Chapter 1 Introduction

■ Particle on a $1-D$ Lattice

$$\mathcal{H}(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mathcal{V}(x)$$

with

$$\mathcal{V}(x + n b) = \mathcal{V}(x) \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

■ Translational Symmetry

Consider the translation operation T_n defined by the coordinate transformation

$$x \longrightarrow x' = T_n x = x + n b$$

Let $\mathbb{T}_n$ be the corresponding Hilbert space operator, we have

$$|x\rangle \longrightarrow |x'\rangle = |T_n x\rangle = |x + n b\rangle = \mathbb{T}_n |x\rangle$$

$$\langle x| \longrightarrow \langle x'| = \langle x + n b| = \langle x| \mathbb{T}_n^{\dagger}$$

Since

$$\langle x | x \rangle = 1 \qquad \forall\ x$$

we have

$$\langle x' | x' \rangle = \left\langle x \middle| \mathbb{T}_n^{\dagger} \mathbb{T}_n \middle| x \right\rangle = 1 \qquad \forall\ x$$

$$\langle x | x \rangle = \left\langle x' \middle| \mathbb{T}_n \mathbb{T}_n^{\dagger} \middle| x' \right\rangle = 1 \qquad \forall\ x'$$

Therefore

$$\mathbb{T}_n^{\dagger} \mathbb{T}_n = \mathbb{T}_n \mathbb{T}_n^{\dagger} = \mathbb{I}$$

ie., $\mathbb{T}_n$ is unitary. This is a general feature of all symmetry operators.

Defining

$$|\psi\rangle \longrightarrow |\psi'\rangle = \mathbb{T}_n |\psi\rangle$$

$$\psi(x) = \langle x | \psi \rangle$$

we have

$$\psi'(x') = \langle x' | \psi' \rangle = \left\langle x \middle| \mathbb{T}_n^{\dagger}\ \mathbb{T}_n \middle| \psi \right\rangle = \langle x | \psi \rangle = \psi(x)$$

The last equality serves to define the transformation $\mathcal{T}_n$ of an arbitrary function, ie.

$$f \longrightarrow f' = \mathcal{T}_n f$$

with

$$f(x) = f'(x') = f'(x + n b) = \mathcal{T}_n f(x')$$

so that

$$f'(x) = f(x - n b) = \mathcal{T}_n(x) f(x) = f\left(T_n^{-1} x\right)$$

The explicit form of $\mathcal{T}_n(x)$ can be found by Taylor expanding $f(x - n b)$:

$$\begin{aligned}
\mathcal{T}_n(x) f(x) &= f(x) - n b \frac{d f(x)}{d x} + \frac{1}{2}(n b)^2 \frac{d^2 f(x)}{d x^2} - \ldots \\
&= \left\{ 1 - n b \frac{d}{d x} + \frac{1}{2}(n b)^2 \frac{d^2}{d x^2} - \ldots \right\} f(x) \\
&= \sum_{m=0} (-n b)^m \frac{d^m}{d x^m} f(x) \\
&= e^{-n b \frac{d}{d x}} f(x)
\end{aligned}$$

so that

$$\mathcal{T}_n(x) = e^{-n b \frac{d}{dx}}$$

or

$$\mathbb{T}_n = e^{-\frac{i}{\hbar} n b \mathbb{p}}$$

In this form, we say that the momentum operator is the generator of the translation operator. It is straightforward to show that the angular momentum & the hamiltonian operators are generators of the rotation & time displacemnt operators, respectively.

The transformation of operators can be found in a similar fashion.

Let

$$\mathbb{A} \,|\psi\rangle = |\phi\rangle$$

$\longrightarrow$

$$\mathbb{T}_n \,\mathbb{A} \,|\psi\rangle = \mathbb{T}_n \,\mathbb{A}\, \mathbb{T}_n^{\dagger}\, \mathbb{T}_n |\psi\rangle = \mathbb{T}_n \mathbb{A}\, \mathbb{T}_n^{\dagger} |\psi'\rangle$$

$$= \mathbb{T}_n |\phi\rangle = |\phi'\rangle$$

Comparing with the definition

$$\mathbb{A}' |\psi'\rangle = |\phi'\rangle$$

we have

$$\mathbb{A}' = \mathbb{T}_n \,\mathbb{A}\, \mathbb{T}_n^{\dagger}$$

In the x – representation

$$\langle x | x' \rangle = \delta(x - x')$$

$$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p x}$$

$$\langle x | \mathbb{p} | x' \rangle = -i\hbar\, \delta(x - x') \frac{\partial}{\partial x}$$

$$\langle x | \mathbb{A} | \psi \rangle = \langle x | \phi \rangle$$

$$= \int d x' \langle x | \mathbb{A} | x' \rangle \langle x' | \psi \rangle$$

For local operators that are functions of $\mathbb{x}$ & $\mathbb{p}$,

$$\langle x | \mathbb{A}(\mathbb{x}, \mathbb{p}) | x' \rangle = \delta(x - x') \langle x | \mathbb{A}(\mathbb{x}, \mathbb{p}) | x \rangle = \delta(x - x')\, \mathcal{A}\left(x, -i\hbar \frac{\partial}{\partial x}\right)$$

For convenience, we'll write $\mathcal{A}\left(x, -i\hbar \frac{\partial}{\partial x}\right)$ simply as $\mathcal{A}(x)$ so that

$$\langle x | \mathbb{A} | \psi \rangle = \langle x | \phi \rangle = \mathcal{A}(x) \langle x | \psi \rangle$$

or

$$\mathcal{A}(x)\, \psi(x) = \phi(x)$$

The transformation under T_n then gives

$$\langle x | \mathbb{T}_n \,\mathbb{A} \,|\psi\rangle = \left\langle x \middle| \mathbb{T}_n \,\mathbb{A}\, \mathbb{T}_n^{\dagger}\, \mathbb{T}_n \middle| \psi \right\rangle = \left\langle x \middle| \mathbb{T}_n \mathbb{A}\, \mathbb{T}_n^{\dagger} \middle| \psi' \right\rangle$$

$$= \langle x | \mathbb{T}_n |\phi\rangle = \langle x | \phi' \rangle$$

$$\mathcal{T}_n(x)\, \mathcal{A}(x)\, \psi(x) = \mathcal{T}_n(x)\, \phi(x) = \phi'(x) = \mathcal{A}'(x)\, \psi'(x)$$

Thus

$$\phi\left(T_n^{-1} x\right) = \mathcal{A}\left(T_n^{-1} x\right) \psi\left(T_n^{-1} x\right)$$

The effect of T_n on $\mathcal{H}$ is therefore

$$\mathcal{H}(x) \quad \longrightarrow \quad \mathcal{H}\left(T_n^{-1} x\right) = \mathcal{H}(x - n b)$$

Writing

$$y = T_n^{-1} x$$

we see that

$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} \qquad \mathcal{V}(y) = \mathcal{V}(x)$$

so that

$$\mathcal{H}(y) = \mathcal{H}(x)$$

as expected.

This invariance of the hamiltonian can of course be expressed as

$$\mathbb{T}_n \, \mathbb{H} \, \mathbb{T}_n^{\dagger} = \mathbb{H}$$

or

$$\mathbb{T}_n \, \mathbb{H} = \mathbb{H} \, \mathbb{T}_n$$

$$[\, \mathbb{T}_n \, , \mathbb{H} \,] = 0$$

which means the eigenstates of $\mathbb{T}_n$ are also that of $\mathbb{H}$. This is basis of the usefulness of group theory in quantum mechanics.