metric tensor g_{ij}, we can raise the indices of an n-form $\tilde{\omega} = \sqrt{|g|}\,\tilde{e}^1 \wedge \cdots \wedge \tilde{e}^n$ in a general, possibly non-orthonormal, basis to get

$$\left(\tilde{\omega}'\right)^{ij\cdots k} = g^{il} g^{jm} \cdots g^{kr} \omega_{lm\cdots r} \quad = g^{il} g^{jm} \cdots g^{kr} \varepsilon_{lm\cdots r} \sqrt{|g|}$$

$$= \varepsilon^{ij\cdots k} \det\left(g^{lm}\right) \sqrt{|g|} \quad = \frac{\sqrt{|g|}}{g} \varepsilon^{ij\cdots k}$$

where we've used $\det\left(g^{lm}\right) = \det\left(\mathbf{g}^{-1}\right) = \frac{1}{g}$. Thus

$$\left(\tilde{\omega}'\right)^{12\cdots n} = \frac{\sqrt{|g|}}{g} \tag{4.41}$$

while

$$\left(\tilde{\omega}\right)^{12\cdots n} = \omega^{12\cdots n} \quad = \frac{1}{\omega_{12\cdots n}} = \frac{1}{\sqrt{|g|}} \tag{4.42}$$

If g is negative, then $\tilde{\omega}' = -\tilde{\omega}$. It is conventional in relativity, where g is negative, to use $\tilde{\omega}'$ in the inverse dual relations so that an extra minus sign is introduced in equations like (4.27).

4.14. The Exterior Derivative

The **exterior derivative operator** $\tilde{d}$ is a derivative operator that increases by 1 the degree of any form it operates on. In particular, for a function, or 0-form, *f*, the quantity $\tilde{d}f$ is a 1-form. Furthermore, we also require that for any *p*-form $\tilde{\alpha}$ and *q*-forms $\tilde{\beta}$ and $\tilde{\gamma}$, it satisfies

1. $\tilde{d}\left(\tilde{\beta}+\tilde{\gamma}\right)=\tilde{d}\tilde{\beta}+\tilde{d}\tilde{\gamma}$.

2. $\tilde{d}\left(\tilde{\alpha}\wedge\tilde{\beta}\right)=\left(\tilde{d}\tilde{\alpha}\right)\wedge\tilde{\beta}+(-)^p\,\tilde{\alpha}\wedge\tilde{d}\tilde{\beta}$.

3. $\tilde{d}\left(\tilde{d}\tilde{\alpha}\right)=0$.

Property (2) is a modified Leibniz rule which, together with (1), characterizes $\tilde{d}$ as an **antiderivation**. It is crucial for maintaining the antisymmetries of all the forms involved. Property(3) is a generalization of the case for a function *f* where

$$\begin{aligned}\tilde{d}\left(\tilde{d}f\right)=\tilde{d}\left(\frac{\partial f}{\partial x^i}\tilde{d}x^i\right)&=\left[\tilde{d}\left(\frac{\partial f}{\partial x^i}\right)\right]\wedge\tilde{d}x^i+\frac{\partial f}{\partial x^i}\tilde{d}\tilde{d}x^i\\&=\frac{\partial^2 f}{\partial x^j\partial x^i}\tilde{d}x^j\wedge\tilde{d}x^i+\frac{\partial f}{\partial x^i}\tilde{d}\tilde{d}x^i=0\end{aligned}$$

where $\tilde{d}\tilde{d}x^i=0$ since it involves 2nd derivatives of x^i, and $\frac{\partial^2 f}{\partial x^j\partial x^i}\tilde{d}x^j\wedge\tilde{d}x^i=0$ because $\frac{\partial^2 f}{\partial x^j\partial x^i}$ is symmetric in the indices *i* and *j*.

Exercise 4.14

(a) Show that

$$\tilde{d}\left(f\,\tilde{d}g\right)=\tilde{d}f\wedge\tilde{d}g \qquad (4.44)$$

(b) Use (a) to show that if

$$\tilde{\alpha}=\frac{1}{p!}\alpha_{i\cdots j}\,\tilde{d}x^i\wedge\cdots\wedge\tilde{d}x^j$$

then

$$\tilde{d}\tilde{\alpha}=\frac{1}{p!}\frac{\partial\alpha_{i\cdots j}}{\partial x^k}\tilde{d}x^k\wedge\tilde{d}x^i\wedge\cdots\wedge\tilde{d}x^j$$

so that

$$\left(\tilde{d}\tilde{\alpha}\right)_{ki\cdots j} = \left(p+1\right)\frac{\partial}{\partial x^{[k}}\alpha_{i\cdots j]} \tag{4.45}$$

4.15. Notation for Derivatives

Partial derivatives will be denoted by a comma, e.g.,

$$\frac{\partial f}{\partial x^i} \equiv f_{,i} \tag{4.46}$$

$$\frac{\partial V_j^{\ i}}{\partial x^k} \equiv V^i_{\ j,k} \tag{4.47}$$

$$\frac{\partial^2 f}{\partial x^i \partial x^j} \equiv f_{,ji} \tag{4.48}$$

Note in particular that $\frac{\partial}{\partial x^j}$ is performed 1st in (4.48). We also emphasize that partial differentiation is **not** a tensor operation unless it is operating on a scalar function [see §2.27]. Thus, $\{f_{,i}\}$ is a 1-form but neither $\{V^i_{\ j,k}\}$ nor $\{f_{,ji}\}$ are tensors. Further examples of the present notation are

$$\left[\bar{U}, \bar{V}\right]^i = U^j V^i_{\ ,j} - V^j U^i_{\ ,j}$$

$$\left(\tilde{d}\tilde{\alpha}\right)_{i\cdots jk} = (-)^p (p+1)\, \alpha_{[i\cdots j,k]} \tag{4.49}$$

where the last is a rewrite of (4.45).

4.16. Familiar Examples of Exterior Differentiation

Consider the 3-D Euclidean space E^3. Let

$$\tilde{a} = a_j \tilde{d}x^j \quad = a_1 \tilde{d}x^1 + a_2 \tilde{d}x^2 + a_3 \tilde{d}x^3$$

$\Rightarrow$

$$\tilde{d}\tilde{a} = \tilde{d}\left(a_j \tilde{d}x^j\right) \quad = \tilde{d}a_j \wedge \tilde{d}x^j + a_j \tilde{d}\tilde{d}x^j \quad = a_{j,k} \tilde{d}x^k \wedge \tilde{d}x^j$$

$$= a_{1,k} \tilde{d}x^k \wedge \tilde{d}x^1 + a_{2,k} \tilde{d}x^k \wedge \tilde{d}x^2 + a_{3,k} \tilde{d}x^k \wedge \tilde{d}x^3$$

$$= a_{1,2} \tilde{d}x^2 \wedge \tilde{d}x^1 + a_{1,3} \tilde{d}x^3 \wedge \tilde{d}x^1 + a_{2,1} \tilde{d}x^1 \wedge \tilde{d}x^2 + a_{2,3} \tilde{d}x^1 \wedge \tilde{d}x^3$$

$$+ a_{3,1} \tilde{d}x^1 \wedge \tilde{d}x^3 + a_{3,2} \tilde{d}x^2 \wedge \tilde{d}x^3$$

$$= \left(a_{1,2} - a_{2,1}\right) \tilde{d}x^2 \wedge \tilde{d}x^1 + \left(a_{3,1} - a_{1,3}\right) \tilde{d}x^1 \wedge \tilde{d}x^3 + \left(a_{2,3} - a_{3,2}\right) \tilde{d}x^3 \wedge \tilde{d}x^2$$

Taking the dual, we have

$$* \tilde{d}\tilde{a} = \left(a_{1,2} - a_{2,1}\right) \; *\left(\tilde{d}x^2 \wedge \tilde{d}x^1\right) + \left(a_{3,1} - a_{1,3}\right) \; *\left(\tilde{d}x^1 \wedge \tilde{d}x^3\right)$$

$$+ \left(a_{2,3} - a_{3,2}\right) \; *\left(\tilde{d}x^3 \wedge \tilde{d}x^2\right)$$

Using the dual of the volume n-form

$$\overline{\varpi} = \frac{1}{3!} \varepsilon^{ijk} \; \overline{e}_i \wedge \overline{e}_j \wedge \overline{e}_k$$

we have

$$*\left(\tilde{d}x^j \wedge \tilde{d}x^k\right) = \overline{\varpi}\left(\tilde{d}x^j \wedge \tilde{d}x^k\right) \quad = \frac{1}{3!} \varepsilon^{ilm} \; \overline{e}_i \wedge \overline{e}_l \wedge \overline{e}_m \left(\tilde{d}x^j \wedge \tilde{d}x^k\right)$$

$$= \frac{2}{3!} \varepsilon^{ilm} \; \overline{e}_i \wedge \overline{e}_l \wedge \overline{e}_m \left(\tilde{d}x^j \otimes \tilde{d}x^k\right) \quad = \frac{2}{3!} \varepsilon^{ilm} \; \delta_i^j \delta_l^k \; \overline{e}_m \; C_2^3$$

$$= \varepsilon^{jkm} \; \overline{e}_m$$

so that

$$* \tilde{d}\tilde{a} = \left(a_{1,2} - a_{2,1}\right)\left(-\overline{e}_3\right) + \left(a_{3,1} - a_{1,3}\right)\left(-\overline{e}_2\right) + \left(a_{2,3} - a_{3,2}\right)\left(-\overline{e}_1\right)$$

$$= \left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}\right)\overline{e}_1 + \left(\frac{\partial a_3}{\partial x^1} - \frac{\partial a_1}{\partial x^3}\right)\overline{e}_2 + \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right)\overline{e}_3$$

$$= \overline{\nabla} \times \overline{a} \tag{4.50}$$

where $\overline{\nabla} \times$ is the ordinary curl operator. Thus, the effect of $* \tilde{d}$ on a 1-form $\tilde{a}$ is the same as that of the curl $\bar{\nabla}\times$ on the vector $\bar{a}$. Note that this conclusion is independent of coordinate system. Next, consider

$$*\bar{a} = *\left(a^i \bar{e}_i\right) = a^i *\left(\bar{e}_i\right) = a^i \frac{1}{3!}\varepsilon_{jkl}\,\tilde{e}^j \wedge \tilde{e}^k \wedge \tilde{e}^l\left(\bar{e}_i\right)$$

$$= a^i \frac{1}{3!}C_1^3 \delta_i^j \varepsilon_{jkl}\,\tilde{e}^k \wedge \tilde{e}^l = \frac{1}{2}\varepsilon_{jkl}\,a^j\,\tilde{e}^k \wedge \tilde{e}^l$$

so that

$$\tilde{d}*\bar{a} = \frac{1}{2}\varepsilon_{jkl}\tilde{d}\left(a^j\,\tilde{e}^k \wedge \tilde{e}^l\right)$$

$$= \frac{1}{2}\varepsilon_{jkl}\left\{a^j_{,i}\,\tilde{d}x^i \wedge \tilde{e}^k \wedge \tilde{e}^l + a^j\left(\tilde{d}\tilde{e}^k\right)\wedge \tilde{e}^l - a^j\,\tilde{e}^k \wedge \left(\tilde{d}\tilde{e}^l\right)\right\}$$

Using $\tilde{e}^j = \tilde{d}x^j$ and $\tilde{d}\tilde{d} = 0$, we have

$$\tilde{d}*\bar{a} = \frac{1}{2}\varepsilon_{jkl}\,a^j_{,i}\,\tilde{d}x^i \wedge \tilde{d}x^k \wedge \tilde{d}x^l$$

$$= \frac{1}{2}\left(\varepsilon_{123}\,a^1_{,1}\,\tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3 + \varepsilon_{132}\,a^1_{,1}\,\tilde{d}x^1 \wedge \tilde{d}x^3 \wedge \tilde{d}x^2\right)$$

$$+\frac{1}{2}\left(\varepsilon_{231}\,a^2_{,2}\,\tilde{d}x^2 \wedge \tilde{d}x^3 \wedge \tilde{d}x^1 + \varepsilon_{213}\,a^2_{,2}\,\tilde{d}x^2 \wedge \tilde{d}x^1 \wedge \tilde{d}x^3\right)$$

$$+\frac{1}{2}\left(\varepsilon_{312}\,a^3_{,3}\,\tilde{d}x^3 \wedge \tilde{d}x^1 \wedge \tilde{d}x^2 + \varepsilon_{321}\,a^3_{,3}\,\tilde{d}x^3 \wedge \tilde{d}x^2 \wedge \tilde{d}x^1\right)$$

$$= \left(a^1_{,1} + a^2_{,2} + a^3_{,3}\right)\tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3$$

$$= a^j_{,j}\,\tilde{\omega} = \left(\bar{\nabla}\cdot\bar{a}\right)\tilde{\omega} \qquad (4.52)$$

where

$$\tilde{\omega} = \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3 = \frac{1}{3!}\varepsilon_{jkl}\,\tilde{d}x^j \wedge \tilde{d}x^k \wedge \tilde{d}x^l$$

is the volume *n*-form. Thus, $\tilde{d}*$ is equivalent to the divergence operator $\bar{\nabla}\cdot$.

4.17. Integrability Conditions for Partial Differential Equations

4.18. Exact Forms

4.19. Proof of the Local Exactness of Closed Forms

4.20. Lie Derivatives of Forms

4.21. Lie Derivatives and Exterior Derivatives Commute

4.22. Stokes' Theorem

4.23. Gauss' Theorem and the Definition of Divergence

4.24. A Glance at Cohomology Theory

4.25. Differential Forms and Differential Equations

4.26. Frobenius' Theorem (Differential Forms Version)

4.27. Proof of the Equivalence of the Two Versions of Frobenius' Theorem

4.28. Conservation Laws

4.29. Vector Spherical Harmonics

4.30. Bibliography