Chapter 3

Group Representations

3.1 Representations

A set of invertible linear transformations, closed with respect to operator multiplication, satisfies the group axioms. Such a set forms a group of linear transformations, or group of operators.

■ Definition 3.1 (**Representations** of a Group):

Let $U(G)$ be a set of operators on V.
If $U(G)$ is homomorphic to G, it forms a **representation** (rep) of G.

To be more specific: the rep is a mapping

$$U: \quad G \longrightarrow U(G)$$
$$\quad\quad g \mapsto U(g)$$

such that

(3.1-1) $$U(g_1)\,U(g_2) = U(g_1\,g_2) \qquad \forall\ g_1, g_2 \in G$$

In other words, a rep is a mapping of G onto a set of operators that preserves the group multiplication.

The dimension of the rep is the dimension of the vector space V.
A rep is said to be faithful if the homomorphism is also an isomorphism (i.e. it is also one- to- one).
A degenerate rep is one which is not faithful.

■ **Matrix Representations**

Consider the case of a finite- dimensional rep.
Choose a set of basis vectors $\{\hat{e}_i\,,\ i = 1, 2, \ldots, n\}$ on V.
The operators $U(g)$ are then realized as $n \times n$ matrices $D(g)$ as follows:

(3.1-2) $$U(g)\,|\,e_i\,\rangle = \big|\,e_j\,\big\rangle\,D(g)^j{}_i \qquad g \in G$$

The group of matrices $D(G)$ forms a matrix representation of G.

- **Proof:**

From (3.1-2), we have

$$U(g_2)\,|e_i\rangle = |e_j\rangle D(g_2)^j{}_i$$

$$U(g_1)\,U(g_2)\,|e_i\rangle = U(g_1)\,|e_j\rangle D(g_2)^j{}_i$$

$$= |e_k\rangle D(g_1)^k{}_j\, D(g_2)^j{}_i$$

$$U(g_1\, g_2)\,|e_i\rangle = |e_k\rangle D(g_1\, g_2)^k{}_i$$

Since $\{e_i\}$ form a basis, we conclude that

$$U(g_1\, g_2) = U(g_1)\, U(g_2)$$

requires

$$D(g_1\, g_2)^k{}_i = D(g_1)^k{}_j\, D(g_2)^j{}_i$$

In other words,

(3.1-3) $D(g_1\, g_2) = D(g_1)\, D(g_2)$ [matrix multiplication]

- ## Examples

- Example 1.

There is a trivial 1-D rep for every group G:

ie. $V = C$, and $U(g) = 1$ for all $g \in G$.

This rep is called the totally symmetric or the identity rep.

- Example 2:

Let G be a group of matrices, $V = C$, and $U(g) = \det g$.

This defines a non- trivial 1-D rep:

$$U(g_1\, g_2) = \det(g_1\, g_2) = \det g_1 \det g_2 = U(g_1)\, U(g_2)$$

- Example 3:

For any given real number ξ in the interval $(-\pi < \xi \leq \pi)$, the numbers

$$\left\{ e^{-i n \xi},\ n = 0, \pm 1, \pm 2, \ldots \right\}$$

form a (1-D) rep of the discrete translation group in one spatial dimension, as discussed in detail in Chap. 1.

- Example 4:

Let G be the dihedral group $D_2 = \{e, h, v, r\}$.

Let $V = E_2$ with basis vectors $\left(\hat{e}_1, \hat{e}_2\right)$.

(3.1-4) $D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $D(h) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$D(v) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $D(r) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

form a 2-D rep.

- Example 5:

Let $G = \{ R(\phi),\ 0 \le \phi \le 2\pi \}$ and $V = E_2$.
Using

(3.1-5) $$\hat{e}_1{}' = U(\phi)\,\hat{e}_1 = \hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi = \hat{e}_1\, D(\phi)^1{}_1 + \hat{e}_2\, D(\phi)^2{}_1$$
$$\hat{e}_2{}' = U(\phi)\,\hat{e}_2 = \hat{e}_1 (-\sin\phi) + \hat{e}_2 \cos\phi = \hat{e}_1\, D(\phi)^1{}_2 + \hat{e}_2\, D(\phi)^2{}_2$$

we have

(3.1-6) $$D(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Note that if x is an arbitrary vector in E_2, $x = \hat{e}_i\, x^i$, then

(3.1-7) $$x' = U(\phi)\, x = \hat{e}_j\, x'^j$$
$$x'^j = D(\phi)^j{}_i\, x^i$$

or,

(3.1-8) $$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

or,

$$\mathbb{x}' = D(\phi)\, \mathbb{x} \qquad \mathbb{x}' = \begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} \qquad \mathbb{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

- Example 6:

Let G be the dihedral group $D_3 = \{ e, \sigma_1, \sigma_2, \sigma_3, C_3, C_3{}^2 \}$ and $V = E_2$.
Expressing the transformed vectors $U(g)\,\hat{e}_i$ in terms of the original basis vectors according to Eq. (3.1-2), one obtains six 2-D matrices $\{D(g)\}$ which form a rep of the group D_3. [Problem 3.1] Since the group D_3 is isomorphic to the symmetric group S_3, these matrices provide a rep of S_3 as well.

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The last three examples illustrate that different groups may be realized on the same vector space. Here, E_2 is seen to provide reps for the two finite groups D_2 , D_3 as well as the continuous (hence infinite) group $R(2)$.

- Example 7:

Let V_f be the space of complex- valued linear homogeneous functions f of two real variables $(x,\ y)$:

(3.1-9) $$f(x,\ y) = a\,x + b\,y \qquad a, b \in C$$

Let
$$\boldsymbol{x} = (x,\ y) \equiv (x^1,\ x^2) \in E_2$$

G can be realized on V_f as a set of mappings

(3.1-10) $$f \overset{g\in G}{\longrightarrow} f'$$

where
$$f'(x') \equiv f(x)$$
$$x' = U(g)\, x$$

and $U(G)$ is a rep of G on E_2.

Note that

$$f\,'(\boldsymbol{x}') = f\left[\,U\!\left(g^{-1}\right)\boldsymbol{x}'\right]$$

so that

$$f\,'(\boldsymbol{x}) = f\left[\,U\!\left(g^{-1}\right)\boldsymbol{x}\right]$$

It is straightforward to show that the mapping (in the function space V_f) defined by Eq. (3.1-10) is a homomorphism; for if $g\,''\,g\,' = g$ then

(3.1-11)

$$f \xrightarrow{g'} f\,' \qquad f\,'[\,U(g')\,x] = f(x)$$

$$f\,' \xrightarrow{g''} f\,'' \qquad f\,''[\,U(g'')\,x] = f\,'(x)$$

With

$$\boldsymbol{x}' = U(g')\,\boldsymbol{x}$$

we have

$$\begin{aligned} & f\,''[\,U(g'')\,\boldsymbol{x}'] \\ = \;& f\,''[\,U(g'')\,U(g')\,\boldsymbol{x}] \\ = \;& f\,''[U(g)\,\boldsymbol{x}\,] \end{aligned}$$

On the other hand

$$\begin{aligned} & f\,''[\,U(g'')\,\boldsymbol{x}'] \\ = \;& f\,'(\boldsymbol{x}') \\ = \;& f\,'[U(g')\,\boldsymbol{x}] \\ = \;& f(\boldsymbol{x}) \end{aligned}$$

Hence

$$f\,''[U(g)\,\boldsymbol{x}\,] = f(\boldsymbol{x})$$

which means

$$f \xrightarrow{g} f\,''$$

Therefore, the set of transformations defined by (3.1-10) forms a rep of G. The function space V_f in this case is 2-D, and (due to the linear nature of the function f) the reps realized on V_f are actually the same as the ones realized on $\{x \in E_2\}$ described in the previous three examples. [Prove!]

There are, of course, other function spaces defined over (x, y) which are of arbitrarily high dimensions. One can obtain rather complicated reps of the group G in question on those higher dimensional spaces. We shall discuss some examples of this kind in later sections. Group reps on function spaces are very important in physical applications. The transformation properties of "Wave Functions" in classical (string, sound, fluids, ...) and quantum (Schrodinger, Dirac, ...) systems as well as "Fields" (Electromagnetic, Gauge, ...) under space- time and "internal" symmetry groups are central to modern physics and will occupy much of our attention.

- Theorem 3.1:

(i) Any rep of $K = G/H$ is also a (degenerate) rep of G.

(ii) Conversely, if $U(G)$ is a degenerate rep of G, then G has at least one invariant subgroup H such that $U(G)$ defines a faithful rep of the factor group G/H.

- Proof:

(i) The mapping $g \in G \to k = g H \in K$ followed by $k \to U(k)$ on V is a homomorphism from G to the group of linear transformations $U(K)$. It, therefore, forms a rep. If H is a non- trivial invariant subgroup, then the first step of the above mapping is many-to-one. Hence the rep is not faithful.

(ii) The proof follows from Theorem 2.5 of Chap. 2. QED

- **Corollary**

All reps (except the trivial one) of simple groups are faithful.

- Example:

S_3 has an invariant subgroup $\{e, (123), (321)\}$. The factor group is isomorphic to $C_2 = \{e, a\}$. The group C_2 has the rather simple rep $(e, a \to 1, -1)$. This induces a one- dimensional rep of S_3 which assigns 1 to the elements $\{e, (123), (321)\}$ and -1 to $\{(12), (23), (31)\}$. The reader should verify for himself that this assignment indeed preserves the operation of group multiplication. (cf Table 2.4)

3.2 Irreducible, Inequivalent Representations

Let $U(G)$ be a rep of the G on V, and S be any invertible operator on V. Then

$U'(G) = S\, U(G)\, S^{-1}$ also form a rep of G on V.

$\dim U'(G) = \dim U(G)$.

$U(G)$ and $U'(G)$ are said to be related by the "similarity transformation" S, which is equivalent to a change of basis.

- Definition 3.2 (Equivalence of **Representations**):

Two reps of a group G related by a similarity transformation are said to be equivalent.

- Definition 3.3 (Characters of a **Representation**):

The character $\chi(g)$ of $g \in G$ in a rep $U(G)$ is defined to be

$$\chi(g) = \mathrm{Tr}\, U(g).$$

If $D(G)$ is a matrix realization of $U(G)$, then

$$\chi(g) = \sum_i D(g)^i{}_i$$

Since the trace is invariant under a similarity transformation, χ is a function of the class- label only.

- Definition 3.4 (Invariant Subspace):

Let $U(G)$ be a rep of G on V, and V_1 be a subspace of V. If

$$U(g) \,|\, x \rangle \in V_1 \qquad \forall\, x \in V_1 \text{ and } g \in G.$$

V_1 is said to be an invariant subspace of V with respect to $U(G)$.

This means there is a basis such that any vector in V can be written as

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \qquad \boldsymbol{x}_1 \in V_1$$

and all $U(g)$ has the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

so that

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A\, x_1 \\ 0 \end{pmatrix} \in V_1$$

An invariant subspace is minimal or proper if it does not contain any non- trivial invariant subspace with respect to $U(G)$.

Examples of trivial invariant subspaces of V with respect to $U(G)$ are:

(i) the space V itself, and

(ii) the subspace consisting only of the null vector.

- Definition 3.5 (Irreducible **Representations**):

$U(G)$ is irreducible if there is no non-trivial invariant subspace in V wrt it.
Otherwise, $U(G)$ is reducible.
In the latter case, if the orthogonal complement of the invariant subspace is also invariant with respect to $U(G)$, then the rep is said to be fully reducible or decomposable.

Note: If V_1 is a subspace of V, the orthogonal complement of V_1 consists of all vectors in V which are orthogonal to every vector in V_1. For finite- dimensional vector spaces, at least, the orthogonal complement of V_1 also forms a subspace, called V_2 say, and we have $V = V_1 \oplus V_2$.

In matrix terms, let

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \qquad \boldsymbol{x}_1 \in V_1 \qquad \boldsymbol{x}_2 \in V_2$$

V_1 is invariant wrt $U(G)$ means

$$U(g) = \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix} \qquad \forall\, g \in G$$

V_2 is invariant wrt $U(G)$ means

$$U(g) = \begin{pmatrix} A(g) & 0 \\ D(g) & C(g) \end{pmatrix}$$

$U(G)$ is decomposable means

$$U(g) = \begin{pmatrix} A(g) & 0 \\ 0 & C(g) \end{pmatrix}$$

- **Examples**

- Example 1:

Consider D_2 on E_2.
The 1-D subspace spanned by $\hat{e}_1$ is invariant under all four group operations.
It is therefore an invariant subspace with respect to D_2.
The same is true for the 1-D subspace spanned by $\hat{e}_2$.
The 2-D rep of the group given by Eq. (3.1-4) is therefore a reducible rep.
The reps defined on the 1-D invariant subspaces are irreducible as these spaces are minimal.

- Example 2:

The 1-D subspace spanned by $\hat{e}_1$ (or $\hat{e}_2$) is not invariant under the group $R(2)$.
Let

(3.2-2) $$\hat{e}_\pm = \frac{1}{\sqrt{2}} \left(\mp \hat{e}_1 + i\, \hat{e}_2 \right)$$

Using

$$U(\phi)\, \hat{e}_1 = \cos\phi\, \hat{e}_1 - \sin\phi\, \hat{e}_2$$
$$U(\phi)\, \hat{e}_2 = \sin\phi\, \hat{e}_1 + \cos\phi\, \hat{e}_2$$

we have

(3.2-3) $$\begin{aligned} U(\phi)\, \hat{e}_+ &= \frac{1}{\sqrt{2}} \left[(-\cos\phi + i \sin\phi)\, \hat{e}_1 + (\sin\phi + i\cos\phi)\, \hat{e}_2 \right] \\ &= \frac{1}{\sqrt{2}} \left[-e^{-i\phi}\, \hat{e}_1 + i\, e^{-i\phi}\, \hat{e}_2 \right] \\ &= \hat{e}_+\, e^{-i\phi} \\ U(\phi)\, \hat{e}_- &= \frac{1}{\sqrt{2}} \left[(\cos\phi + i\sin\phi)\, \hat{e}_1 + (-\sin\phi + i\cos\phi)\, \hat{e}_2 \right] \\ &= \frac{1}{\sqrt{2}} \left[e^{i\phi}\, \hat{e}_1 + i\, e^{i\phi}\, \hat{e}_2 \right] \\ &= \hat{e}_-\, e^{i\phi} \end{aligned}$$

Therefore, the 1-D spaces spanned by $\hat{e}_\pm$ are individually invariant under $R(2)$.
According to (3.2-3), we have

(3.2-4) $$D'(\phi) = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{+i\phi} \end{pmatrix}$$

The $D'(\phi)$ matrices can also be obtained from the $D(\phi)$ matrices of Eq. (3.1-6) by a similarity transformation S, which is just the transformation from the original basis $\left(\hat{e}_1 ,\ \hat{e}_2 \right)$ to the new basis $\left(\hat{e}_+ ,\ \hat{e}_- \right)$ given by Eq. (3.2-2), ie.

$$\hat{e}_+ = \hat{e}_1\, S_{1+} + \hat{e}_2\, S_{2+}$$
$$\hat{e}_- = \hat{e}_1\, S_{1-} + \hat{e}_2\, S_{2-}$$

so that

$$S = \begin{pmatrix} S_{1+} & S_{2+} \\ S_{1-} & S_{2-} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}$$

$$S^{-1} = S^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix}$$

$$D'(\phi) = S\, D(\phi)\, S^{-1}$$

Thus, using

$$D(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

we have

$$\begin{aligned} S\,D(\phi)\,S^{-1} &= \frac{1}{2}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -i & -i \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}\begin{pmatrix} -e^{-i\phi} & e^{i\phi} \\ -i\,e^{-i\phi} & -i\,e^{i\phi} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2\,e^{-i\phi} & 0 \\ 0 & 2\,e^{i\phi} \end{pmatrix} \\ &= D'(\phi) \end{aligned}$$

as advertised.

- Example 3:

The reader should be able to convince himself (after working out Problem 3.l) that the 2-D space E_2 is minimal with respect to the dihedral group D_3. Therefore, the 2-D rep of D_3 (hence S_3) described in Example 6 of Sec. 1 is an irreducible rep.

3.3 Unitary Representations

- Definition 3.6 (Unitary **Representations**):

If the group representation space is an inner product space (cf. Appendix II.5), and if the operators $U(g)$ are unitary for all $g \in G$, then the rep $U(G)$ is said to be a unitary representation.

- Theorem 3.2:

If a unitary rep is reducible, then it is also decomposable (i.e. fully reducible).

- Proof:

Let $U(G)$ be a reducible unitary rep of G on the inner product space V.
Write $V = V_1 \oplus V_2$ where V_1 is the invariant subspace (of dimension n_1) with respect to $U(G)$ and V_2 its orthogonal complement.

Choose an orthonormal basis of V

$$\{\hat{e}_i\,,\ i = 1, 2, \ldots\}$$

such that

$$\begin{aligned} \hat{e}_i \in V_1 \text{for} \quad & i = 1, 2, \ldots, n_1 \\ \hat{e}_i \in V_2 \text{for} \quad & i = n_1 + 1, n_1 + 2, \ldots \end{aligned}$$

We need to prove that V_2 is also invariant with respect to $U(G)$. This can be established in two steps:

(i) Since V_1 is invariant,

$$| e_i(g) \rangle \equiv U(g) | e_i \rangle \in V_1 \qquad \text{for } i = 1, \ldots, n$$

(ii) Since $U(G)$ is unitary,

$$\begin{aligned} \langle e^j(g) | e_i(g) \rangle &= \langle e^j | U(g)^\dagger U(g) | e_i \rangle \\ &= \langle e^j | e_i \rangle \\ &= \delta_i^j \end{aligned}$$

In particular,

$$\langle e^j(g) | e_i(g) \rangle = 0 \qquad \text{for all } j = n_1 + 1, n_1 + 2, \ldots \quad \text{and } i = 1, 2, \ldots, n_1.$$

This means $\hat{e}_j(g)$ are in V_2, the orthogonal complement of V_1. Since any vector $x \in V_2$, is a linear combination of $\{ \hat{e}_j , \ j = n_1 + 1, \ldots \}$, $U(g) | x \rangle$ must also be in V_2. QED

- Theorem 3.3:

Every rep $D(G)$ of a finite group on an inner product space is equivalent to a unitary rep.

- **Proof:**

In order to prove this theorem, we need to find a non-singular operator S such that $S\, D(g)\, S^{-1} = U(g)$ is unitary for all $g \in G$. S can be chosen to be one of those operators which satisfy the following equation:

(3.3-1)

$$\begin{aligned} (x, y) &\equiv \langle S x | S y \rangle \\ &= \sum_g \langle D(g)\, x | D(g)\, y \rangle \qquad \forall\, x, y \in V \end{aligned}$$

The existence of S is easily established by noting that

(i) (x, y) satisfies the axioms of the definition for a new scalar product (Problem 3.4); and

(ii) S represents the transformation from a basis orthonormal with respect to the scalar product $\langle \ | \ \rangle$ another basis orthonormal with respect to the new scalar product (,).

To show that $U(g)$ is unitary for such a choice of S, note:

(3.3.-2)

$$\begin{aligned} \langle U(g)\, x | U(g)\, y \rangle &= \langle S\, D(g)\, S^{-1} x | S\, D(g)\, S^{-1} y \rangle \\ &= \langle S X | S Y \rangle \qquad [\, X = D(g)\, S^{-1} x,\ Y = D(g)\, S^{-1} y \,] \\ &= \sum_{g'} \langle D(g')\, X | D(g')\, Y \rangle \\ &= \sum_{g'} \langle D(g')\, D(g)\, S^{-1} x | D(g')\, D(g)\, S^{-1} y \rangle \\ &= \sum_{g'} \langle D(g' g)\, S^{-1} x | D(g' g)\, S^{-1} y \rangle \\ &= \sum_{g''} \langle D(g'')\, S^{-1} x | D(g'')\, S^{-1} y \rangle \qquad [\, g'' = g' g \,] \\ &= (S^{-1} x, S^{-1} y) \\ &= \langle S(S^{-1} x) | S(S^{-1} y) \rangle \\ &= \langle x | y \rangle \end{aligned}$$

Since Eq. (3.3-2) holds for all $x, y \in V$ and $g \in G$, the rep $U(G)$ is indeed a unitary one.

Note that the change of summation index from g' to g'' is allowed because of the rearrangement theorem.

Alternative Proof:

See M.Tinkham, "Group Theory & Quantum Mechanics", §3.2, McGraw Hill (1964).

Let

$$H = \sum_g D(g)\, D(g)^\dagger$$

Obviously, $H^\dagger = H$.
Therefore, there exists a unitary U such that

$$U^\dagger H\, U = \text{diag}\,(\,\lambda_i\,) \equiv d$$

where λ_i are the (real) eigenvalues of H.

Hence

$$\begin{aligned} d &= \sum_g U^\dagger D(g)\, D(g)^\dagger\, U \\ &= \sum_g U^\dagger D(g)\, U\, U^\dagger D(g)^\dagger\, U \\ &= \sum_g D'(g)\, D'(g)^\dagger \qquad [\, D' = U^\dagger D\, U\,] \end{aligned}$$

Now, λ_i are all real and positive ($\lambda_i > 0$) [see next section for proof].
Therefore,

$$d^{1/2} = \text{diag}\left(\sqrt{\lambda_i}\right)$$

$$d^{-1/2} = \text{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right)$$

both exist and are real.

We shall show that the matrices

$$\begin{aligned} D''(g) &= d^{-1/2}\, D'(g)\, d^{1/2} \\ &= d^{-1/2}\, U^\dagger D(g)\, U\, d^{1/2} \end{aligned}$$

are unitary so that the similarity transform by

$$S = d^{-1/2}\, U^\dagger \qquad S^{-1} = U\, d^{1/2}$$

will make $D(g)$ unitary.

Since $d^{1/2}$ and $d^{-1/2}$ are real & diagonal, we have

$$D''(g)^\dagger = d^{1/2}\, D'(g)^\dagger\, d^{-1/2}$$

so that

$$\begin{aligned} D''(g)\, D''(g)^\dagger &= d^{-1/2}\, D'(g)\, d^{1/2}\, d^{1/2}\, D'(g)^\dagger\, d^{-1/2} \\ &= d^{-1/2}\, D'(g)\, d\, D'(g)^\dagger\, d^{-1/2} \\ &= d^{-1/2}\, D'(g) \sum_{g'} D'(g')\, D'(g')^\dagger\, D'(g)^\dagger\, d^{-1/2} \\ &= d^{-1/2} \sum_{g'} D'(g)\, D'(g')\, [D'(g)\, D'(g')]^\dagger\, d^{-1/2} \\ &= d^{-1/2} \sum_{g'} D'(g\, g')\, D'(g\, g')^\dagger\, d^{-1/2} \\ &= d^{-1/2} \sum_{g''} D'(g'')\, D'(\,g'')^\dagger\, d^{-1/2} \qquad [\, g'' = g\, g'\,] \\ &= d^{-1/2}\, d\, d^{-1/2} \\ &= \text{I} \qquad \text{QED} \end{aligned}$$

Note that the change of summation index from g' to g'' is allowed because of the rearrangement theorem.

Proof of $\lambda_i > 0$

Consider 1st a hermitian matrix

$$h = A^\dagger A$$

where A is arbitrary.

Hence, the eigenvalues l_i of h are all real.

Furthermore, for any $\boldsymbol{x} \in V$,

$$\begin{aligned} \boldsymbol{x}^\dagger h \boldsymbol{x} &= \boldsymbol{x}^\dagger A^\dagger A \boldsymbol{x} \\ &= (A\boldsymbol{x})^\dagger (A\boldsymbol{x}) \\ &= \boldsymbol{y}^\dagger \boldsymbol{y} \qquad \qquad \boldsymbol{y} = A\boldsymbol{x} \in V \\ &\geq 0 \end{aligned}$$

This means h is non-negative and its eigenvalues are all non-negative ($l_i \geq 0$).
In other words, if $\boldsymbol{x}_i$ is an eigenvector of h with eigenvalue l_i, we have

$$\boldsymbol{x}_i^\dagger h \boldsymbol{x}_i = l_i \boldsymbol{x}_i^\dagger \boldsymbol{x}_i \geq 0 \quad \longrightarrow \quad l_i \geq 0 \text{ since } \boldsymbol{x}_i^\dagger \boldsymbol{x}_i > 0.$$

Now, if A is invertible, we have $\det A \neq 0$ and

$$\begin{aligned} \det\left(A^\dagger A\right) &= \det A^\dagger \cdot \det A \\ &= (\det A)^* \det A \\ &= |\det A|^2 \\ &\neq 0 \end{aligned}$$

Hence, $l_i > 0$.

Applying the same maneuver to the sum

$$H = \sum_i h_i = \sum_i A_i^\dagger A_i$$

we have

$$\boldsymbol{x}^\dagger H \boldsymbol{x} = \sum_i \boldsymbol{x}^\dagger A_i^\dagger A_i \boldsymbol{x} \geq 0$$

so that its eigenvalues $\lambda_i > 0$ as advertised.

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Although we restricted the statement of the Theorem 3.3 to finite groups, the proof suggests that it remains valid for any group provided the summation over group elements in Eq. (3.3-1) can be properly defined and such that the Rearrangement Lemma holds.

Examples of continuous groups (which are necessarily infinite with the required property which occur in physical applications are rotation groups in E^n, the unitary group $U(n)$, and the special unitar groups SU(n).

Collaries

All reducible reps of finite groups are fully reducible.

Let V_1 and V_2 be complementary invariant subspaces wrt $U(G)$.
$U_1(G)$, $U_2(G)$ denote the operator which coincide with $U(G)$ on these subspaces.
Then

$V = V_1 \oplus V_2$, in the sense of vector spaces,

$U(g) = U_1(g) \oplus U_2(g)$ in the sense of operators (see Appendix II) for all $g \in G$.

■ Definition 3.7 (Direct Sum Representations):

Given the above situation, the rep $U(G)$ is said to be the direct sum rep of $U_1(G)$ (on V_1) and $U_2(G)$ (on V_2).

In general
(3.3-3)

$$U(G) = \underbrace{U^1(G) \oplus \ldots \oplus U^1(G)}_{a_1 \text{ terms}} \underbrace{U^2(G) \oplus \ldots \oplus U^2(G)}_{a_2 \text{ terms}} \oplus \ldots$$

$$= \sum_{\mu \oplus} a_\mu U^\mu(G)$$

where $U^\mu(G)$ denotes inequivalent IRs labelled by μ.

With the proper choice of bases, $U(G)$ will appear in block-diagonal form with $U^\mu(G)$ appearing as diagonal blocks.

3.4 Schur's Lemma

■ Schur's Lemma 1:

Let $U(G)$ be an IR of G on V, and A an arbitrary operator on V.
If

$$A\,U(g) = U(g)\,A \qquad \forall\; g \in G$$

then

$$A = \lambda E$$

where λ is a number.

■ Proof:

(i) Without loss of generality, we can take $U(G)$ to be unitary and A to be hermitian.
If $U(G)$ is not unitary, we can make it into one by a similarity transformation (Theorem 3.3).
If A is not hermitian, we can always decompose it into two hermitian operators

$$A_+ = \left(A + A^\dagger\right)/2$$

$$A_- = \left(A - A^\dagger\right)/2\,i$$

and consider these separately before combining them again in

$$A\,(= \;A_+ + i\,A_-)$$

(ii) A basis of V, $\{\hat{u}_{\alpha,i}\}$, can be chosen to consist of the eigenvectors of A, i.e.

$$A \left| u_{\alpha,i} \right\rangle = \left| u_{\alpha,i} \right\rangle \lambda_i$$

where λ_i are the eigenvalues of A,
α represents additional labels needed to specify the basis vectors fully.
The set $\{\hat{u}_{\alpha,i}\}$ may be chosen to be orthonormal;

(iii) For any given i, denote by V^i the subspace spanned by

$$\{\hat{u}_{\alpha,i}\;;\;\alpha = 1, 2, \ldots\}$$

Now,

$$A\,U(g) \left| u_{\alpha,i} \right\rangle = U(g)\,A \left| u_{\alpha,i} \right\rangle = U(g) \left| u_{\alpha,i} \right\rangle \lambda_i$$

Therefore

$$\left| U(g)\,u_{\alpha,i} \right\rangle \in V^i$$

That is, V^i are invariant subspaces with respect to $U(G)$.

(iv) But $U(G)$ is irreducible on V.
That is, V does not contain any nontrivial invariant subspace (Definition 3.5).
Consequently, the invariant subspace V^i must coincide with V itself.
i.e. $V^i = V$.
It follows that A has only one eigenvalue, and $A = \lambda E$. QED

Theorem 3.4:

- IRs of any abelian group must be 1-D.

- Proof:

Let $U(G)$ be an IR of the abelian group G.
Denote by p a fixed element of G.
Due to the abelian nature of the group, we have

$$U(p)\,U(g) = U(g)\,U(p) \qquad \forall g \in G.$$

According to Schur's Lemma,

$$U(p) = \lambda_p E.$$

This applies to all $p \in G$. Hence, the rep $U(G)$ is equivalent to the 1-D rep

$$p \to \lambda_p \in C \qquad \text{for all } p \in G. \qquad \text{QED}$$

- Schur's Lemma 2:

Let $U(G)$ and $U'(G)$ be two IRs of G on the vector spaces V and V' respectively.
Let A be a linear transformation from V' to V which satisfies

$$A\,U'(g) = U(g)\,A \qquad \forall g \in G.$$

It follows then, either
(i) $A = 0$,
or
(ii) V and V' are isomorphic and $U(G)$ is equivalent to $U'(G)$.

Graphically, we have,

$$\begin{array}{ccc} V & \xrightarrow{U} & V \\ A\uparrow & & \uparrow A \\ V' & \xrightarrow[U']{} & V' \end{array}$$

$$\begin{array}{ccc} \boldsymbol{x} = A\,\boldsymbol{x}' & \overset{U}{\mapsto} & U(g)\,\boldsymbol{x} = U(g)\,A\,\boldsymbol{x}' \overset{?}{=} A\,U'(g)\,\boldsymbol{x}' \\ A\uparrow & & \uparrow A \\ \boldsymbol{x}' & \underset{U'}{\mapsto} & U'(g)\,\boldsymbol{x}' \end{array}$$

The lemma says that if $\overset{?}{=}$ is true, then either trivially $A = 0$ or $U(G)$ and $U'(G)$ are equivalent.

- Proof:

(i) Denote by R the range of A, i.e.

$$R = \{\mathbf{x} \in V\,;\ \mathbf{x} = A\mathbf{x}' \text{ for some } \mathbf{x}' \in V'\}. \qquad \text{(cf. Fig. 3.2b)}$$

Now,

$$U(g)\,|\,\mathbf{x}\,\rangle = U(g)\,A\,|\,\mathbf{x}'\,\rangle \qquad \forall\ |\,\mathbf{x}\,\rangle \in V,\ g \in G$$
$$= A\,U'(g)\,|\,\mathbf{x}'\rangle$$
$$= A\,|\,U'(g)\,\mathbf{x}'\rangle \ \in R$$

Therefore, R is an invariant subspace of V with respect to $U(G)$.

But if $U(G)$ is an IR, we must have either

$R = 0$ (hence $A = 0$)

or $R = V$ (i.e, the mapping A is "onto")

[cf Definition 3.5].

(ii) Now, consider (in V') the null space N' of A:

$$N' = \{\mathbf{x}' \in V'\,;\ A\mathbf{x}' = 0\}. \qquad \text{(see Fig. 3.2c)}$$

Since

$$A\,U'(g)\,|\,\mathbf{x}'\,\rangle = U(g)\,A\,|\,\mathbf{x}'\,\rangle \qquad \forall\ |\,\mathbf{x}'\rangle \in N',\ g \in G$$
$$= U(g)\,|\,0\,\rangle$$
$$= 0$$

Hence

$$U'(g)\,|\,x'\,\rangle \in N' \qquad \forall\ g \in G.$$

ie., N' is an invariant subspace of V' with respect to $U'(G)$.

Since $U'(G)$ is irreducible, we must have either

$N' = V'$ (hence $A = 0$)

or $N' = 0$.

In the second case,

$$A\,|\,\mathbf{x}'\,\rangle = A\,|\,\mathbf{y}'\,\rangle \quad \text{implies} \quad |\,\mathbf{x}'\,\rangle = |\,\mathbf{y}'\,\rangle.$$

Hence the mapping A is one-to-one.

Combining (i) and (ii), we see that either

$$A = 0$$

or it establishes an isomorphism between V and V'.

In the latter case, we have also

$U(G) = A\,U'(G)\,A^{-1}$. QED

3.5 Orthonormality and Completeness Relations of Irreducible Representation Matrices

■ Notation

n_G order of G;
μ, ν labels for inequivalent, IRs of G;
n_μ the dimension of the μ-rep;
$D^\mu(g)$ the matrix corresponding to $g \in G$ in the μ-rep wrt an orthonormal basis;
$\chi^\mu{}_i$ character of class ζ_i elements in the μ-rep [Tung's];
$\chi^\mu(\zeta)$ character of class ζ elements in the μ-rep [Mine];
n_i number of elements in class ζ_i ($i = 1, 2, \ldots, n_c$) [Tung's];
n_ζ number of elements in class ζ ($\zeta = 1, 2, \ldots, n_c$) [Mine];
n_c number of classes in G.

■ Theorem 3.5 (Orthonormality of IR Matrices):

With the symbols defined above, the following orthonormality condition holds,

(3.5-1) $$\frac{n_\mu}{n_G} \sum_g D_\mu{}^{-1}(g)^k{}_i \, D^\nu(g)^j{}_l = \delta_\mu{}^\nu \, \delta_i{}^j \, \delta_l{}^k$$

which for unitary IRs become

$$\frac{n_\mu}{n_G} \sum_g D_\mu{}^\dagger(g)^k{}_i \, D^\nu(g)^j{}_l = \delta_\mu{}^\nu \, \delta_i{}^j \, \delta_l{}^k$$

where $D_\mu{}^\dagger(g)^k{}_i = \left[D^\mu(g)^i{}_k\right]^*$. [see Appendix I.]

Note: this applies only to IRs.

For fixed (ν, j, l), we regard $D^\nu(g)^j{}_l \sqrt{n_\nu / n_G}$ as a n_G component vector (with g ranging over G), then this equation is just the usual orthonormality relation for vectors.

For abelian groups, we have,

(3.5-2) $$\frac{1}{n_G} \sum_g d_\mu{}^\dagger(g) \, d^\nu(g) = \delta_\mu{}^\nu$$

where $d^\nu(g)$ are c-numbers.

■ Proof:

(i) Let X be any $n_\mu \times n_\nu$ matrix and define

(3.5-3) $$M_X = \sum_g D_\mu{}^\dagger(g) \, X \, D^\nu(g) \qquad \left[D_\mu{}^\dagger(g) = D_\mu{}^{-1}(g)\right]$$

Then,

$$D_\mu{}^{-1}(p)\, M_X\, D^\nu(p) = \sum_g D_\mu{}^\dagger(p)\, D_\mu{}^\dagger(g)\, X\, D^\nu(g)\, D^\nu(p)$$

$$= \sum_g \left[D_\mu(g)\, D_\mu(p)\right]^\dagger X\, D^\nu(g\, p)$$

$$= \sum_g D_\mu{}^\dagger(g\, p)\, X\, D^\nu(g\, p)$$

$$= \sum_{g'} D_\mu{}^\dagger(g')\, X\, D^\nu(g') \qquad [g' = g\, p]$$

$$= \mathrm{M}_X$$

ie. $M_X\, D^\nu(p) = D_\mu(p)\, M_X \qquad \forall\ p \in G$

According to the second Schur's Lemma,
either $\mu \neq \nu$ and $M_X = 0$,
or $\mu = \nu$ and $M_X = c_X\, E$
where c_x is a constant and E is the unit matrix.

(ii) Choose X to be one of the $n_\nu\, n_\mu$ matrices X_l^k with matrix elements

$$\left(X_l^k\right)^i{}_j = \delta_j^k\, \delta_l^i \qquad \left(k = 1, \ldots, n_\nu;\ l = 1, \ldots, n_\mu\right)$$

We obtain

$$\left(M_l^k\right)^m{}_n = \sum_g D_\mu{}^\dagger(g)^m{}_i \left(X_l^k\right)^i{}_j\, D^\nu(g)^j{}_n$$

$$= \sum_g D_\mu{}^\dagger(g)^m{}_l\, D^\nu(g)^k{}_n$$

According to (i), $M_X = 0$ if $\mu \neq \nu$. Hence

$$\sum_g D_\mu{}^\dagger(g)^m{}_l\, D^\nu(g)^k{}_n \propto \delta_\mu^\nu$$

For $\mu = \nu$,

$$M_l^k = c_l^k\, E$$

or

$$\left(M_l^k\right)^m{}_n = c_l^k\, \delta_n^m$$

so that

$$\sum_g D_\mu{}^\dagger(g)^m{}_l\, D^\mu(g)^k{}_n = c_l^k\, \delta_n^m$$

To determine c_l^k, we take the trace of both sides:

$$\sum_g D_\mu{}^\dagger(g)^m{}_l\, D^\mu(g)^k{}_m = c_l^k\, \delta_m^m$$

whereupon

$$\text{LHS} = \sum_g \left(D^{\mu}(g)\, D_{\mu}{}^{\dagger}(g) \right)^k_l$$
$$= \sum_g \delta^k_l$$
$$= n_G\, \delta^k_l$$
$$\text{RHS} = n_\mu\, c^k_l$$

Therefore,

$$c^k_l = \frac{n_G}{n_\mu}\, \delta^k_l$$

so that

$$\sum_g D_\mu{}^{\dagger}(g)^m{}_l\; D^\mu(g)^k{}_n = \frac{n_G}{n_\mu}\, \delta^k_l\, \delta^m_n$$

and, incorporating the $\mu \neq \nu$ case,

$$\sum_g D_\mu{}^{\dagger}(g)^m{}_l\; D^\nu(g)^k{}_n = \frac{n_G}{n_\mu}\, \delta^k_l\, \delta^m_n\, \delta^\nu_\mu$$

■ Example 1:

Consider group C_2 (cf. Table 2.1).
The identity rep d_1 is given by

$$(e,\, a) \xrightarrow{d_1} (1,\, 1).$$

which stands for:

$d_1(e) = 1$ and $d_1(a) = 1$.

This short-hand notation is natural since we want to regard $d_1(e)$ and $d_1(a)$ as two components of a "vector".

A second, inequivalent IR d_2 must be orthonormal to $(1,\, 1)$.
The only two- component vector which is orthogonal to $(1,\, 1)$ and also properly normalized is $(1,\, -1)$.

Therefore, the only candidate for a second IR is

$$(e,\, a) \xrightarrow{d_2} (1,\, -1).$$

It is easily verified that this is indeed an IR. There is no other IR of C_2. We can summarize the results in Table 3.1.

$\mu \setminus g$	e	a
1	1	1
2	1	−1

Table 3.1 Inequivalent IRs of C_2.

■ ✸

Thus, theorem 3.5 can be used to deduce new, inequivalent IRs of a group from knowledge of the simpler ones − by the requirement of orthonormality. When combined with other handles on the simpler reps (such as Theorem 3.1), we can obtain a wealth of information about IRs of the simpler groups.

■ Example 2:

Consider the dihedral group D_2 [cf. Table 2.3].

The trivial IR is

$$\{e, a, b, c\} \longrightarrow \{1, 1, 1, 1\}.$$

The elements $\{e, a\}$ form an invariant subgroup.

The factor group consists of $\{(e, a), (b, c)\}$ and is isomorphic to C_2. (There is only one group of order 2.)

According to Example 1, this factor group has two inequivalent IRs.
Using Theorem 3.1, we obtain two IRs of the full group D_2.
The first one is again the identity rep.
The second one is

$$\{e, a, b, c\} \xrightarrow{d_2} \{1, 1, -1, -1\}.$$

Applying the same procedure to the invariant subgroups with elements (e, b), and (e, c), we obtain two more irreducible reps:

$$\{e, a, b, c\} \xrightarrow{d_3} \{1, -1, 1, -1\}$$

and $$\xrightarrow{d_4} \{1, -1, -1, 1\}.$$

It is easily seen that these rep vectors do satisfy the required orthonormality condition, and no other IR is allowed by that condition. These reps are summarized in Table 3.2.

$\mu\backslash g$	e	a	b	c
1	1	1	1	1
2	1	1	−1	−1
3	1	−1	1	−1
4	1	−1	−1	1

Table 3.2 Inequivalent IRs of D_2.

- Example 3:

We have already seen (in Chap. l) the orthonormality relation at work in the case of the infinite abelian group $T^d = \{ T(n), n = 1, 2, \ldots \}$ consisting of all discrete translations. The reps are given by $T(n) \to e^{-i n \xi}$, where ξ serves as the rep label. Eq. (1.4-2), then, is the expression of the orthonormality relation (3.5-2) with the correspondence $\mu, \nu \to \xi, \xi'$ and $g \to n$.

We pointed out in Chap. 1 that this relation coincides with the classical Fourier Theorem for periodic functions. Theorem 3.5 represents a significant extension of that important and powerful result. We shall emphasize this central result a number of times throughout this book.

- Corollary 1:

The number of inequivalent IRs of a finite group is restricted by the condition:

$$\sum_\mu {n_\mu}^2 \le n_G.$$

■ Proof:

As mentioned earlier, we can regard $D^\mu(g)^i{}_j$, $g \in G$ as the n_G components of a set of orthogonal "vectors" labelled by (μ, i, j). Since the labels (i, j) take $n_\mu{}^2$ values, $\sum_\mu n_\mu{}^2$ represents the number of "vectors" in this set, whereas n_G is the number of components of each such vector. The inequality then follows from the well-known fact that the number of mutually orthogonal (hence linearly independent) vectors must be less than the dimension of the vector space (which is just n_G).
QED

It is this important result which makes it possible to set the main objective for group representation theory as that of finding all the possible inequivalent IRs for the groups of interest. The most remarkable fact is, in fact, that the above inequality is always saturated: i.e. $\sum_\mu n_\mu{}^2 = \mathrm{n}_G$. Thus, the n_G-component vectors $D^\mu(g)^i{}_j$ labelled by (μ, i, j) are a complete set in addition to being orthogonal.

■ Theorem 3.6 (Completeness of IR Matrices):

(i) The dimensionality parameters $\{n_\mu\}$ for the inequivalent IRs satisfy:

(3.5-4) $$\sum_\mu n_\mu{}^2 = \mathrm{n}_G$$

(ii) The corresponding rep matrices satisfy the Completeness Relation:

(3.5-5) $$\frac{n_\mu}{n_G} D^\mu(g)^l{}_k \, D_\mu{}^\dagger(g')^k{}_l = \delta_{g g'}$$

The proof of Eq. (3.5-4) will be deferred until we introduce the notion of the regular rep later in this chapter (See Theorem 3.1 1). If this result is accepted, then the set of orthonormal vectors labelled by (μ, l, k) and with components equal to $D^\mu(g)^l{}_k$ must be complete –because the number of independent vectors in the set is equal to the dimension of the vector space. The completeness relation, according to Theorem 11.13, can be written in the form of Eq. (3.5-5).

■ ✸

For abelian groups, all IRs are 1-D, hence $n_\mu = 1$ for all μ.
Thus, there are n_G inequivalent IRs.

This result can also be generalized to infinite groups if the required summation can be meaningfully carried out. We again mention the example of the discrete transformation group T_d. Eq.(1.4-1) is the expression of completeness for the rep functions for that particular case.

The orthonormality and completeness relations for $D^\mu(g)^i{}_j$ are very important theoretical results. However, the rep matrices themselves are basis dependent, and hence bring in much detail which is not intrinsic to the IR $(U^\mu(G))$ itself. For this reason, it is much more fruitful to move on to the corresponding relations concerning group characters, which are basis independent.

■ C_{3v}

A 2-D IR can be obtained for $V = E_2$.

The configuration is as shown below:

3

$\hat{y}$
$\uparrow$
$o \longrightarrow \hat{x}$

1 2

where o is center of gravity of the equilateral triangle 123.

It is easily verified that

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D(\sigma_1) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad D(\sigma_2) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad D(\sigma_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D(C_3) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad D\left(C_3{}^2\right) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

For example,

$$U(\sigma_1)\,\hat{x} = \hat{x}\ D(\sigma_1)^1{}_1 + \hat{y}\ D(\sigma_1)^2{}_1$$
$$= \frac{1}{2}\,\hat{x} + \frac{\sqrt{3}}{2}\,\hat{y}$$
$$U(\sigma_1)\,\hat{y} = \hat{x}\ D(\sigma_1)^1{}_2 + \hat{y}\ D(\sigma_1)^2{}_2$$
$$= \frac{\sqrt{3}}{2}\,\hat{x} - \frac{1}{2}\,\hat{y}$$

We can construct a representation table for C_{3v}:

C_{3v}	e	σ_1	σ_2	σ_3	C_3	$C_3{}^2$
D^1	1	1	1	1	1	1
D^2	1	−1	−1	−1	1	1
$[D^3]^1{}_1$	1	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	$-\frac{1}{2}$
$[D^3]^1{}_2$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$[D^3]^2{}_1$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$[D^3]^2{}_2$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$

where D^1 & D^2 are 1-Dim IRs.

To check the orthogonality & completeness theorems, we convert the table to

C_{3v}	e	σ_1	σ_2	σ_3	C_3	$C_3{}^2$
$D^{-1}{}_1$	1	1	1	1	1	1
$D^{-1}{}_2$	1	−1	−1	−1	1	1
$[D^{-1}{}_3]^1{}_1$	1	$\frac{1}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	$-\frac{1}{2}$
$[D^{-1}{}_3]^2{}_1$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$[D^{-1}{}_3]^1{}_2$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$[D^{-1}{}_3]^2{}_2$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$

Note that the rows are listed according to $[\mathrm{D}^3]^{\mathrm{i}}{}_{\mathrm{j}}$ and $[\mathrm{D}^{-1}{}_3]^{\mathrm{j}}{}_{\mathrm{i}}$ so that the *m*th row of each table is dual to the *m*th row of the other table.

For example, the 4th row gives

$$\sum_g D_3{}^{-1}(g)^2{}_1\, D^3(g)^1{}_2$$
$$= 0\cdot 0 + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right) + 0\cdot 0 + \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}$$
$$= 3$$
$$= \frac{n_G}{n_\mu} = \frac{6}{2} = 3$$

Similarly, the 2nd column gives

$$\frac{n_\mu}{n_G} D^\mu(\sigma_1)^l{}_k\, D_\mu{}^{-1}(\sigma_1)^k{}_l$$
$$= \frac{1}{6}\left\{1\cdot 1\cdot 1 + 1\cdot(-1)\cdot(-1) + 2\cdot\left[\frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)\right]\right\}$$
$$= 1$$

3.6 Orthonormality and Completeness Relations of Irreducible Characters [Tung's Notation]

Recall that the characters $\chi(G) \equiv \mathrm{Tr}\, U(G)$ are independent of the choice of basis.
All group elements in a given class have the same character in a given rep.

- Lemma:

Let $U^\mu(G)$ be an IR of G, and ζ_i be any class,

(3.6-1) $$\sum_{h\in\zeta_i} U^\mu(h) = \frac{n_i}{n_\mu}\chi_i^\mu\, E$$

- Proof:

Let

$$A_i = \sum_{h \in \zeta_i} U^\mu(h)$$

By definition, if $h \in \zeta_i$, we have

$$g\, h\, g^{-1} \in \zeta_i \qquad \forall\; g \in G$$

Therefore

$$U^\mu(g)\, A_i\, U^\mu(g)^{-1} = \sum_{h \in \zeta_i} U^\mu(g)\, U^\mu(h)\, U^\mu(g)^{-1}$$

$$= \sum_{h' \in \zeta_i} U^\mu(h') \qquad [\, h' = g\, h\, g^{-1} \in \zeta_i \,]$$

$$= \mathrm{A}_i$$

Hence

$$A_i = c_i\, E \qquad \text{[Schur's Lemma 1]}$$

To evaluate c_i, we take the trace on both sides

$$\mathrm{Tr}\, A_i = c_i\, \mathrm{Tr}\, E$$

Hence

$$\mathrm{LHS} = \sum_{h \in \zeta_i} \mathrm{Tr}\, U^\mu(h)$$

$$= \sum_{h \in \zeta_i} \chi_i^\mu$$

$$= n_i\, \chi_i^\mu \qquad (\text{ no sum over } i\,)$$

$$\mathrm{RHS} = c_i\, n_\mu$$

so that

$$c_i = \frac{n_i}{n_\mu}\, \chi_i^\mu \qquad \text{QED}$$

- Theorem 3.7
  (Orthonormality and Completeness of Group Characters):

The characters of inequivalent IRs of G satisfy the following relations:

(3.6-2) $$\frac{n_i}{n_G}\, \chi_\mu{}^{\dagger\, i}\, \chi^\nu{}_i = \delta_\mu{}^\nu \qquad (\text{ orthonormality })$$

(3.6-3) $$\frac{n_i}{n_G}\, \chi^\mu{}_i\, \chi_\mu{}^{\dagger\, j} = \delta^j{}_i \qquad (\text{ no sum over } i\,)\ (\text{ completeness })$$

where, by convention, $\chi_\mu{}^{\dagger\, i} = \left(\chi^\mu{}_i\right)^*$. The summation on i is over the distinct classes of the group; and the summation on μ is over all the inequivalent IRs.

- Proof:

(i) We start from Theorem 3.5.

$$\frac{n_\mu}{n_G} \sum_g D_\mu{}^\dagger(g)^k{}_i \; D^\nu(g)^j{}_l = \delta_\mu{}^\nu \, \delta_i{}^j \; \delta_l{}^k$$

Setting $i = k$, $j = l$, and summing over both indices, we have

$$\frac{n_\mu}{n_G} \sum_g D_\mu{}^\dagger(g)^i{}_i \; D^\nu(g)^j{}_j = \delta_\mu{}^\nu \, \delta_i{}^j \; \delta_j{}^i$$

whereupon

$$\text{LHS} = \frac{n_\mu}{n_G} \sum_g \chi_\mu{}^\dagger(g) \, \chi^\nu(g)$$

$$= \frac{n_\mu}{n_G} n_i \, \chi_\mu{}^{\dagger\, i} \, \chi^\nu{}_i$$

and

$$\text{RHS} = \delta_\mu{}^\nu \, \delta_i{}^i$$

$$= \delta_\mu{}^\nu \, n_\mu$$

Hence

$$\frac{n_i}{n_G} \chi_\mu{}^{\dagger\, i} \, \chi^\nu{}_i = \delta_\mu{}^\nu \qquad \text{(orthonormality)}$$

(ii) Turning to Theorem 3.6.

$$\frac{n_\mu}{n_G} D^\mu(g)^l{}_k \, D_\mu{}^\dagger(g')^k{}_l = \delta_{g\, g'}$$

Summing g over elements of the class ζ_i, and g' over the class ζ_j , we have

$$\text{LHS} = \frac{n_\mu}{n_G} \sum_{g \in \zeta_i} D^\mu(g)^l{}_k \sum_{g' \in \zeta_j} D_\mu{}^\dagger(g')^k{}_l$$

$$= \frac{n_\mu}{n_G} \frac{n_i}{n_\mu} \chi_i^\mu \, \delta_k^l \, \frac{n^j}{n_\mu} \chi_\mu{}^{\dagger\, j} \, \delta_l^k \qquad \text{[lemma]}$$

$$= \frac{n_i \, n^j}{n_G \, n_\mu} \chi_i^\mu \, \chi_\mu{}^{\dagger\, j} \, \delta_l^l$$

$$= \frac{n_i \, n^j}{n_G} \chi_i^\mu \, \chi_\mu{}^{\dagger\, j} \qquad \text{(no sum over } i, \; j \text{)}$$

Note that we write n^j instead of n_j simply to indicate the fact that there is no summation j.

$$\text{RHS} = \sum_{g \in \zeta_i} \sum_{g' \in \zeta_j} \delta_{g\, g'}$$

$$= \sum_{g \in \zeta_i} \delta_i^j \qquad \text{(elements of different classes are distinct)}$$

$$= n_i \, \delta_i^j \qquad \text{(no sum over } i \text{)}$$

Hence

$$\frac{n_i}{n_G} \chi^\mu{}_i \, \chi_\mu{}^{\dagger\, j} = \delta^j{}_i \qquad \text{(no sum over } i \text{) (completeness)}$$

■ ✸

The orthonormality and completeness relations become even more transparent if we define normalized characters,

(3.6-4) $$\tilde{\chi}_i \equiv \sqrt{\frac{n_i}{n_G}}\ \chi_i$$

Using implicit summation convention [cf. Appendix I], Theorem 3.7 appears simply as:

(3.6-5) $$\tilde{\chi}_\mu{}^{\dagger i}\, \tilde{\chi}^\nu{}_i = \delta^\nu{}_\mu$$

(3.6-6) $$\tilde{\chi}^\mu{}_i\, \tilde{\chi}_\mu{}^{\dagger j} = \delta^j{}_i$$

(Compare with Theorem II.I3.)

If $\left\{ \tilde{\chi}_i\,,\ i = 1,\ 2,\ \ldots,\ n_c \right\}$ are interpreted as components of a vector $\tilde{\chi}$, Eq. (3.6-5) can be further simplified to $\tilde{\chi}_\mu{}^\dagger \cdot \tilde{\chi}^\nu = \delta^\nu{}_\mu$, where · indicates that a "scalar product" in the n_c-dimensional vector space is taken. These notational conveniences will prove useful in some of the subsequent discussions.

An important consequence of Theorem 3.7 is the following corollary.

■ Corollary:

The number of inequivalent IRs for any finite group G is equal to the number of distinct classes of G (i.e. n_c). In other words, $\chi^\mu{}_i$ is a $n_c \times n_c$ square matrix if we designate μ as the row index and i as the column index. A table of this matrix for any given G is called the character table.

■ Example:

For abelian groups, each group element forms a class by itself and all IRs are 1-D, hence $D^\mu(g) = \chi^\mu{}_i$. Therefore, tables of $D^\mu(g)$ (such as Tables 3.1 and 3.2) are also character tables for the corresponding group (C_2 and D_2 respectively).

In general, because the characters $\chi^\mu{}_i$ are much simpler than the rep matrices $D^\mu(g)^k{}_l$, and because they are intrinsic to the reps (i.e. independent of the arbitrary choice of basis), they are much more useful in the studying of IRs of groups.

■ Example:

Let us find all the IRs of the non- abelian group S_3.

(i) S_3 has three classes:

the 1-cycle	$\{e\}$,
the 2-cycles	{ (12), (23), (31) },
and the 3-cycles	{ (123), (321) }.

Hence, there are three inequivalent IRs.

(ii) We know, a priori, that there is the trivial (identity) rep $p \to E$ for all $p \in S_3$.
Let us label it as $\mu = 1$.
The three characters are (1, 1, 1).

(iii) From the example following Theorem 3.1, we know of a second 1-D IR of S_3 which has the characters (1, −1, 1) for the 1-, 2-, and 3-cycles respectively. We label this rep by $\mu = 2$.

(iv) The last IR ($\mu = 3$) must be 2-D, since Theorem 3.6 requires

$$n_G = 6 = 1 + 1 + n_3^2.$$

The three characters $\left(\chi_i^3,\ i = 1, 2, 3 \right)$ can be determined as follows:

(a) $\chi_1^3 = \operatorname{Tr} D(e) = \operatorname{Tr} E = 2;$

(b) from the orthonormality and completeness relations, one can deduce that $\chi_2^3 = 0$, and $\chi_3^3 = -1$.

The character table for S_3 is summarized in the following:

$\mu \backslash i$	1	2	3
1	1	1	1
2	1	−1	1
3	2	0	−1

Table 3.3 Character Table of the Group S_3

■ ✸

Note that for any rep, the character for the identity element (say, $i = 1$) is equal to the dimension of the rep. Hence the first column of the character table reveals the dimension of all the IRs. The most important uses of the character table for practical applications, however, are derived from the following Theorems.

■ Theorem 3.8:

In the reduction of a given rep $U(G)$ of G into its irreducible components, the number of times (a_ν) that the IR $U^\nu(G)$ occurs [cf Eq. (3.3-3)] can be determined from the formula:

(3.6-7) $$a_\nu = \sum_i \frac{n_i}{n_G} \chi_\nu^{\dagger i} \chi_i = \tilde{\chi}_\nu^\dagger \cdot \tilde{\chi}$$

■ Proof:

In terms of its irreducible components, $U(G)$ can be written as [Eq. (3.3-3)]:

$$U(g) = \sum_{\mu \oplus} a_\mu U^\mu(g) \qquad \forall\ g \in G$$

Taking the trace on both sides gives

$$\chi(g) = a_\mu \chi^\mu(g) \qquad \text{(sum over } \mu \text{ implied)}$$

or, for $g \in \zeta_i$,

$$\chi_i = a_\mu \chi_i^\mu$$

so that

$$\sqrt{\frac{n_i}{n_G}}\ \chi_i \equiv \tilde{\chi}_i = a_\mu \tilde{\chi}_i^\mu$$

Using the orthonormality theorem, we have

$$\tilde{\chi}_\nu^\dagger \cdot \tilde{\chi}_i = a_\mu \tilde{\chi}_\nu^\dagger \cdot \tilde{\chi}_i^\mu$$

$$= a_\mu \delta_\nu^\mu$$

$$= \mathrm{a}_\nu \qquad \text{QED}$$

■ Example:

Consider the following matrix rep $D(G)$ or the group C_2 :

$$e \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad a \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(Verify that this is indeed a rep!)

The characters are $\chi_i = (2, 0)$.
The character table for the IRs is given by Table 3.1.
In fact,

$$\chi_i^{\mu=1} = (1, 1), \qquad \chi_i^{\mu=2} = (1, -1).$$

Thus, according to the above theorem, $a_1 = 1$ and $a_2 = 1$, i.e. each of the IRs occurs once in the reduction of the 2-D rep.

We leave as an exercise to prove that, through a similarity transformation, this rep can be brought to the fully reduced (diagonal) form

$$e \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad a \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{[Problem 3.6]}$$

This form explicitly shows that $D(G)$ is a direct sum of the two IRs.

■ Theorem 3.9 (Condition for Irreducibility):

A necessary and sufficient condition for a rep $U(G)$ with characters $\{ \chi_i \}$ to be irreducible is that

$$\sum_i n_i \, |\chi_i|^2 = n_G \qquad \text{ie} \qquad \tilde{\chi}^\dagger \cdot \tilde{\chi} = 1$$

■ Proof:

If a_μ denotes the number of times that the irreducible rep $U^\mu(G)$ is contained in $U(G)$, we have:

$$\begin{aligned} \tilde{\chi}^\dagger \cdot \tilde{\chi} &= \left(a_\mu \, \tilde{\chi}^\mu\right)^\dagger \cdot \left(a_\nu \, \tilde{\chi}^\nu\right) \\ &= a^{\mu *} a_\nu \, \tilde{\chi}_\mu^{\ \dagger} \cdot \tilde{\chi}^\nu \\ &= a^{\mu *} a_\nu \, \delta_\mu^{\ \nu} \\ &= a^{\mu *} a_\mu \\ &= \sum_\mu |a_\mu|^2 \end{aligned}$$

If $U(G)$ is equivalent to an IR ν, then $a_\nu = 1$ and $a_\mu = 0$ for $\mu \neq \nu$.
Hence $\tilde{\chi}^\dagger \cdot \tilde{\chi} = 1$ and the condition of the theorem is satisfied.

Conversely, if this condition is satisfied, we must have $\sum_\mu |a_\mu|^2 = 1$.

Since $a_\mu = 0, 1, 2, \ldots$, the only way this can be fulfilled is if $a_\nu = 1$ and $a_\mu = 0$ for $\mu \neq \nu$. QED

■ ✸

Because of the many uses of group characters, character tables for all useful symmetry groups have been evaluated and are readily available. In particular, character tables of all crystallographic point- groups are given in all books which concern the application of group theory to solid state physics. [Hamermesh, Tinkham]

3.6 Orthonormality and Completeness Relations of Irreducible Characters [My Notation]

Recall that the characters $\chi(G) \equiv \mathrm{Tr}\, U(G)$ are independent of the choice of basis.
All group elements in a given class have the same character in a given rep.

- Lemma:

Let $U^{\mu}(G)$ be an IR of G, and ζ be any class,

(3.6-1) $$\sum_{h\in\zeta} U^{\mu}(h) = \frac{n_\zeta}{n_\mu}\, \chi^{\bar{\mu}}(\zeta)\; E$$

- Proof:

Let

$$A_\zeta = \sum_{h\in\zeta} U^{\mu}(h)$$

By definition, if $h \in \zeta$, we have

$$g\, h\, g^{-1} \in \zeta \qquad \forall\; g \in G$$

Therefore

$$\begin{aligned} U^{\mu}(g)\, A_\zeta\, U^{\mu}(g)^{-1} &= \sum_{h\in\zeta} U^{\mu}(g)\, U^{\mu}(h)\, U^{\mu}(g)^{-1} \\ &= \sum_{h'\in\zeta} U^{\mu}(h') \qquad [\; h' = g\, h\, g^{-1} \in \zeta \;] \\ &= \mathrm{A}_\zeta \end{aligned}$$

Hence

$$A_\zeta = c_\zeta\, E \qquad [\text{ Schur's Lemma 1 }]$$

To evaluate c_ζ, we take the trace on both sides

$$\mathrm{Tr}\, A_\zeta = c_\zeta\, \mathrm{Tr}\, E$$

Hence

$$\begin{aligned} \mathrm{LHS} &= \sum_{h\in\zeta} \mathrm{Tr}\, U^{\mu}(h) \\ &= \sum_{h\in\zeta} \chi^{\mu}(\zeta) \\ &= \mathrm{n}_\zeta\, \chi^{\mu}(\zeta) \end{aligned}$$

$$\mathrm{RHS} = c_\zeta\, n_\mu$$

so that

$$c_\zeta = \frac{n_\zeta}{n_\mu}\, \chi^{\bar{\mu}}(\zeta) \qquad \text{QED}$$

- Theorem 3.7
  (Orthonormality and Completeness of Group Characters):

The characters of inequivalent IRs of G satisfy the following relations:

(3.6-2) $$\frac{1}{n_G}\sum_{\zeta} n_\zeta\, \chi_\mu{}^{-1}(\zeta)\, \chi^\nu(\zeta) = \delta_\mu{}^\nu$$ (orthonormality)

(3.6-3) $$\frac{n_\zeta}{n_G}\, \chi^\mu(\zeta)\, \chi_\mu{}^{-1}(\zeta') = \delta_{\zeta\zeta'}$$ (completeness)

where, by convention, $\chi_\mu{}^{\dagger i} = \left(\chi^\mu{}_i\right)^*$. The summation on i is over the distinct classes of the group; and the summation on μ is over all the inequivalent IRs.

- Proof:

(i) We start from Theorem 3.5.

$$\frac{n_\mu}{n_G}\sum_g D_\mu{}^{-1}(g)^k{}_i\; D^\nu(g)^j{}_l = \delta_\mu{}^\nu\, \delta_i{}^j\; \delta_l{}^k$$

Setting $i = k$, $j = l$, and summing over both indices, we have

$$\frac{n_\mu}{n_G}\sum_g D_\mu{}^{-1}(g)^i{}_i\; D^\nu(g)^j{}_j = \delta_\mu{}^\nu\, \delta_i{}^j\; \delta_j{}^i$$

whereupon

$$\text{LHS} = \frac{n_\mu}{n_G}\sum_g \chi_\mu{}^{-1}(g)\, \chi^\nu(g)$$
$$= \frac{n_\mu}{n_G}\sum_\zeta n_\zeta\, \chi_\mu{}^{-1}(\zeta)\, \chi^\nu(\zeta)$$

and

$$\text{RHS} = \delta_\mu{}^\nu\, \delta_i{}^i$$
$$= \delta_\mu{}^\nu\, n_\mu$$

Hence

$$\frac{1}{n_G}\sum_\zeta n_\zeta\, \chi_\mu{}^{-1}(\zeta)\, \chi^\nu(\zeta) = \delta_\mu{}^\nu$$ (orthonormality)

(ii) Turning to Theorem 3.6.

$$\frac{n_\mu}{n_G}\, D^\mu(g)^l{}_k\, D_\mu{}^{-1}(g')^k{}_l = \delta_{g g'}$$

Summing g over elements of the class ζ, and g' over the class ζ', we have

$$\text{LHS} = \frac{n_\mu}{n_G} \sum_{g\in\zeta} D^\mu(g)^l{}_k \sum_{g'\in\zeta'} D_\mu{}^{-1}(g')^k{}_l$$

$$= \frac{n_\mu}{n_G} \frac{n_\zeta}{n_\mu} \chi^\mu(\zeta)\, \delta^l_k \frac{n_{\zeta'}}{n_\mu} \chi_\mu{}^{-1}(\zeta')\, \delta^k_l \qquad [\text{ lemma }]$$

$$= \frac{n_\zeta\, n_{\zeta'}}{n_G\, n_\mu} \chi^\mu(\zeta)\, \chi_\mu{}^{-1}(\zeta')\, \delta^l_l$$

$$= \frac{n_\zeta\, n_{\zeta'}}{n_G} \chi^\mu(\zeta)\, \chi_\mu{}^{-1}(\zeta')$$

Note that we write n^j instead of n_j simply to indicate the fact that there is no summation j.

$$\text{RHS} = \sum_{g\in\zeta} \sum_{g'\in\zeta'} \delta_{g\,g'}$$

$$= \sum_{g\in\zeta} \delta_{\zeta\,\zeta'} \qquad (\text{ elements of different classes are distinct })$$

$$= \mathrm{n}_\zeta\, \delta_{\zeta\,\zeta'}$$

Hence

$$\frac{n_\zeta}{n_G} \chi^\mu(\zeta)\, \chi_\mu{}^{-1}(\zeta') = \delta_{\zeta\,\zeta'} \qquad (\text{ completeness })$$

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The orthonormality and completeness relations become even more transparent if we define normalized characters,

(3.6-4) $$\tilde{\chi}(\zeta) \equiv \sqrt{\frac{n_\zeta}{n_G}}\, \chi(\zeta)$$

Using implicit summation convention [cf. Appendix I], Theorem 3.7 appears simply as:

(3.6-5) $$\sum_\zeta \tilde{\chi}_\mu{}^{-1}(\zeta)\, \tilde{\chi}^\nu(\zeta) = \delta^\nu{}_\mu$$

(3.6-6) $$\tilde{\chi}^\mu(\zeta)\, \tilde{\chi}_\mu{}^{-1}(\zeta') = \delta_{\zeta\,\zeta'}$$

(Compare with Theorem II.I3.)
If $\{\, \tilde{\chi}(\zeta),\ \zeta = 1, 2, \ldots, n_c \,\}$ are interpreted as components of a vector $\tilde{\boldsymbol{\chi}}$, Eq. (3.6-5) can be further simplified to $\tilde{\boldsymbol{\chi}}_\mu{}^{-1} \cdot \tilde{\boldsymbol{\chi}}^\nu = \delta^\nu{}_\mu$, where · indicates that a "scalar product" in the n_c-D vector space is taken. These notational conveniences will prove useful in some of the subsequent discussions.

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The number of inequivalent IRs for any finite group G is equal to the number of distinct classes of G (i.e. n_c). In other words, $\chi^\mu(\zeta)$ is a $n_c \times n_c$ square matrix if we designate μ as the row index and ζ as the column index. A table of this matrix for any given G is called the character table.

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For abelian groups, each group element forms a class by itself and all IRs are 1-D, hence $D^\mu(g) = \chi^\mu(\zeta)$. Therefore, tables of $D^\mu(g)$ (such as Tables 3.1 and 3.2) are also character tables for the corresponding group (C_2 and D_2 respectively).

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 and the 3-cycles $\{(123), (321)\}$.
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(ii) We know, a priori, that there is the trivial (identity) rep $p \to E$ for all $p \in S_3$.
 Let us label it as $\mu = 1$.
 The three characters are $(1, 1, 1)$.

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(iv) The last IR ($\mu = 3$) must be 2-D, since Theorem 3.6 requires
 $n_G = 6 = 1 + 1 + n_3{}^2$.
 The three characters $\left(\chi^3(\zeta), \zeta = 1, 2, 3\right)$ can be determined as follows:
 (a) $\chi^3(1) = \text{Tr}\, D(e) = \text{Tr}\, E = 2$;
 (b) from the orthonormality and completeness relations, one can deduce that
 $\chi^3(2) = 0$, and $\chi^3(3) = -1$.

The character table for S_3 is summarized in the following:

$\mu\backslash\zeta$	1	2	3
1	1	1	1
2	1	−1	1
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Table 3.3 Character Table of the Group S_3

- ✸

Note that for any rep, the character for the identity element (say, $\zeta = 1$) is equal to the dimension of the rep. Hence the first column of the character table reveals the dimension of all the IRs. The most important uses of the character table for practical applications, however, are derived from the following Theorems.

- Theorem 3.8:

In the reduction of a given rep $U(G)$ of G into its irreducible components, the number of times (a_ν) that the IR $U^{\nu}(G)$ occurs [cf Eq. (3.3-3)] can be determined from the formula:

(3.6-7) $$a_\nu = \sum_\zeta \frac{n_\zeta}{n_G}\, \chi_\nu{}^{-1}(\zeta)\, \chi(\zeta) = \tilde{\boldsymbol{\chi}}_\nu{}^{-1} \cdot \tilde{\boldsymbol{\chi}}$$

Proof:

In terms of its irreducible components, $U(G)$ can be written as [Eq. (3.3-3)]:

$$U(g) = a_\mu\, U^\mu(g) \qquad \forall\ g \in G \qquad \text{[direct sum]}$$

Taking the trace on both sides gives

$$\chi(g) = a_\mu\, \chi^\mu(g)$$

or, for $g \in \zeta$,

$$\chi(\zeta) = a_\mu\, \chi^\mu(\zeta)$$

so that

$$\sqrt{\frac{n_\zeta}{n_G}}\ \chi(\zeta) \equiv \tilde{\chi}\,(\zeta) = \ a_\mu\, \tilde{\chi}^\mu(\zeta)$$

Using the orthonormality theorem, we have

$$\tilde{\boldsymbol{\chi}}_\nu^{\ -1} \cdot \tilde{\boldsymbol{\chi}} = a_\mu\, \tilde{\boldsymbol{\chi}}_\nu^{\ -1} \cdot \tilde{\boldsymbol{\chi}}^\mu$$

$$= \ \mathrm{a}_\mu\, \delta^\mu_\nu$$

$$= \ \mathrm{a}_\nu \qquad \text{QED}$$

■ Example:

Consider the following matrix rep $D(G)$ or the group C_2 :

$$e \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad a \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(Verify that this is indeed a rep!)

The characters are $\chi = (\,2\,,\ 0\,)$.

The character table for the IRs is given by Table 3.1.

In fact,

$$\chi^{\,\mu=1} = \ (1,\ 1), \qquad \chi^{\,\mu=2} = \ (1,\ -1).$$

Thus, according to the above theorem, $a_1 = 1$ and $a_2 = 1$, i.e. each of the IRs occurs once in the reduction of the 2-D rep.

We leave as an exercise to prove that, through a similarity transformation, this rep can be brought to the fully reduced (diagonal) form

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This form explicitly shows that $D(G)$ is a direct sum of the two IRs.

■ Theorem 3.9 (Condition for Irreducibility):

A necessary and sufficient condition for an unitary rep $U(G)$ with characters $\{\ \chi(\zeta)\ \}$ to be irreducible is that

$$\sum_\zeta n_\zeta\, |\ \chi(\zeta)\ |^2 = n_G \quad \text{ie} \quad \tilde{\boldsymbol{\chi}}^\dagger \cdot \tilde{\boldsymbol{\chi}} = 1$$

- Proof:

If a_μ denotes the number of times that the irreducible rep $U^\mu(G)$ is contained in $U(G)$, we have:

$$\begin{aligned}\tilde{\chi}^\dagger \cdot \tilde{\chi} &= \left(a_\mu \tilde{\chi}^\mu\right)^\dagger \cdot \left(a_\nu \tilde{\chi}^\nu\right) \\ &= a^{\mu *} a_\nu \; \tilde{\chi}_\mu{}^\dagger \cdot \tilde{\chi}^\nu \\ &= a^{\mu *} a_\nu \; \delta_\mu{}^\nu \\ &= a^{\mu *} a_\mu \\ &= \sum_\mu \left|a_\mu\right|^2\end{aligned}$$

If $U(G)$ is equivalent to an IR ν, then $a_\nu = 1$ and $a_\mu = 0$ for $\mu \neq \nu$.
Hence $\tilde{\chi}^\dagger \cdot \tilde{\chi} = 1$ and the condition of the theorem is satisfied.

Conversely, if this condition is satisfied, we must have $\sum_\mu \left|a_\mu\right|^2 = 1$.

Since $a_\mu = 0, 1, 2, \ldots$, the only way this can be fulfilled is if $a_\nu = 1$ and $a_\mu = 0$ for $\mu \neq \nu$. QED

- ✱

Because of the many uses of group characters, character tables for all useful symmetry groups have been evaluated and are readily available. In particular, character tables of all crystallographic point- groups are given in all books which concern the application of group theory to solid state physics. [Hamermesh, Tinkham]

3.7 The Regular Representation

The regular **representation** defined on the group algebra [See Appendix III] plays an important role in the development of the group representation theory.

Let $G = \{ g_i \,,\, i = 1, \ldots, n_G \}$.
The group multiplication rule $g_i\, g_j = g_k$ can be written formally as
(3.7-1) $g_i\, g_j = g_m\, \Delta^m{}_{i\,j}$
where

$$\Delta^m{}_{i\,j} = \begin{cases} 1 & \text{for} \quad m = k \\ 0 & \phantom{\text{for}} \quad m \neq k \end{cases}$$

- Theorem 3.10 (The Regular **Representation**):

The matrices
$$(\,\Delta_i\,)^k{}_j = \Delta^k{}_{i\,j} \qquad i = 1, \ldots, n_G$$
form a rep of G.
It is called the regular **representation**.

- Proof:

The group algebra $\mathcal{G} = \left\{ g_i\, c^i \;\middle|\; c^i \in C \right\}$ is also a vector space spanned by the basis $\{ g_i \}$.

Thus, we can interpret
(3.7-1) $g_i\, g_j = g_m\, \Delta^m{}_{i\,j} = g_m\, (\Delta_i)^m{}_j$
as the definition of the operator g_i on the basis vector g_j.

If this interpretation is acceptible, Δ_i will be the matrix rep of g_i on $\mathcal{G}$.

What need to be shown is that Δ_i preserves the group multiplication, ie.

$$g_i\, g_j = g_k \qquad \Longrightarrow \qquad (\Delta_i)^l{}_m \left(\Delta_j\right)^m{}_n = (\Delta_k)^l{}_n$$

Now, the operator eq.

$$g_i\, g_j = g_k$$

implies

$$g_i\, g_j\, g_m = g_k\, g_m \qquad \text{(b)}$$

where g_m is considered as a vector.

Applying eq. (3.7-1) to (b), we have

$$\text{LHS} = g_i\, g_l\, \Delta^l{}_{j\,m}$$
$$= \;\; g_n\, \Delta^n{}_{i\,l}\, \Delta^l{}_{j\,m}$$
$$\text{RHS} = g_n\, \Delta^n{}_{k\,m}$$

Treating g_n as basis vectors, we have

$$\Delta^n{}_{i\,l}\, \Delta^l{}_{j\,m} = \Delta^n{}_{k\,m}$$

or

$$(\Delta_i)^n{}_l \left(\Delta_j\right)^l{}_m = (\Delta_k)^n{}_m$$

QED.

■ ✸

Note that Theorem 3.10 is really just a different incarnation of Cayley's Theorem [every group of order n is isomorphic to a subgroup of S_n].

Recall that the mapping

$$g^i \in G \;\rightarrow\; p^i \in S_n$$

requires

$$p^i_m = j \qquad \text{where } \; g^i\, g^m = g^j$$

Reminder:

$$p^i = \begin{pmatrix} 1 & \dots & l & \dots & n \\ p^i{}_1 & \dots & p^i{}_l & \dots & p^i{}_n \end{pmatrix}$$

$$p^i\, p^j = \begin{pmatrix} 1 & \dots & l & \dots & n \\ p^i{}_1 & \dots & p^i{}_l & \dots & p^i{}_n \end{pmatrix} \begin{pmatrix} 1 & \dots & l & \dots & n \\ p^j{}_1 & \dots & p^j{}_l & \dots & p^j{}_n \end{pmatrix}$$

$$= \begin{pmatrix} p^j{}_1 & \dots & p^j{}_l & \dots & p^j{}_n \\ p^i{}_{p^j{}_1} & \dots & p^i{}_{p^j{}_l} & \dots & p^i{}_{p^j{}_n} \end{pmatrix} \begin{pmatrix} 1 & \dots & l & \dots & n \\ p^j{}_1 & \dots & p^j{}_l & \dots & p^j{}_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \dots & l & \dots & n \\ p^i{}_{p^j{}_1} & \dots & p^i{}_{p^j{}_l} & \dots & p^i{}_{p^j{}_n} \end{pmatrix}$$

$$\left(p^i\, p^j\right)_l = p^i{}_{p^j{}_l} = p^i{}_m \quad \text{where } \; g^j\, g^l = g^m$$
$$= \;\; \text{k} \qquad\qquad \text{where } \; g^i\, g^m = g^k$$

Thus

$$g^k = g^i\, g^j\, g^l$$
$$= \;\; g^n\, g^l \qquad\qquad \text{where } \; g^n = g^i\, g^j$$

so that we can write

$$k = p^n{}_l = \left(p^i\, p^j\right)_l$$

which shows $g^i \rightarrow p^i$ is indeed a rep of G.

Comparing with the regular rep

$$g^i \in G \ \to \ \Delta_i$$

where

$$(\Delta_i)^k{}_j = \Delta^k{}_{i\,j} = \begin{cases} 1 & \text{for} \quad g^k = g^i g^j \\ 0 & \text{otherwise} \end{cases}$$

we have

(3.7-3) $$(\Delta_i)^k{}_j = \Delta^k{}_{i\,j} = \delta^k_{p^i_j}$$

since

$$p^i_j = k \qquad \text{if} \qquad g^k = g^i g^j$$

This provides us with an alternative proof of Theorem 3.10:

$$\begin{aligned}(\Delta_i)^l{}_m \left(\Delta_j\right)^m{}_h = \delta^l_{p^i_m}\, \delta^m_{p^j_h} \\ &= \delta^l_{p^i_{p^j_h}} \\ &= \delta_{(p^i p^j)^l_h} \\ &= \delta^m_{p^k_h} \qquad \text{where} \quad p^k = p^i\, p^j \\ &= (\Delta_k)^l{}_h\end{aligned}$$

- Theorem 3.11 (Decomposition of the Regular **Representation**):

(i) The regular rep contains every inequivalent IR μ precisely n_μ times

(ii)

(3.7-5) $$\sum_\mu n_\mu{}^2 = \text{n}_G$$

Note: (ii) is just the missing element in the proof of Theorem 3.6 on the completeness relation for the IR matrices. It restores the logical completeness of our presentation.

- Proof (i):

The characters of the regular rep is

$$\chi_i{}^R = \text{Tr}(\Delta_i) = \Delta^k{}_{i\,k}$$

where

$$\Delta^i{}_{j\,k} = \begin{cases} 1 & \text{for} \quad \text{g}_\text{i} = \text{g}_\text{j}\, \text{g}_\text{k} \\ 0 & \text{otherwise} \end{cases}$$

For $i = 1$ (the identity),

$$\Delta^k{}_{1\,j} = \delta^k{}_j \qquad [\ g_k = g_1\, g_k\ ]$$

so that

$$\chi_1{}^R = \Delta^k{}_{1\,k} = \delta^k{}_k \ = n_G$$

For $i \neq 1$, $g_i\, g_k \neq g_k$.

Therefore

$$\Delta^k{}_{i\,j} = 0 \qquad \text{for} \qquad k = j$$

so that

$$\chi_i{}^R = \Delta^k{}_{i\,k} = 0$$

Applying Theorem 3.8, we have

(3.7-4) $$a_\mu{}^R = \frac{n_i}{n_G} \chi_\mu{}^{\dagger i} \chi_i{}^R = \frac{n_1}{n_G} \chi_\mu{}^{\dagger 1} \chi_1{}^R = \frac{1}{n_G} n_\mu n_G = \mathrm{n}_\mu$$

- **Proof (ii):**

Using the above result, we have

$$\Delta_e{}^R = \sum_\mu n_\mu U^\mu(e)$$

Taking the trace, we obtain

$$\chi_e{}^R = n_G = \sum_\mu n_\mu{}^2. \qquad \text{QED}$$

- Example:

Consider $C_2 = \{e, a\}$ with multiplication table

C_2	e	a
e	e	a
a	a	e

The regular rep matrices are

(3.7-6) $$\Delta_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Delta_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The characters are:

$$\chi_e{}^R = 2 \qquad \chi_a{}^R = 0$$

Making use of the character table of the IRs of C_2,

C_2	e	a
D_1	1	1
D_2	1	−1
D_R	2	0

and

$$a_\mu{}^R = \frac{n_i}{n_G} \chi_\mu{}^{\dagger i} \chi_i{}^R$$

we have,

$$a_1{}^R = \frac{1}{2} (1 \cdot 2 + 1 \cdot 0) = 1$$

$$a_2{}^R = \frac{1}{2} [1 \cdot 2 + (-1) \cdot 0] = 1$$

Hence

$$D_R = D_1 + D_2$$

By a similarity transformation, $\Delta_i{}' = S\,\Delta_i\,S^{-1}$, the matrices of Eq. (3.7-6) can be diagonalized:

(3.7-7) $$\Delta_e{}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Delta_a{}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where $$S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is used. Eq. (3.7-7) makes explicit the reduction of the regular rep into its irreducible components.

Group Algebra

As a consequence of Theorem 3.11, one can obtain all the inequivalent IRs of any finite group G, by systematically reducing the corresponding regular rep into its irreducible components. With an appropriate choice of basis vectors, as illustrated in the above example, the rep matrices in the regular rep can be all brought to the block- diagonal form

$$\begin{pmatrix} 1 & & & & & & & \\ & D^2 & & & & & & \\ & & D^2 & & & & & \\ & & & \ddots & & & & \\ & & & & D^{n_c} & & & \\ & & & & & & \ddots & \\ & & & & & D^{n_c} & & \\ & n_2 \text{ Blocks} & \dots & n_c \text{ Blocks} & & & & \end{pmatrix}$$

Note that the first IR ($\mu = 1$) is always the 1-D identity rep. The above expression explicitly displays the content of Theorem 3.11.

In the first part of Chap. 5, the reduction of the regular rep will be worked out in detail for a non-trivial example – the permutation group S_3. To prepare our way for that study, and for the general analysis, we need to examine the regular rep space a little more closely. By the way the rep is introduced, the space has as its basis the group elements $\{ g_m \in G \}$ —the right- hand side of Eq. (3.7-1) contains an explicit summation over the basis element label m. [Although there is actually no sum since all but one coefficients vanish.] Conceptually, we have thereby introduced a new mathematical structure which, on the one hand, is a linear vector space (with respect to forming linear combinations) and, on the other hand, has a well-defined multiplication rule (based on the original group structure). This new mathematical construct is called the group algebra. A systematic method for the reduction of the regular rep on the group algebra space can be formulated in terms of irreducible projection operators called idempotents. This method is employed to derive all IRs of the general permutation group S_n in the second part of Chap. 5. An introduction to the basic theory of group algebra necessary for that analysis is presented in Appendix III rather than here because it represents a somewhat more advanced topic than the rest of this chapter. The group algebra is also called the group ring.

3.8 Direct Product Representations, Clebsch- Gordan Coefficients

- Definition 3.8 (Direct Product Vector Space):

Let U and V be inner product spaces.
The respective orthonormal bases are
$$\{ \hat{u}_i \,;\, i = 1, \ldots, n_u \} \text{ and } \{ \hat{v}_j \,;\, j = 1, \ldots, n_v \}$$
Then the direct product space $W = U \otimes V$ consists of all linear combinations of the orthonormal basis vectors
$$\{ \hat{w}_k \,;\; k = (i, j) \,;\, i = 1, \ldots, n_u \,;\; j = 1, \ldots, n_v \}$$
where $\hat{w}_k$ can be regarded as the "formal product" $\hat{w}_k = \hat{u}_i \otimes \hat{v}_j$. (see Note)

By definition :

(i) $\langle w^{k'} | w_k \rangle = \delta^{k'}{}_k = \delta^{i'}{}_i \, \delta^{j'}{}_j$
where $k' = (i', j')$ and $k = (i, j)$

(ii) $W = \{ \boldsymbol{x} \,;\, | \boldsymbol{x} \rangle = | w_k \rangle \, x^k \}$
where the complex numbers x^k are the components of x.

(iii) $\langle \boldsymbol{x} | \boldsymbol{y} \rangle \equiv x_k{}^\dagger \, y^k$
where $x_k{}^\dagger = \left(x^k \right)^*$

Note: It is possible to define the direct product without reference to specific bases. One makes use of the space consisting of all linear transformations from V to U. [Halmos] The above formulation is adopted because it is just as general, and it stays closer to the way such products arise in physical applications.

- Example:

Consider two- particle system.

Let x_1 be the coordinate vector of "particle 1",
x_2 be the coordinate vector of "particle 2".

The two- particle system are characterized classically by the coordinates (x_1 , x_2).

In quantum mechanics, let $| \psi_s \rangle$, $s = 1, 2$, denote the state vectors of particle s in the quantum mechanical Hilbert spaces $\tilde{H}_s$, and let $\{ | \boldsymbol{x}_s \rangle \}$ be the basis vectors in the "coordinate representation". We have,

$$| \psi_s \rangle = \int | \boldsymbol{x} \rangle \, \psi_s(\boldsymbol{x}) \, d^3 x \qquad s = 1, 2$$

where $\psi_s(\boldsymbol{x})$ is the Schrodinger wave function of particle s. [Messiah, Schiff]

Then the states of the combined two- particle system are elements of the direct product space $\tilde{H}_1 \times \tilde{H}_2$ with coordinate basis vectors $\{ | \boldsymbol{x}_1 , \boldsymbol{x}_2 \rangle \}$ and state vectors

$$| \Psi \rangle = \int | \boldsymbol{x}_1 , \, x_2 \rangle \, \Psi(x_1 , \, x_2) \, d^3 x_1 \, d^3 x_2$$

where $\Psi(x_1 , x_2)$ is the c-number Schrodinger wave function of the two- particle system.

- **Direct Product Operator**

Let A and B be operators on U and V respectively.
The direct product operator, $D \equiv A \times B$, defined on $W = U \times V$, is defined as,

(3.8-1) $\quad D \, | w_k \rangle = | w_{k'} \rangle \, D^{k'}{}_k$

$\quad D^{k'}{}_k \equiv A^{i'}{}_i \, B^{j'}{}_j$

where $A^{i'}{}_i \left(B^{j'}{}_j \right)$ are matrix elements of $A \, (B)$ on the subspace $U \, (V)$ with respect to to the bases $\{ \hat{u}_i \} \left(\{ \hat{v}_j \} \right)$ and $k = (i, j)$, $k' = (i', j')$.

If A and B correspond to the same physical operator realized U and V, the operator D also corresponds to the same operator realized on $U \times V$.
For instance, in the example given above, the operators A, B, and D could be the quantum mechanical momentum operator p_z for particles 1, 2 and the combined two- particle system respectively.
Likewise, they could be the Hamiltonian, or one of the angular momentum operators for the respective systems.

- Definition 3.9 (Direct Product **Representation**):

Let $D^{\mu}(G)$ and $D^{\nu}(G)$ be reps of G on U and V respectively.
Then the operators $D^{\mu\times\nu}(G) = D^{\mu}(G)\times D^{\nu}(G)$ on W with $g\in G$ also form a rep of the group G.
It is called the direct product rep of $D^{\mu}(G)$ (on U) and $D^{\nu}(G)$ (on V).

- **Characters**

(3.8-2) $\chi^{\mu\times\nu} = \chi^{\mu}\,\chi^{\nu}$

- **Proof:**

$$\begin{aligned}\chi^{\mu\times\nu}(g) &= \mathrm{Tr}\, D^{\mu\times\nu}(g)\\ &= D^{\mu\times\nu}(g)^{k}{}_{k}\\ &= D^{\mu}(g)^{i}{}_{i}\, D^{\nu}(g)^{j}{}_{j} \qquad k=(i,\ j)\\ &= \chi^{\mu}(g)\,\chi^{\nu}(g)\end{aligned}$$

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Suppose $D^{\mu}(G)$ and $D^{\nu}(G)$ are IRs of G of dimension n_{μ} and n_{ν}, respectively, then $D^{\mu\times\nu}(G)$ is a rep of dimension $n_{\mu}\times n_{\nu}$, and it is usually reducible. The number of times a_{λ} that a given IR $D^{\lambda}(G)$ occurs in $D^{\mu\times\nu}(G)$ is given by Theorem 3.8:

$$a_{\lambda}{}^{\mu\times\nu} = \tilde{\chi}_{\lambda}{}^{\dagger}\,\tilde{\chi}^{\mu\times\nu} = \sum_{\zeta} \chi^{\lambda *}(\zeta)\,\chi^{\mu}(\zeta)\,\chi^{\nu}(\zeta)\,\frac{n_{\zeta}}{n_G}$$

- Example:

Consider the product reps $D^{\mu}\times D^{\nu}$ of the symmetric group S_3 where D^{μ} and D^{ν} are the three IRs discussed previously. By consulting the character table

S_3	e	ζ_2	ζ_3
D_1	1	1	1
D_2	1	−1	1
D_3	2	0	−1

$\zeta_2 = \{(12),\ (13),\ (23)\}$
$\zeta_3 = \{(123),\ (132)\}$

we have

S_3	e	ζ_2	ζ_3
$D^{1\times1}$	1	1	1
$D^{1\times2}$	1	−1	1
$D^{1\times3}$	2	0	−1
$D^{2\times2}$	1	1	1
$D^{2\times3}$	2	0	−1
$D^{3\times3}$	4	0	1

By inspection,

$D^{1\times1}\sim D^{1}$ $\quad D^{1\times2}\sim D^{2}$ $\quad D^{2\times2}\sim D^{1}$

$D^{1\times3}\sim D^{3}$ $\quad D^{2\times3}\sim D^{3}$

where $\sim$ stands for "equivalent to".

What about $D^{3\times3}$?
This is a 4-D rep; hence it must be reducible.
Applying the formula

$$a_\lambda{}^\mu = \frac{1}{n_G}\sum_\zeta n_\zeta\,\chi_\lambda{}^*(\zeta)\,\chi^\mu(\zeta)$$

we obtain:

$$a_1 = \frac{1}{6}[\,1\cdot4 + 3\,(1\cdot0) + 2\,(1\cdot1)] = \frac{1}{6}\cdot6 = 1$$
$$a_2 = \frac{1}{6}[\,1\cdot4 + 3\,(-1\cdot0) + 2\,(1\cdot1)] = \frac{1}{6}\cdot6 = 1$$
$$a_3 = \frac{1}{6}[\,2\cdot4 + 3\,(0\cdot0) + 2\,(-1\cdot1)] = \frac{1}{6}\cdot6 = 1$$

Hence

$$D^{3\times3} = D^1 + D^2 + D^3.$$

- Definition 3.10 (C-GCs):

Let D^μ (defined on U) and D^ν (defined on V) be IRs and

$$D^{\mu\times\nu} \sim \sum_{\lambda\oplus} a_\lambda\,D^\lambda$$

This means

$$W = U\times V = \sum_{\lambda,\alpha\,\oplus} W^{\lambda_\alpha}$$

where λ is the label for IRs, and $\alpha\,(=\; 1,\,\ldots,\,a_\lambda\,)$ distinguishes the spaces which correspond to the same λ.

Thus, there is an orthonormal basis of W, namely,

$$\left\{\,\hat{e}_l^{\lambda_\alpha} = \left|\begin{smallmatrix}\lambda\\ l\,\alpha\end{smallmatrix}\right\rangle;\; l = 1,\,\ldots,\,n_\lambda\,;\;\alpha = 1,\,\ldots,\,a_\lambda\right\}$$

with respect to which $D^{\mu\times\nu}$ are all in block- diagonal form.

It is related to the original basis $\left\{\,\hat{e}_{i\,j}^{\mu\,\nu} = \hat{e}_i^\mu\,\hat{e}_j^\nu = \left|\begin{smallmatrix}\mu\,\nu\\ i\,j\end{smallmatrix}\right\rangle\right\}$ by a unitary transformation,

(3.8-3) $$\left|\begin{smallmatrix}\lambda\\ l\,\alpha\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}\overline{\mu\,\nu}\\ i\,j\end{smallmatrix}\right\rangle\left\langle\begin{smallmatrix}i\,j\\ \mu\,\nu\end{smallmatrix}\middle|\begin{smallmatrix}\lambda\\ l\,\alpha\end{smallmatrix}\right\rangle$$

where $\left\langle\begin{smallmatrix}i\,j\\ \mu\,\nu\end{smallmatrix}\middle|\begin{smallmatrix}\lambda\\ l\,\alpha\end{smallmatrix}\right\rangle$ are (complex number) elements of the transformation matrix with (i, j) as the "row index" and (λ, α, l) as the "column index".

The matrix elements $\left\langle\begin{smallmatrix}i\,j\\ \mu\,\nu\end{smallmatrix}\middle|\begin{smallmatrix}\lambda\\ l\,\alpha\end{smallmatrix}\right\rangle$ are called Clebsch- Gordan Coefficients (C-GCs).

Note: there're numerous notations for C-GCs in the literature.
Choose one that is easiest for you to memorize.

In most applications, the labels ($\mu\,\nu$) are fixed so that they are exempted from summation.

- Theorem 3.12 (Orthonormality and Completeness of C-GCs):

The C-GCs satisfy the following orthonormality and completeness relations:

(3.8-4) $\left\langle \begin{smallmatrix} i' & j' \\ \bar{\mu} & \bar{\nu} \end{smallmatrix} \middle| \begin{smallmatrix} \lambda \\ l\,\alpha \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} l \\ \lambda_\alpha \end{smallmatrix} \middle| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle = \delta^{i'}{}_i\ \delta^{j'}{}_j$

and

(3.8-5) $\left\langle \begin{smallmatrix} l' \\ \lambda_{\alpha'} \end{smallmatrix} \middle| \begin{smallmatrix} \bar{\mu}\,\bar{\nu} \\ i\,j \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} i\,j \\ \mu\,\nu \end{smallmatrix} \middle| \begin{smallmatrix} \lambda \\ l\,\alpha \end{smallmatrix} \right\rangle = \delta^{\lambda}_{\lambda_{\alpha'}\alpha}\ \delta^{l'}_{l}$

where

$$\left\langle \begin{smallmatrix} l \\ \lambda_\alpha \end{smallmatrix} \middle| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle \equiv \left\langle \begin{smallmatrix} i\,j \\ \mu\,\nu \end{smallmatrix} \middle| \begin{smallmatrix} \lambda \\ l\,\alpha \end{smallmatrix} \right\rangle^*$$

- Proof:

These relations are direct consequences of the orthonormality and completeness of the bases { $\hat{e}^{\mu\nu}_{i\,j}$ } and { $\hat{e}^{\lambda_\alpha}_{l}$ }. [cf (i) and (ii) of Theorem II.13] QED

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Clearly, the inverse of Eq. (3.8-3) is:

(3.8-6) $\left| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle = \left| \begin{smallmatrix} \lambda \\ l\,\alpha \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} l \\ \lambda_\alpha \end{smallmatrix} \middle| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle$

The defining properties of the two bases are:

(3.8-7) $U(g) \left| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle = \left| \begin{smallmatrix} \mu\ \ \nu \\ i'\,j' \end{smallmatrix} \right\rangle D^{\mu}(g)^{i'}{}_i\ D^{\nu}(g)^{j'}{}_j$

and

(3.8-8) $U(g) \left| \begin{smallmatrix} \lambda \\ l\,\alpha \end{smallmatrix} \right\rangle = \left| \begin{smallmatrix} \lambda \\ l'\,\alpha \end{smallmatrix} \right\rangle D^{\lambda}(g)^{l'}{}_l$

Note: the most general form of (3.8-8) is

$$U(g) \left| \begin{smallmatrix} \lambda \\ l\,\alpha \end{smallmatrix} \right\rangle = \left| \begin{smallmatrix} \lambda \\ l'\,\alpha \end{smallmatrix} \right\rangle D^{\lambda_\alpha}(g)^{l'}{}_l$$

where the D^{λ_α} (g) 's are all equivalent to D^{λ} (g).
By a suitable choice of basis, one can always use (3.8-8) instead.

Let us use Eq. (3.8-6) on the left-hand side of Eq. (3.8-7):

$$U(g) \left| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle = U(g) \left| \begin{smallmatrix} \lambda \\ l\,\alpha \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} l \\ \lambda_\alpha \end{smallmatrix} \middle| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle$$

$$= \left| \begin{smallmatrix} \lambda \\ l'\,\alpha \end{smallmatrix} \right\rangle D^{\lambda}(g)^{l'}{}_l \left\langle \begin{smallmatrix} l \\ \lambda_\alpha \end{smallmatrix} \middle| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle$$

$$= \left| \begin{smallmatrix} \bar{\mu}\ \bar{\nu} \\ i'\,j' \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} i'\ j' \\ \mu\ \ \nu \end{smallmatrix} \middle| \begin{smallmatrix} \lambda \\ l'\,\alpha \end{smallmatrix} \right\rangle D^{\lambda}(g)^{l'}{}_l \left\langle \begin{smallmatrix} l \\ \lambda_\alpha \end{smallmatrix} \middle| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle$$

Comparing the last result with the right- hand side of Eq. (3.8-7), and making use of the linear independence of the basis vectors { $\hat{e}^{\mu\nu}_{i\,j}$ }, we obtain a useful theorem.

- Theorem 3.13 (Reduction of Product **Representation**):

The similarity transformation composed of Clebsch- Gordan coefficients decomposes the direct product rep $D^{\mu\times\nu}$ into its irreducible components. The following reciprocal relations hold:

(3.8-9) $D^{\mu}(g)^{i'}{}_i\ D^{\nu}(g)^{j'}{}_j = \left\langle \begin{smallmatrix} i'\ j' \\ \bar{\mu}\ \bar{\nu} \end{smallmatrix} \middle| \begin{smallmatrix} \lambda \\ l'\,\alpha \end{smallmatrix} \right\rangle D^{\lambda}(g)^{l'}{}_l \left\langle \begin{smallmatrix} l \\ \lambda_\alpha \end{smallmatrix} \middle| \begin{smallmatrix} \mu\,\nu \\ i\,j \end{smallmatrix} \right\rangle$

and

(3.8-10) $\delta^{\lambda}_{\lambda_{\alpha'}\alpha} D^{\lambda}(g)^{l'}{}_l = \left\langle \begin{smallmatrix} l' \\ \lambda'_{\alpha'} \end{smallmatrix} \middle| \begin{smallmatrix} \bar{\mu}\ \bar{\nu} \\ i'\,j' \end{smallmatrix} \right\rangle D^{\mu}(g)^{i'}{}_i\ D^{\nu}(g)^{j'}{}_j \left\langle \begin{smallmatrix} i\,j \\ \mu\,\nu \end{smallmatrix} \middle| \begin{smallmatrix} \lambda \\ l\,\alpha \end{smallmatrix} \right\rangle$

The last equation makes explicit the block- diagonal form of the direct product rep in the new basis: the similarity- transformed matrix (right- hand side) is diagonal in two of the three labels – λ , α (left- hand side).

The proof of Eq. (3.8-9) is provided in the discussion preceding the theorem. The reciprocal formula, Eq. (3.8-10), can be derived in a similar way – by substituting Eq. (3.8-3) in Eq. (3.8-8). (It also follows from Eq. (3.8-9) in conjunction with the orthonormality and completeness property of the Clebsch- Gordan coefficients, Theorem 3.12.)

Physically, direct products arise when one considers the behavior of systems involving two or more degrees of freedom under symmetry operations, or when one explores the regularities of transition amplitudes of physical processes implied by the underlying symmetry (or, sometimes, "broken- symmetry"). The "addition" of angular momenta of a two- particle system, or of the orbital- and spin- angular momenta of a single particle, is the most often encountered example of a direct product rep (of the rotation group). We shall study it in detail in Chap. 7. The Wigner- Eckart theorem (to be introduced in the next chapter) which provides the most important method to uncover symmetry- related regularities in matrix elements of physical observables, is derived from the reduction of direct product reps to their irreducible parts. For those who have not had previous experience with these applications, it may be desirable to come back to this section after studying Chap. 7, so that the content and the power of the general results may be better appreciated.