Chapter 5
Symmetric Groups

■ References

The following notes are based on Chap 5, Tung & Chap 15, Inui.
Caution: the conventions used by Tung & Inui are very different (see section group product).
Basic concepts of the idempotents are described in note GroupAlgebra , which was based on App III, Tung.
Other noteworthy references are:

1. J.Q.Chen, "Group Representation Theory for Physicists" , Sec 1.2, World Scientific (87).
   This book also describes the Eigenfunction Method for finding group representations.
2. Chapter 7, Hamermeash.

5.0 Basics

■ S_n

The $n!$ permutations of n objects form a **symmetric (permutation) group** S_n of degree n & order $n!$.

The group elements are written as

$$p = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \equiv \begin{pmatrix} i \\ p_i \end{pmatrix}$$

Obviously, the order of the elements in the permutation symbol is immaterial, eg.:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} = \dots$$

■ Group Product

There're 2 inequivalent ways to define the group product.
The active view will be adopted in this note.
In either view, 2 permutations commute if they do not involve permutations of the same index.

Active Point of View

The **active way** used by Tung (see p.18, Tung) interprets a permutation

$$p = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} = (p_i \leftarrow i)$$

as taking the object originally in box i to box p_i.

The product $p\,q$ of 2 permutations p & q then denotes 2 consecutive actions:

1st, take objects in box i to box q_i ,

then, take objects in box i to box p_i.

Thus, an object that is originally in box i is 1st taken to box q_i , then to box p_{q_i} .

Symbolically:

$$\begin{aligned} p\,q &= (p_i \leftarrow i)(q_i \leftarrow i) \\ &= \left(p_{q_i} \leftarrow q_i \right)(q_i \leftarrow i) \\ &= \left(p_{q_i} \leftarrow i \right) \end{aligned}$$

or

$$p\,q = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$$

$$= \begin{pmatrix} q_1 & q_2 & \dots & q_n \\ p_{q_1} & p_{q_2} & \dots & p_{q_n} \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \dots & n \\ p_{q_1} & p_{q_2} & \dots & p_{q_n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \dots & n \\ (p\,q)_1 & (p\,q)_2 & \dots & (p\,q)_n \end{pmatrix}$$

where $\begin{pmatrix} q_1 & q_2 & \dots & q_n \\ p_{q_1} & p_{q_2} & \dots & p_{q_n} \end{pmatrix}$ is the rearrangement of the columns of $\begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$ so that the 1st row $(1 \;\; 2 \;\; \dots \;\; n)$ becomes $(q_1 \;\; q_2 \;\; \dots \;\; q_n)$.

Thus $(p\,q)_j = p_{q_j}$.

The inverse p^{-1} of p is:

$$p^{-1} = \begin{pmatrix} 1 & 2 & \dots & n \\ \left(p^{-1}\right)_1 & \left(p^{-1}\right)_2 & \dots & \left(p^{-1}\right)_n \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & \dots & p_n \\ 1 & 2 & \dots & n \end{pmatrix} = (i \leftarrow p_i) = (p_i \rightarrow i)$$

so that

$$\begin{aligned} p\,p^{-1} &= (p_i \leftarrow i)(i \leftarrow p_i) = (p_i \leftarrow p_i) = e \\ p^{-1}\,p &= (i \leftarrow p_i)(p_i \leftarrow i) = (i \leftarrow i) = e \end{aligned}$$

$$p\,p^{-1} = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}\begin{pmatrix} p_1 & p_2 & \dots & p_n \\ 1 & 2 & \dots & n \end{pmatrix} = \begin{pmatrix} p_1 & p_2 & \dots & p_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} = e$$

$$p^{-1}\,p = \begin{pmatrix} p_1 & p_2 & \dots & p_n \\ 1 & 2 & \dots & n \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} = e$$

as expected.

Obviously, $\quad p_{\left(p^{-1}\right)_j} = \left(p\,p^{-1} \right)_j = e_j = j = \left(p^{-1}\,p \right)_j = \left(p^{-1}\right)_{p_j}$

Passive Point of View

The **passive way** used by Inui is (see p.16, Inui) interprets a permutation

$$p = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} = (\, i \to p_i)$$

as relabeling object i as p_i .

The product $p\,q$ of 2 permutations p & q then denotes 2 consecutive re-labeling:

1st, relabeling object i as q_i ,

then, relabeling object i as p_i .

Thus, object i is finally labeled q_{p_i} .

Symbolically:

$$\begin{aligned} p\,q &= (\, i \to p_i\,)\,(\, i \to q_i\,) \\ &= (\, i \to p_i\,)\left(p_i \to q_{p_i} \right) \\ &= \left(i \to q_{p_i} \right) \end{aligned}$$

or

$$p\,q = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}\begin{pmatrix} p_1 & p_2 & \dots & p_n \\ q_{p_1} & q_{p_2} & \dots & q_{p_n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \dots & n \\ q_{p_1} & q_{p_2} & \dots & q_{p_n} \end{pmatrix}$$

where $\begin{pmatrix} p_1 & p_2 & \dots & p_n \\ q_{p_1} & q_{p_2} & \dots & q_{p_n} \end{pmatrix}$ is the rearrangement of the columns of $\begin{pmatrix} 1 & 2 & \dots & n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$ so that the 1st row $(\; 1 \quad 2 \quad \dots \quad n \;)$ becomes $(\; p_1 \quad p_2 \quad \dots \quad p_n \;)$.

Thus $(\, p\,q\,)_j = q_{p_j}$.

This means products in the passive view correspond to products in inverse order in the active view, & vice versa. ie.

passive $p\,q \dots r\,s \quad \Longleftrightarrow \quad$ active $s\,r \dots q\,p$

Since we shall adopt the active point of view in the rest of this note, further development of the passive way will be left as exercise for those interested.

- **Example**

Let

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The active view gives

$$p\,q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$q\,p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

The passive view gives

$$p\,q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$q\,p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

■ Example S_3

Permutations of 3 objects form the group S_3.
There are $3! = 6$ group elements:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

■ S_3 & C_{3v}

Consider the group C_{3v} , which is isomorphic to S_3.

To facilitate the correspondence between elements of the 2 groups, we identify the objects under permutation to be the vertices of the triangle & the boxes to be positions in space.
Both vertices & positions are labeled 1, 2, 3 in a counterclockwise sense.
Originally, vertex labelled i is at postion (box) i .

In the **active point of view**, the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$ moves the vertex 1 at position 1 to position 2, vertex 2 at position 2 to position 3, and vertex 3 at position 3 to position 1. The result is that vertices 1, 2, 3 are now at positions 2, 3, 1, respectively. Clearly, this corresponds to the C_3 rotation of the triangle if the boxes or positions are fixed in space.

In the **passive point of view**, the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$ relabels the vertex 1 as vertex 2, vertex 2 as vertex 3, and vertex 3 as vertex 1. The result is that vertices 1, 2, 3 are now at positions 3, 1, 2, respectively. Clearly, this corresponds to the C_3^2 rotation of the triangle if the positions are fixed in space. Alternatively, we can also say that the boxes are rotated by C_3 while the triangle is held fixed.

Similarly, the elements of $S_3 = \{e, a, b, c, d, f\}$ can be identified with those of $C_{3v} = \{E, C_3, C_3^2, \sigma_1, \sigma_2, \sigma_3\}$, respectively, if we adopt the convention that in the active (passive) point of view, operations of C_{3v} represent rotations of the triangle (positions).
Obviously, it is also correct to identify $S_3 = \{e, a, b, c, d, f\}$ with $C_{3v} = \{E, C_3^2, C_3, \sigma_1, \sigma_2, \sigma_3\}$, respectively, if we adopt the convention that in the active (passive) point of view, operations of C_{3v} represent rotations of the positions (triangle).

This freedom of interpretation may create confusion to novices & is a major source of computational error, especially when results from different authors are quoted.
We shall henceforth adopt the active point of view for all groups, which means elements of the point groups are treated as actions on the geometric figure.
The active point of view is usually preferred by physicists while the passive one, by mathematicians.

The group multiplication table for S_3 can be obtained from that of C_{3v} (p.12, Inui) :

S_3	e	$a = C_3$	$b = C_3^2$	$c = \sigma_1$	$d = \sigma_2$	$f = \sigma_3$
e	e	a	b	c	d	f
a	a	b	e	f	c	d
b	b	e	a	d	f	c
c	c	d	f	e	a	b
d	d	f	c	b	e	a
f	f	c	d	a	b	e

This table is the same as that in p.15, Tung but the roles of a & b are interchanged in p.17, Inui.
Both authors adopt the active view on rotation operators but Tung used the active, Inui the passive view for permutations.

▪ Cycles

An $n-$ cycle $(i_1\, i_2 \ldots i_n)$ is defined by

$$(i_1\, i_2 \ldots i_n) = \begin{pmatrix} i_1 & i_2 & \ldots & i_n \\ i_2 & i_3 & \ldots & i_1 \end{pmatrix}$$

▪ Examples

$1-$ cycle:

$$(i) = \begin{pmatrix} i \\ i \end{pmatrix}$$

$2-$ cycle, also called a transposition:

$$(i,\ j) = \begin{pmatrix} i & j \\ j & i \end{pmatrix}$$

$3-$ cycle :

$$(i\ j\ k) = \begin{pmatrix} i & j & k \\ j & k & i \end{pmatrix}$$

The generator of a cyclic group C_n is the $n-$ cycle.

Any permutation can be written as a product of cycles with no common indices, eg.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 1 & 5 & 8 & 2 & 6 \end{pmatrix} = (14)\,(237)\,(5)\,(68)$$

▪ Example S_3

In cycle notations, elements of S_3 becomes:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)\,(2)\,(3) = e a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)\, b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

$$c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)\,(23) = (23) \qquad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13) \qquad f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)$$

For convenience, the multiplication table is reproduced in cycle notations (cf p.15, Tung):

S_3	e	(123)	(132)	(23)	(13)	(12)
e	e	(123)	(132)	(23)	(13)	(12)
(123)	(123)	(132)	e	(12)	(23)	(13)
(132)	(132)	e	(123)	(13)	(12)	(23)
(23)	(23)	(13)	(12)	e	(123)	(132)
(13)	(13)	(12)	(23)	(132)	e	(123)
(12)	(12)	(23)	(13)	(123)	(132)	e

For comparison purposes, the following table may be more useful:

S_3 C_{3v}	e	$a = (123) = C_3$	$b = (132) = C_3^2$	$c = (23) = \sigma_1$	$d = (13) = \sigma_2$	$f = (12) = \sigma_3$
e	e	$a = (123) = C_3$	$b = (132) = C_3^2$	$c = (23) = \sigma_1$	$d = (13) = \sigma_2$	$f = (12) = \sigma_3$
$a = (123) = C_3$	$a = (123) = C_3$	$b = (132) = C_3^2$	e	$f = (12) = \sigma_3$	$c = (23) = \sigma_1$	$d = (13) = \sigma_2$
$b = (132) = C_3^2$	$b = (132) = C_3^2$	e	$a = (123) = C_3$	$d = (13) = \sigma_2$	$f = (12) = \sigma_3$	$c = (23) = \sigma_1$
$c = (23) = \sigma_1$	$c = (23) = \sigma_1$	$d = (13) = \sigma_2$	$f = (12) = \sigma_3$	e	$a = (123) = C_3$	$b = (132) = C_3^2$
$d = (13) = \sigma_2$	$d = (13) = \sigma_2$	$f = (12) = \sigma_3$	$c = (23) = \sigma_1$	$b = (132) = C_3^2$	e	$a = (123) = C_3$
$f = (12) = \sigma_3$	$f = (12) = \sigma_3$	$c = (23) = \sigma_1$	$d = (13) = \sigma_2$	$a = (123) = C_3$	$b = (132) = C_3^2$	e

■ Cycle Structures

Every element of S_n can be expressed as a product of cycles.
If an element contains k_i i-cycles ($i = 1 \ldots n$), its cycle structure is denoted by:

$$\left(1^{k_1}\, 2^{k_2} \ldots i^{k_i} \ldots n^{k_n}\right)$$

with the convention that terms with $k_i = 0$ can be omitted.
Some authors, eg. Inui, uses the n-tuple notation $k = (k_1 \ldots k_n)$.

Since each element is a permutation of n objects, the ' total length ' of the cycles must be n. Thus

$$\sum_{j=1}^{n} j\, k_j = \mathrm{n}$$

As an example, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 1 & 5 & 8 & 2 & 6 \end{pmatrix} = (14)\,(237)\,(5)\,(68)$$

has a cycle structure $\left(1^1\, 2^2\, 3^1\right)$ with $k = (1 \times 2 \times 1 \times 0 \times 0 \times 0 \times 0 \times 0)$.

$$\sum_{j=1}^{n} j\, k_j = 1\times 1 + 2\times 2 + 3\times 1 + 4\times 0 + \ldots = 8$$

Classes

Let

$$p = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} = (p_i \leftarrow i) \qquad q = \begin{pmatrix} 1 & 2 & \dots & n \\ q_1 & q_2 & \dots & q_n \end{pmatrix} = (q_i \leftarrow i)$$

$$\begin{aligned} q\, p\, q^{-1} &= (q_i \leftarrow i)(p_i \leftarrow i)(i \leftarrow q_i) \\ &= (q_i \leftarrow i)(p_i \leftarrow q_i) \\ &= \left(q_{p_i} \leftarrow p_i\right)(p_i \leftarrow q_i) \\ &= \left(q_{p_i} \leftarrow q_i\right) \\ &= \begin{pmatrix} q_1 & q_2 & \dots & q_n \\ q_{p_1} & q_{p_2} & \dots & q_{p_n} \end{pmatrix} \\ &= \text{q[p]} \end{aligned}$$

where

$$q[p] \equiv \begin{pmatrix} q_1 & q_2 & \dots & q_n \\ q_{p_1} & q_{p_2} & \dots & q_{p_n} \end{pmatrix}$$

is the permutation of the numbers in the array $p = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$.

Thus, $q\, p\, q^{-1}$ has the same cycle structure as p.
Conversely, the cycle structure can be used to characterize classes.

The number of elements in a class is therefore equal to the number of distinct elements with the same cycle structure.
Given a cycle structure $\left(1^{k_1} 2^{k_2} \dots i^{k_i} \dots n^{k_n}\right)$, there'll be $n!$ ways to fill it with the numbers $1 \dots n$.
Obviously, permutations within each of the sets of k_i i-cycle result in the same class element. The number of such permutations is $k_1! \dots k_n! = \prod_j k_j!$.

Furthermore, cyclic permutations within each i-cycle also results in the same class element. For an i-cycle, there are i such permutations. Hence, the total number of such permutations is $1^{k_1} \dots i^{k_i} \dots n^{k_n} = \prod_j j^{k_j}$.

Thus, the number of elements in a class is

$$\frac{n!}{k_1! \dots k_n!\; 1^{k_1} \dots i^{k_i} \dots n^{k_n}} = \frac{n!}{\prod_j k_j!\; j^{k_j}}$$

■ Example

Let

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1 \times 2) \qquad\qquad q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (12 \times 3)$$

By direct multiplication, we have:

$$q\,p\,q^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

Alternatively,

$$q\,p\,q^{-1} = q[\,p\,] = q\left[\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right] = \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_1 & q_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix} = (2 \times 3)$$

■ Notations & Properties of Cycle Structures

1. In writing a permutation in terms of products of cycles, the 1 – cycle is usually omitted, eg.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 1 & 5 & 8 & 2 & 6 \end{pmatrix} = (14)\,(237)\,(68)$$

2. The order of the cycles in the product is immaterial since they do not have common indices.

3. Any cyclic permutations of the indices of a cycle are equivalent, eg.
$(\,i\ j\ k\,) = (\,j\ k\ i\,) = (\,k\ i\ j\,) = \ldots$

4. Any transposition can be written as product of adjacent transpositions according to the following recursive formula:
$(i,\ i+v) = (i+1,\ i+v)\,(i,\ i+1)\,(i+1,\ i+v)$
eg.
For the group S_4:
$(1\times 3) = (2\times 3)\,(1\times 2)\,(2\times 3)$ $(v = 2)$
$(2\times 4) = (3\times 4)\,(2\times 3)\,(3\times 4)$ $(v = 2)$
$(1\times 4) = (2\times 4)\,(1\times 2)\,(2\times 4)$ $(v = 3)$
$= (3\times 4)\,(2\times 3)\,(3\times 4)\,(1\times 2)\,(3\times 4)\,(2\times 3)\,(3\times 4)$

5. Some useful relations:
$(\,a\,b\,c\,d\,e\,) = (\,a\,b\,)\,(\,b\,c\,d\,e\,) = (\,a\,b\,c\,)\,(\,c\,d\,e\,) = (\,a\,b\,c\,d\,)\,(\,d\,e\,) = (\,a\,b\,)\,(\,b\,c\,)\,(\,c\,d\,)\,(\,d\,e\,)$
$(a\,b\,c\,d\,e\,)^{-1} = (\,e\,d\,c\,b\,a\,)$

6. The generators of S_n are the $n-1$ adjacent transpositions
$(i,\ i+1),\ \ i = 1,\ 2\ \ldots\ n-1.$

7. The parity δ_P of a permutation is defined by
$\delta_P = (-)^N$
where N is the number of transpositions in the permutation.
The permutation is even (odd) if $\delta_P = 1\ (-1)$.

8. An n – cycle is a product of $n-1$ transpositions. (see 5.)
Thus, an n – cycle is an even (odd) permutation if n = odd (even).

9. Consider the sum of 2 integers m & n:

$$n+m = \begin{cases} \text{even} & \text{if} \quad \text{n, m are both odd or both even} \\ \text{odd} & \quad\ \ \text{1 of n, m is even, the other odd} \end{cases}$$

Hence, the product of 2 permutations p & q is

$$p\,q = \begin{cases} \text{even} & \text{if} \quad \text{p, q are both odd or both even} \\ \text{odd} & \quad\ \ \text{1 of p, q is even, the other odd} \end{cases}$$

10. The inverse of a cycle is just the cycle written in reverse order.
ie. $p = (\,i\ j \ldots l\ m\,) \longrightarrow p^{-1} = (\,m\ l\ \ldots\ j\ i\,)$
This can be proved by writing the cycle in terms of a product of transpositions.
Writing an arbitrary permutation as products of cycles, its inverse is obtained by writing each cycle in reversed order.
Thus, a permutation & its inverse have the same parity & cycle structure.

11. Similarity transforms $p\,q\,p^{-1}$ do not change cycle structures.
Elements with the same cycle structure must belong to the same class & vice versa.
A similarity transform will be even (odd) if q is even (odd) since the $p\,p^{-1}$ part is always even.

■ Example S_3

In cycle structure notations, elements of S_3 becomes:

$e = (1^3\,2^0\,3^0) = (1^3)$ $a = (1^0\,2^0\,3^1) = (3^1)$ $b = (1^0\,2^0\,3^1) = (3^1)$

$c = (1^1\,2^1\,3^0) = (1^1\,2^1)$ $d = (1^1\,2^1\,3^0) = (1^1\,2^1)$ $f = (1^1\,2^1\,3^0) = (1^1\,2^1)$

There're therefore 3 classes with cycle structures

$\{e\} = (1^3)$ $\{a, b\} = (3^1)$ $\{c, d, f\} = (1^1\,2^1)$

or, in the k notation:

$\{e\} = (3 \times 0 \times 0)$ $\{a, b\} = (0 \times 0 \times 1)$ $\{c, d, f\} = (1 \times 1 \times 0)$

The # of elements in these classes can be calculated using the formula

$$\frac{n!}{k_1! \ldots k_n!\; 1^{k_1} \ldots i^{k_i} \ldots n^{k_n}}$$

Thus

$(1^3) = (1^3\,2^0\,3^0) \longrightarrow 3!/(3! \times 0! \times 0!\; 1^3\,2^0\,3^0) = 1$

$(3^1) = (1^0\,2^0\,3^1) \longrightarrow 3!/(0! \times 0! \times 1!\; 1^0\,2^0\,3^1) = 2$

$(1^1\,2^1) = (1^1\,2^1\,3^0) \longrightarrow 3!/(1! \times 1! \times 0!\; 1^1\,2^1\,3^0) = 3$

as expected.

Since S_3 is isomorphic to C_{3v}, its character table is:

S_3	(1^3)	$2(3^1)$	$3(1^1\,2^1)$
Γ_1	1	1	1
Γ_2	1	1	−1
Γ_3	2	−1	0

5.1 1-D Representations

■ Alternating Group A_n

Since the product of 2 even permutations is even, the set of all even permutations of n objects is a group called the **alternating group** A_n.

Since an even permutation remains even under a similarity transform, A_n is an invariant subgroup of S_n with $S_n / A_n = S_2$ where

S_2	(1^2)	(2^1)
χ^1	1	1
χ^2	1	−1

Thus, there are always two 1-D representations for S_n.
The 1st is obtained by assigning 1 to all elements & is called the identity representation.

The other is obtained by assigning ± 1 to elements with $\begin{matrix}\text{even}\\\text{odd}\end{matrix}$ permutations.

This also implies that the number of even & odd permutations in S_n are equal.

Definition:

$s = \sum_p p$ is called the **symmetrizer**

$a = \sum_p (-)^p p$ is called the **anti-symmetrizer**

The left coset decomposition of S_n is

$$S_n = \sum_{i=1}^{k} R_i S_2 \qquad k = \frac{n_{S_n}}{n_{S_2}} = \frac{n!}{2}$$

- **Theorem:** ***s*** **&** ***a*** **are essentially idempotent & primitive**

- **Proof:**

Rearrangement theorem $\longrightarrow$

$$q s = q \sum_p p = \sum_{p'} p' = s = s q \qquad \forall\ q \in S_n$$

$\therefore$ $s s = \sum_p p s = \sum_p s = n!\, s \qquad \longrightarrow \qquad s$ is essentially idempotent.

$s q s = s s = n!\, s \qquad \longrightarrow \qquad s$ is primitive

Similarly:

$$q a = q \sum_p (-)^p p = \sum_{p'} (-)^{p'+q} p' = (-)^q a = a q$$

$\therefore$ $a a = \sum_p (-)^p p a = \sum_p (-)^{p+p} a = \sum_p a = n!\, a \qquad \longrightarrow \qquad a$ is essentially idempotent.

$a q a = a (-)^q a = (-)^q n!\, a \qquad \longrightarrow \qquad a$ is primitive

Since there are equal number of even & odd permutations in S_n,

$$s a = \sum_p p a = \sum_p (-)^p a = 0$$

Hence:

$s q a = s (-)^q a = s a = 0 \qquad \longrightarrow \qquad s$ & a are inequivalent.

5.2 Partitions & Young Diagrams

- **Partition of *n***

A **partition** λ of an integer n is a set $\lambda = \{ \lambda_1 \ldots \lambda_r \}$ such that

1. λ_i = positive integers.

2. $\sum_{i=1}^{r} \lambda_i = \text{n}$

3. $\lambda_i \geq \lambda_{i+1}$ (decending order)

Some basic definitions:

a. $\lambda = \mu \qquad \Longleftrightarrow \qquad \lambda_i = \mu_i \qquad \forall\ i$

b. $\lambda > \mu$ if 1st non-zero number in the sequence $\{\lambda_i - \mu_i\}$ is positive.

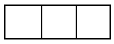
$\lambda < \mu$ if negative.

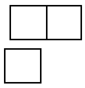
Young Diagram

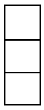
The **Young diagram** of a partition $\lambda = \{\lambda_1 \dots \lambda_r\}$ is an arrangement of n squares into r rows such that the i th row has λ_i squares.

■ Example   Partition of 3

Possible partitions are $\{3\}$, $\{2, 1\}$, $\{1, 1, 1\} = \{1^3\}$.
The corresponding Young diagrams are:

$\{3\}$ $\longrightarrow$ 

$\{2, 1\}$ $\longrightarrow$ 

$\{1, 1, 1\}$ $\longrightarrow$ 

■ Theorem: Young Diagrams & IRs of S_n

of distinct Young diagrams of partition of n = # of classes of S_n
= # of IR's of S_n.

Given a Young diagram:
Each column represents a cycle.
# of vertical blocks in each column is equal to the length of the cycle.

■ Proof:

Every class of S_n is characterized by its cycle structure $\left(1^{k_1} \dots i^{k_i} \dots n^{k_n}\right)$ where k_j is the number of j – cycles.
Since the ' total length ' of the cycles must be n , we have

$$n = \sum_j j\, k_j = k_1 + 2\, k_2 + \dots + n\, k_n$$

$$= (k_1 + k_2 + \dots + k_n) + (k_2 + \dots + k_n) + \dots + (k_{n-1} + k_n) + k_n$$

$$= \sum_{j=1}^{n} \sum_{i=j}^{n} k_i$$

$$= \sum_{j=1}^{n} \lambda_j$$

where

$$\lambda_j = \sum_{i=j}^{n} k_i \qquad j = 1 \dots n$$

Since k_j are all non-negative integers, $\lambda_j \le \lambda_{j-1}$ $\forall$ $j = 2 \dots n$.
Thus, $\{\lambda_j\}$ is a partition of n.

Specifically,

$\lambda_1 = k_1 + k_2 + \ldots + k_n$

= # of cycles in each permutation.

= # of blocks in the 1st row of the Young diagram.

# of blocks in the 1st column = # of non-zero λ_j 's

= length of the longest cycle

The reason for the last statement is as follows:

Let m be the length of the longest cycle.

This implies λ_m is the last non-zero number in the partition.

ie. $\lambda_m = \sum_{i=m}^{m} k_i = k_m$

Obviously,

$\lambda_j \neq 0 \qquad \forall\ j \leq m$

$\lambda_j = 0 \qquad \forall\ j > m$

Thus, the # of blocks in the 1st and longest column is just m.

By the same token, we see that

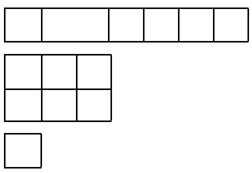
# of blocks in the 2nd column = length of the 2nd longest cycle

or, in general,

# of blocks in each column = length of a cycle

To summarize, given a cycle structure, the corresponding Young diagram is constructed by writing each $j-$ cycle present as a column of j blocks. The columns are placed adjacently with longer ones to the left of shorter ones.

For example, the class $\left(1^3\, 3^2\, 4^1\right) = \left(1^3\, 2^0\, 3^2\, 4^1\right)$ with $k = (3 \times 0 \times 2 \times 1)$ corresponds to the following Young diagram with $\lambda = \{6, 3, 3, 1, 0, \ldots, 0\}$.



6
3
3
1

4 3 3 1 1 1

$\left(1^3\, 2^0\, 3^2\, 4^1\right)$

Thus, there is a $1-1$ correspondence between the partitions (Young diagrams) of n to the classes of S_n.
The number of distinct Young diagrams is therefore equal to the number of classes of S_n.

- **Example:** S_3

The cycle structures of the classes of S_3 are

$\{e\} = \left(1^3\right) \qquad \{a, b\} = \left(3^1\right) \qquad \{c, d, f\} = \left(1^1\, 2^1\right)$

With $k = (k_1 \ldots k_n)$, we have

$k = (3, 0, 0)$ for $\{e\}$

$k = (0, 0, 1)$ for $\{a, b\}$

$k = (1, 1, 0)$ for $\{c, d, f\}$

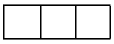
Thus, the partitions $\lambda = \left\{\lambda_j = \sum_{i=j}^{n} k_i\right\}$ are

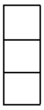
$\lambda = (3, 0, 0)$ for $\{e\}$

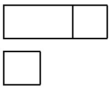
$\lambda = (1, 1, 1)$ for $\{a, b\}$

$\lambda = (2, 1, 0)$ for $\{c, d, f\}$

The corresponding Young diagrams are:

 for $\{e\} = (1^3)$

 for $\{a, b\} = (3^1)$

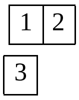
 for $\{c, d, f\} = (1^1\, 2^1)$

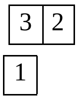
Note that given n , its partitions are easily constructed.
So are the Young diagrams.
The classes & their cycle structures of S_n can then be read off directly from the Young diagrams.

■ Young Tableau

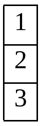
When the squares of a Young diagram are filled with the numbers $\{1, 2, \ldots, n\}$, it becomes a **Young tableau** θ_λ.
Since each Young diagram represents a class of S_n, an element (permutation) in the class is just the cyclic permutations of the numbers within each column in the Young tableau. In this sense, each Young tableau is an element of a class in S_n if we treat tableaux that differ by mere cyclic permutations in any column as equivalent.

Using S_3 as an example,

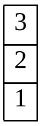


or


represents $(1 \times 3) = d \in (1^1\, 2^1)$



or


or


represents $(1 \times 2 \times 3) = b \in (3^1)$

Obviously, permutations of numbers in a row or non-cyclic permutations within a column will in general result in a different member of the class.

When the numbers are filled sequentially from top left to right bottom, the resultant Young tableau is called **normal** & denoted by Θ_λ.

If the magnitudes of the numbers in every row increase from left to right, & those in every column increase from top to bottom, the tableau is called **standard**.
We state without proof that the number of distinct standard tableaux of a Young diagram is equal to the dimension of the IR generated by that diagram.

Any Young tableau θ_λ can be obtained from Θ_λ by an appropriate permutation p.

This is denoted by $\theta_\lambda = \Theta_\lambda^p = p\,\Theta_\lambda$.

Obviously $q\,\Theta_\lambda^p = q\,p\,\Theta_\lambda = \Theta_\lambda^{q\,p}$.

- **Example** **Partition of 3**

Normal	1	2
	3	
Standard	1	3
	2	

5.3 Symmetrizers & Anti-Symmetrizers of Young Tableau

- **Horizontal Permutations** $\{h_\lambda^p\}$

The horizontal permutations $\{h_\lambda^p\}$ of a Young tableau Θ_λ^p are permutations which leaves invariant the sets of numbers in each row of Θ_λ^p.

- **Vertical Permutations** $\{v_\lambda^p\}$

The vertical permutations $\{v_\lambda^p\}$ of a Young tableau Θ_λ^p are permutations which leaves invariant the sets of numbers in each column of Θ_λ^p.

- **Symmetrizers** s_λ^p

The symmetrizer s_λ^p of a Young tableau Θ_λ^p is the sum of all horizontal permutations h_λ^p:

$$s_\lambda^p = \sum_h h_\lambda^p$$

- **Anti–Symmetrizers** a_λ^p

The anti – symmetrizer a_λ^p of a Young tableau Θ_λ^p is the signed sum of all vertical permutations v_λ^p:

$$a_\lambda^p = \sum_v (-)^v v_\lambda^p$$

- **Irreducible / Young Symmetrizers** e_λ^p

The irreducible symmetrizer e_λ^p of a Young tableau Θ_λ^p is the sum of the products of the horizontal & vertical symmetrizers.

$$e_\lambda^p = s_\lambda^p a_\lambda^p = \sum_{h\,v} (-)^v h_\lambda^p v_\lambda^p$$

■ Example S_3

In cycle notations, elements of S_3 are:

$e = e \qquad a = (123) \qquad b = (132)$

$c = (23) \qquad d = (13) \qquad f = (12)$

The classes of S_3 have cycle structures:

$\{e\} = (1^3) \qquad \{a, b\} = (3^1) \qquad \{c, d, f\} = (1^1\, 2^1)$

The corresponding partitions are found using the formula

$$\lambda_i = \sum_{j=i}^{n} k_j \qquad k_j = \#\text{ of } j-\text{cycle}$$

For class $\{e\}$:

$k = \{k_1, k_2, k_3\} = \{3, 0, 0\}$

$\lambda = \{\lambda_1, \lambda_2, \lambda_3\} = \{3+0+0,\ 0+0,\ 0\} = \{3, 0, 0\} = \{3\}$

For class $\{a, b\}$:

$k = \{k_1, k_2, k_3\} = \{0, 0, 1\}$

$\lambda = \{\lambda_1, \lambda_2, \lambda_3\} = \{0+0+1,\ 0+1,\ 1\} = \{1, 1, 1\}$

For class $\{c, d, f\}$:

$k = \{k_1, k_2, k_3\} = \{1, 1, 0\}$

$\lambda = \{\lambda_1, \lambda_2, \lambda_3\} = \{1+1+0,\ 1+0,\ 0\} = \{2, 1, 0\} = \{2, 1\}$

The **normal Young tableau** are:

$\Theta_1 =$ | 1 | 2 | 3 | :

$h_\lambda: \quad \{e, a, b, c, d, f\}$

$v_\lambda: \quad e$

$$s_1 = \sum_p p = e + a + b + c + d + f \equiv S$$

$a_1 = e$

$e_1 = s_1\, a_1 = S$

$\Theta_2 =$ | 1 | 2 |
| 3 |

$h_\lambda: \quad \{e, f = (12)\}$

$v_\lambda: \quad \{e, d = (13)\}$

$s_2 = e + f = e + (12)$

$a_2 = e - d = e - (13)$

$e_2 = s_2\, a_2 = (e + f)(e - d) = e - d + f - f\, d = e - d + f - b$

$\quad = e - (13) + (12) - (132)$

$$\Theta_3 = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

h_λ : e
v_λ : $\{e,\ a,\ b,\ c,\ d,\ f\}$
$s_3 = e$
$a_3 = \sum_p (-)^p\, p = e - a - b + c + d + f \equiv A$
$e_3 = A = e - a - b + c + d + f$

The only **standard Young tableau** is:

$$\Theta_2^c = \Theta_2^{(23)} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

h_λ : $\{e,\ d = (13)\}$
v_λ : $\{e,\ f = (12)\}$
$s_2 = e + d = e + (13)$
$a_2 = e - f = e - (12)$
$e_2 = s_2\, a_2 = (e + d)(e - f) = e + d - f - d\, f = e + d - f - a$
$= e + (13) - (12) - (123)$

5.4 IR of S_n

■ $\{h_\lambda\}$ & $\{v_\lambda\}$ are subgroups of S_n

See Lemma IV.2 , Tung

e must belong to $\{h_\lambda\}$.
h_λ^{-1} must be a horizontal permutations & hence in $\{h_\lambda\}$.
Products of two horizontal permutations must be another horizontal permutations & hence in $\{h_\lambda\}$.
Therefore, $\{h_\lambda\}$ is a group.
Since h_λ are permutations of n numbers, they must be elements of S_n.
Thus $\{h_\lambda\}$ is a subgroup of S_n.

Same argument is applicable to $\{v_\lambda\}$.

■ s_λ & a_λ are essentially idempotent

See Lemma IV.2 & p.67–8, Tung.

Using the rearrangement theorem on the subgroups $\{h_\lambda\}$ & $\{v_\lambda\}$, we have

$$s_\lambda h_\lambda = \sum_{h'} h_\lambda' h_\lambda = \sum_{h''} h_\lambda'' = s_\lambda \qquad \text{where } h_\lambda'' = h_\lambda' h_\lambda$$

Similarly,

$$h_\lambda s_\lambda = s_\lambda.$$

$$a_\lambda v_\lambda = \sum_{v'} (-)^{v_\lambda'} v_\lambda' v_\lambda = \sum_{v''} (-)^{v_\lambda'' + v_\lambda} v_\lambda'' = (-)^{v_\lambda} a_\lambda \qquad \text{where } v_\lambda'' = v_\lambda' v_\lambda$$

Similarly,

$$v_\lambda a_\lambda = (-)^{v_\lambda} a_\lambda$$

$$s_\lambda s_\lambda = \sum_h h_\lambda \sum_{h'} h_\lambda' = \sum_{h h''} h_\lambda'' = \sum_h s_\lambda = n_\lambda s_\lambda$$

$$a_\lambda a_\lambda = \sum_v (-)^{v_\lambda} v_\lambda \sum_{v'} (-)^{v_\lambda'} v_\lambda' = \sum_{v, v''} (-)^{v_\lambda''} v_\lambda'' = \sum_v a_\lambda = m_\lambda a_\lambda$$

where n_λ = # of horizontal permutations in θ_λ & m_λ = # of vertical permutations in θ_λ.

Now, the diagram θ_λ consists of rows of λ_j blocks. Thus the # of horizontal permutations n_λ is simply

$$n_\lambda = \lambda_1! \ldots \lambda_n!$$

The diagram θ_λ consists of k_j columns of j blocks. Thus the # of vertical permutations m_λ is

$$m_\lambda = (1!)^{k_1} \ldots (j!)^{k_j} \ldots (n!)^{k_n}$$

To summarize,

$$s_\lambda h_\lambda = h_\lambda s_\lambda = s_\lambda \qquad s_\lambda s_\lambda = n_\lambda s_\lambda \qquad n_\lambda = \lambda_1! \ldots \lambda_n!$$

$$a_\lambda v_\lambda = v_\lambda a_\lambda = (-)^{v_\lambda} a_\lambda \qquad a_\lambda a_\lambda = m_\lambda a_\lambda \qquad m_\lambda = (1!)^{k_1} \ldots (n!)^{k_n}$$

Hence, s_λ & a_λ are essentially idempotent.

■ $x_\lambda^p = p\, x_\lambda\, p^{-1}$ $\qquad x = h, v, s, a, e$

See Lemma IV.1 , Tung.

Consider a tableau θ_λ & its permutation $\theta_\lambda^p = p\,\theta_\lambda$.

An operation x_λ on θ_λ gives a tableau $\theta_\lambda' = x_\lambda \theta_\lambda$.

The problem is to find the corresponding operation x_λ^p on θ_λ^p so that $\theta_\lambda^p{}' = x_\lambda^p \theta_\lambda^p$ is the permutation of θ_λ', ie., $\theta_\lambda^p{}' = p\,\theta_\lambda'$.

Thus

$$p\,\theta_\lambda' = p\, x_\lambda \theta_\lambda = p\, x_\lambda\, p^{-1}\, p\,\theta_\lambda = p\, x_\lambda\, p^{-1}\, \theta_\lambda^p$$

Setting this to

$$\theta_\lambda^p{}' = x_\lambda^p \theta_\lambda^p$$

We have

$$x_\lambda^p = p\, x_\lambda\, p^{-1}.$$

- **Example**

Let $p = (123)$

$$\theta_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \longrightarrow \quad \theta_\lambda^p = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Let $x_\lambda = h_\lambda = (12)$

$\longrightarrow$

$$h_\lambda \, \theta_\lambda = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$h_\lambda^p = p \, h_\lambda \, p^{-1} = p\left[\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right] = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix} = (23)$$

$$h_\lambda^p \, \theta_\lambda^p = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} = \qquad p \, h_\lambda \, \theta_\lambda = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

- **$p \neq h_\lambda \, v_\lambda \quad \Longleftrightarrow$**
  $\exists$ 2 nos in 1 row of θ_λ = those in 1 column of θ_λ^p.

See Lemma IV.3 , Tung

- **Proof** $\Longleftarrow$

Assume $\quad p = h_\lambda \, v_\lambda$.

$\longrightarrow \quad \theta_\lambda^p = p \, \theta_\lambda = h_\lambda \, v_\lambda \, \theta_\lambda = h_\lambda \, v_\lambda \, h_\lambda^{-1} \, h_\lambda \, \theta_\lambda = v_{\lambda \lambda}^h \, \theta_{\lambda \lambda}^h$

where $v_{\lambda\lambda}^h = h_\lambda \, v_\lambda \, h_\lambda^{-1}$ is the vertical permutation on tableau $\theta_{\lambda\lambda}^h$ that corresponds to v_λ on θ_λ.

Consider 2 numbers a & b in the same row of θ_λ.
They will remain in the same row in $\theta_{\lambda\lambda}^h$ since h_λ is a permutation of numbers in each row of θ_λ.
Thus, a & b must be in different columns of $\theta_{\lambda\lambda}^h$.
The operation $v_{\lambda\lambda}^h$ is a permutation of numbers in each column of $\theta_{\lambda\lambda}^h$.
Hence, a & b must remain in different columns of θ_λ^p.

To summarize,

$p = h_\lambda \, v_\lambda \quad \Longrightarrow$ 2 numbers in the same row of θ_λ must be in different columns of θ_λ^p.

Taking the negation,

$p \neq h_\lambda \, v_\lambda \quad \Longleftarrow$ At least 2 numbers in the same row of θ_λ appear in same columns of θ_λ^p.

- **Proof** $\Longrightarrow$

Let every number in the same row of θ_λ be in different columns of θ_λ^p.
Tableau θ_λ^p can be obtained from θ_λ in the following manner.

Starting with the numbers in the 1st column of θ_λ^p, they must belong to different rows in θ_λ.
Hence, they can be brought to the 1st column by a horizontal permutation on θ_λ.
The same procedure can be applied to all columns of θ_λ^p.
Subsequently, we have a tableau $\theta_{\lambda\lambda}^h = h_\lambda \theta_\lambda$ each of whose columns contains the same numbers of the corresponding columns in θ_λ^p.
Obviously, $\theta_{\lambda\lambda}^h$ can be brought to θ_λ^p by pure vertical permutations, ie.,

$$\theta_\lambda^p = v_{\lambda\lambda}^h \theta_{\lambda\lambda}^h = h_\lambda v_\lambda h_\lambda^{-1} h_\lambda \theta_\lambda = h_\lambda v_\lambda \theta_\lambda = p \theta_\lambda$$

$\longrightarrow \quad p = h_\lambda v_\lambda$

To summarize:

Every number in the same row of θ_λ be in different columns of $\theta_\lambda^p \quad \Longrightarrow \quad p = h_\lambda v_\lambda$

Taking the negation:

At least 2 numbers in the same row of θ_λ appear in same columns of $\theta_\lambda^p \quad \Longleftarrow \quad p \neq h_\lambda v_\lambda$

- **$p \neq h_\lambda v_\lambda \quad \Longrightarrow$**
  $\exists\ \tilde{h}_\lambda \ \& \ \tilde{v}_\lambda \ \ni \ p = \tilde{h}_\lambda \ p \ \tilde{v}_\lambda$

$p \neq h_\lambda v_\lambda \quad \Longrightarrow$ At least 2 numbers in the same row of θ_λ appear in same columns of θ_λ^p

Let the transposition of these 2 numbers be t.

$\longrightarrow \quad t \in \{ h_\lambda \} \quad \& \quad t \in \left\{ v_\lambda^p \right\}$

Thus we can write

$$t = \tilde{h}_\lambda = \tilde{v}_\lambda^p$$

where $\tilde{h}_\lambda$ is the transition in $\{ h_\lambda \}$ that equals to t & simularly for $\tilde{v}_\lambda^p$.

Using $\quad \tilde{v}_\lambda^p = p \tilde{v}_\lambda p^{-1} \quad$ where $\tilde{v}_\lambda \in \{ v_\lambda \}$

we have

$$\tilde{h}_\lambda \ p \ \tilde{v}_\lambda = \tilde{h}_\lambda \ p \ p^{-1} \tilde{v}_\lambda^p \ p = \tilde{h}_\lambda \ \tilde{v}_\lambda^p \ p = t^2 \ p = p$$

since any transition is its own inverse, ie. $t^2 = e$.

- **$h_\lambda r v_\lambda = (-)^{v_\lambda} r \ \forall h_\lambda, v_\lambda$**
  $\Longrightarrow \quad r \propto e_\lambda$

Let $\quad r = \sum_{p \in G} \alpha_p \, p \qquad G = S_n$

$h_\lambda r v_\lambda = (-)^{v_\lambda} r \quad \longrightarrow \quad \sum_{p \in G} \alpha_p h_\lambda p v_\lambda = (-)^{v_\lambda} \sum_{p \in G} \alpha_p \, p$

Now,

$$\sum_{p \in G} \alpha_p h_\lambda p v_\lambda = \sum_{q \in G} \left(\alpha_{h_\lambda^{-1} q v_\lambda^{-1}} \right) q \qquad \text{where } q = h_\lambda p v_\lambda \text{ so that } p = h_\lambda^{-1} q v_\lambda^{-1}$$

$$= (-)^{v_\lambda} \sum_{q \in G} \alpha_q \, q$$

$\longrightarrow \quad \alpha_{h_\lambda^{-1} q v_\lambda^{-1}} = (-)^{v_\lambda} \alpha_q \qquad \forall h_\lambda \ \& \ v_\lambda$

If $q \neq h_\lambda' v_\lambda'$

$\longrightarrow$ $\exists$ transpositions $\tilde{h}_\lambda$ & $\tilde{v}_\lambda \ni q = \tilde{h}_\lambda\, q\, \tilde{v}_\lambda$

$\alpha_{\tilde{h}_\lambda^{-1} q \tilde{v}_\lambda^{-1}} = \alpha_q = (-)^{\tilde{v}_\lambda} \alpha_q = -\alpha_q$ where $(-)^{\tilde{v}_\lambda} = -1$ for transpositions.

Thus $\alpha_q = 0$.

If $q = h_\lambda' v_\lambda'$

$\alpha_{h_\lambda'^{-1} q v_\lambda'^{-1}} = \alpha_e = (-)^{v_\lambda'} \alpha_q$

ie. $\alpha_q = (-)^{v_\lambda'} \alpha_e = (-)^{v_\lambda'} \xi$ where $\xi = \alpha_e$ is a constant independent of q.

Thus

$$r = \sum_{p \in G} \alpha_p\, p = \xi \sum_{h_\lambda v_\lambda} (-)^{v_\lambda} h_\lambda v_\lambda = \xi\, e_\lambda$$

■ $x_\mu^p x_\nu^q \propto \delta_{\mu\nu}$ $x = s, a, e$

See Lemma IV.6 , Tung

Without loss of generality, we can assume $\mu > \nu$, ie. the 1st non-zero $\mu_i - \nu_i$ is positive.
Since tableau θ_μ^q has rows of lengths $\{\mu_1 \ldots \mu_n\}$ & similarly for θ_ν^p , the longer rows in both tableau are equal in length until the i th row corresponding to the 1st non-zero $\mu_i - \nu_i$. The i th row of θ_μ^q will be longer than any row in θ_ν^p with label $j \geq i$.

Consider 1st the normal tableaux Θ_μ & Θ_ν. The i th in Θ_μ will contain all the numbers in the i th row of Θ_ν plus at least 1 extra number, say a. Now, a must appear in a row of Θ_ν with label $j \geq i$.
In other words, there must be a number b in the i th row of Θ_μ such that it is in the same column with a in Θ_ν.

Since $\theta_\mu^q = q\,\Theta_\mu$, $\theta_\nu^q = q\,\Theta_\nu$, we see that the numbers q_a & q_b will be in the same row of θ_μ^q & the same column in θ_ν^q. So do any $h_\mu \theta_\mu^q = \theta_\mu^{h_\mu q}$ & $v_\nu \theta_\nu^q = \theta_\nu^{v_\nu q}$.

Now, given $\theta_\mu^q = q\,\Theta_\mu$, $\theta_\nu^p = p\,\Theta_\nu$, there exists h_μ , v_ν , r such that $q = h_\mu r$ & $p = v_\nu r$.
Hence, $\theta_\mu^q = h_\mu \theta_\mu^r$ & $\theta_\nu^p = v_\nu \theta_\nu^r$ so that there is at least 2 numbers appearing in both in a row of θ_μ^q & a column of θ_ν^p.

Now, let the transposition of these 2 numbers be $t = \tilde{h}_\mu = \tilde{v}_\nu$.

$\longrightarrow$ $t\, s_\mu^q = s_\mu^q\, t = s_\mu^q$

$t\, a_\nu^p = a_\nu^p\, t = (-)^t a_\nu^p = -a_\nu^p$

Hence

$a_\mu^p s_\nu^q = a_\mu^p\, t\; t\; s_\nu^q = -a_\mu^p s_\nu^q = 0$

$s_\nu^q a_\mu^p = s_\nu^q\; t\; t\; a_\mu^p = -s_\nu^q a_\mu^p = 0$

$e_\mu^q e_\nu^p = s_\mu^q a_\mu^q\; s_\nu^p a_\nu^p = 0$

■ $s_\lambda\, r\, a_\lambda = \xi_r\, e_\lambda$

See Theorem 5.3, Tung

Let $u = s_\lambda\, r\, a_\lambda$

$h_\lambda\, u\, v_\lambda = h_\lambda\, s_\lambda\, r\, a_\lambda\, v_\lambda = s_\lambda\, r\, a_\lambda\, (-)^{v_\lambda} = u\,(-)^{v_\lambda}\ \forall\, h_\lambda, v_\lambda$

$\longrightarrow$ $u = s_\lambda\, r\, a_\lambda = \xi_r\, e_\lambda$

$e_\lambda{}^2 = \eta\, e_\lambda$

See Theorem 5.3, Tung

$$e_\lambda{}^2 = s_\lambda\, a_\lambda\, s_\lambda\, a_\lambda = \xi_{a_\lambda s_\lambda}\ e_\lambda = \eta\ \ e_\lambda$$

Thus, e_λ is essentially idempotent.

- **e_λ are primitive idempotents**

See Theorem 5.4, Tung

$$e_\lambda\, r\, e_\lambda = s_\lambda\, a_\lambda\, r\, s_\lambda\, a_\lambda = \xi_{a_\lambda r s_\lambda}\, e_\lambda$$

Hence, e_λ are primitive idempotents.

- **IRs generated by e_λ & e_λ^p are equivalent**

IRs generated by 2 primitive idempotents e_1 & e_2 are equivalent iff $e_1\, r\, e_2 \neq 0$ for some $r \in \mathcal{G}$.

Now, using $e_\lambda^p = p\, e_\lambda\, p^{-1}$, we have

$$e_\lambda\, p^{-1}\ e_\lambda^p = e_\lambda\, p^{-1}\ p\, e_\lambda\, p^{-1} = e_\lambda{}^2\ p^{-1} = \eta\, e_\lambda\, p^{-1} \neq 0$$

- **IRs generated by e_μ & e_ν are inequivalent if $\mu \neq \nu$**

$$\forall\ p \in G \qquad e_\mu\, p\, e_\nu = e_\mu\, p\, e_\nu\, p^{-1}\, p = e_\mu\, e_\nu^p\ p\ = 0 \qquad \text{if} \qquad \mu \neq \nu$$

$$\forall\ r \in \mathcal{G} \qquad r = \sum_p c_p\ p$$

$$\longrightarrow \qquad e_\mu\, r\, e_\nu = 0$$

Hence, the IRs generated by e_μ & e_ν are inequivalent.

- **e_λ of all normal Young tableaux generate all IRs of the group**

See Theorem 5.7, Tung

Since

1. \# of Young diagrams = \# of normal Young tableaux
   = \# of classes
   = \# of inequivalent IRs.
2. There's one e_λ for each normal tableau Θ_λ.
3. IRs generated by e_λ & e_μ are inequivalent if $\lambda \neq \mu$.

Hence, the set of all e_λ's generates all inequivalent IRs of S_n.

- **Example: S_3**

As shown before, the partitions of 3 are

$$\lambda = \{3\},\ \{2,\ 1\},\ \{1,\ 1,\ 1\}$$

For $\lambda = \{3\}$, there is only one standard tableau Θ_1
Hence, the IR is $1 - D$.
The Young symmetrizer is

$$e_1 = S$$

Since $e_1\, g = e_1 \;\; \forall \;\; g \in S_3$, the IR is the identity representation, as expected.

For $\lambda = \{2,\; 1\}$, there are 2 standard tableaux Θ_2 & $\Theta_2^{(23)}$.
The IRs are $2 - D$.
For Θ_2, the Young symmetrizer is

$$e_2 = e - (13) + (12) - (132)$$

■ e_λ of all standard tableaux generate the complete IR decomposition of the Regular Representation

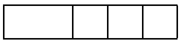
The # of standard tableaux for each Young diagram is equal to the dimension of the IR generated from it. Since $\Gamma^{\text{reg}} = \sum_\mu n_\mu \, \Gamma^\mu$, the theorem is proved.

■ S_4

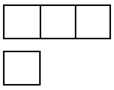
The partitions of 4 are

$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

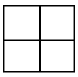
The corresponding Young diagrams & cycle structures are are:

$\lambda = \{4\}$  $\left(1^4\right)$ $k = (4 \times 0 \times 0 \times 0)$

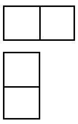
# of elements: $4\,! \big/ 1^4 \times 4\,! = 1$
Elements: e

$\lambda = \{3 \times 1\}$  $\left(1^2\, 2^1\right)$ $k = (2 \times 1 \times 0 \times 0)$

# of elements: $4\,! \big/ 1^2\;\; 2\;\; 2\,! = 6$
Elements: (1×2), (1×3), (1×4), (2×3), (2×4), (3×4)

$\lambda = \left\{2^2\right\}$  $\left(2^2\right)$ $k = (0 \times 2 \times 0 \times 0)$

# of elements: $4\,! \big/ 2^2\;\; 2\,! = 3$
Elements: $(1 \times 2)\,(3 \times 4)$, $(1 \times 3)\,(2 \times 4)$, $(1 \times 4)\,(2 \times 3)$

$\lambda = \left\{2 \times 1^2\right\}$  $\left(1^1\, 3^1\right)$ $k = (1 \times 0 \times 1 \times 0)$

# of elements: $4\,! / 3 = 8$
Elements: $(1 \times 2 \times 3)$, $(1 \times 2 \times 4)$, $(1 \times 3 \times 2)$, $(1 \times 3 \times 4)$, $(1 \times 4 \times 2)$, $(1 \times 4 \times 3)$, $(2 \times 3 \times 4)$, $(4 \times 3 \times 2)$

$\lambda = \{ 1^4 \}$ $\quad \begin{array}{|c|} \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array} \quad$ (4^1) $\quad k = (0 \times 0 \times 0 \times 1)$

\# of elements: $4\,!\,/\,4 = 6$

Elements: $(1 \times 2 \times 3 \times 4)$, $(1 \times 2 \times 4 \times 3)$, $(1 \times 3 \times 2 \times 4)$, $(1 \times 3 \times 4 \times 2)$, $(1 \times 4\ 2 \times 3)$, $(1 \times 4 \times 3 \times 2)$

Thus, there are 5 classes, namely, (1^4), $(1^2\, 2^1)$, (2^2), $(1^1\, 3^1)$, & (4^1).

$\lambda = \{ 4 \}$

There's only 1 standard tableaux:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

$\{ h_\lambda \} = S_n \qquad s_\lambda = \sum_g g$

$\{ v_\lambda \} = e \qquad a_\lambda = e$

$e_\lambda = s_\lambda$

Since $e_\lambda\, g = e_\lambda$, this gives the identity representation.

$\lambda = \{ 3 \times 1 \}$

There are 3 standard tableaux so that the IRs are $3 - D$.

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

For the normal tableau:

$\{ h_\lambda \} = e,\ (1 \times 2 \times 3),\ (1 \times 3 \times 2),\ (1 \times 2),\ (1 \times 3),\ (2 \times 3)$

$s_\lambda = e + (1 \times 2 \times 3) + (1 \times 3 \times 2) + (1 \times 2) + (1 \times 3) + (2 \times 3)$

$\{ v_\lambda \} = e,\ (1 \times 4)$

$a_\lambda = e - (1 \times 4)$

$$\begin{aligned} e_\lambda &= \{ e + (1 \times 2 \times 3) + (1 \times 3 \times 2) + (1 \times 2) + (1 \times 3) + (2 \times 3) \} \{ e - (1 \times 4) \} \\ &= e + (1 \times 2 \times 3) + (1 \times 3 \times 2) + (1 \times 2) + (1 \times 3) + (2 \times 3) \\ &\quad - (1 \times 4) - (1 \times 4 \times 2 \times 3) - (1 \times 4 \times 3 \times 2) - (1 \times 4 \times 2) - (1 \times 4 \times 3) - (2 \times 3)(1 \times 4) \end{aligned}$$

5.5 Symmetry Classes of Tensors

Let V_m be a m-D vector space.

The set $\{g\}$ of all invertible linear transformations on V_m forms a group called the general linear group GL(m, C).

In this chapter, we call it simply G_m.

Given any basis $\{ \,|\, i \,\rangle,\ i = 1, 2, \ldots, m \}$ on V_m, a natural matrix rep of G_m is:

(5.5-1) $\qquad g \,|\, i \,\rangle = |\, j \,\rangle\, g^j{}_i$

where $g^j{}_i$ are elements of an invertible $m \times m$ matrix (i.e. $\det g \neq 0$).

■ Definition 5.5 (**Tensor** Space):

The direct product space $V_m \times V_m \times \ldots \times V_m$, involving n factors of V shall be referred to as the **tensor** space and denoted by $V_m{}^n$.

✸

Given a basis $\{ | \, i \rangle \}$ on V_m, a natural basis for $V_m{}^n$ is obtained in the form:

(5.5-2) $| \, i_1 \, i_2 \ldots i_n \, \rangle = | \, i_1 \rangle | \, i_2 \rangle \ldots | \, i_n \rangle$

$= | \, \mathrm{i} \rangle_n$ Tung's notation

$= | \, I \rangle$ My notation

An arbitrary element x of the tensor space $V_m{}^n$ has the decomposition,

(5.5-3) $| \, x \, \rangle = | \, i_1 \, i_2 \ldots i_n \, \rangle \, x^{i_1 \, i_2 \cdots i_n}$

(5.5-4) $= | \, I \rangle \, x^I$

where $x^I = x^{i_1 \, i_2 \cdots i_n}$ are the **tensor components** of x.

Elements of G_m (defined on V_m) induce the following linear transformations on $V_m{}^n$

(5.5-5) $g | \, I \, \rangle = | \, J \rangle \, D(g)^J{}_I$

where

(5.5-6) $D(g)^J{}_I = g^{j_1}{}_{i_1} \, g^{j_2}{}_{i_2} \ldots g^{j_n}{}_{i_n} \qquad \forall \; g \in G_m$

It can easily be verified that $\{D(g)\}$ forms a $(n \cdot m)$-D rep of G_m, and that for any $| \, x \, \rangle \in V_m{}^n$,

(5.5-7) $g | \, x \, \rangle = | \, x_g \, \rangle = | \, J \, \rangle \, x_g{}^J$

where

(5.5-8) $x_g{}^J = D(g)^J{}_I \; x^I$

Independently, S_n also has a natural realization on $V_m{}^n$.
In particular, consider the mapping

$p \in S_n \longrightarrow p =$ linear transformation on $V_m{}^n$

defined by,

(5.5-9) $p | \, x \, \rangle = | \, x_p \, \rangle$

where $| \, x \, \rangle$, $| \, x_p \, \rangle \in V_m{}^n$ and

(5.5-10) $x_p{}^I = x^{I_p}$

with

$I = i_1 \, i_2 \ldots i_n$

$I_p \equiv i_{p_1} \, i_{p_2} \ldots i_{p_n}$

In terms of the basis vectors $\{ | \, I \rangle \}$, the action of p goes as

(5.5-11)

$p | \, I \rangle = | \, I_{p^{-1}} \rangle$

where

$I_{p^{-1}} = i_{p^{-1}{}_1} \, i_{p^{-1}{}_2} \ldots i_{p^{-1}{}_n}$

Therefore, if we write

(5.5-12) $p | \, I \, \rangle = | \, J \rangle \, D(p)^J{}_I$

then

(5.5-13) $D(p)^J{}_I = \delta_{I^J_{p^{-1}}} = \delta^J_{I \, p}$

$= \delta^{j}_{i_{p^{-1}{}_1}{}_1} \ldots \delta^{j}_{i_{p^{-1}{}_n}{}_n} = \delta^{j}_{i_{1 \, p_1}} \ldots \delta^{j}_{i_{n \, p_n}}$

The last equality involves permuting the n δ-factors by p. The reader should verify that Eq. (5.5-9), or equivalently (5.5-11), does provide a rep for S_n. [Problem 5.5]

Both $D[G_m]$ and $D[S_n]$ are in general reducible.

As S_n is a finite group, $D[S_n]$ can be decomposed into IRs.
This will be achieved through the e_μ's.

G_m is an infinite group.
A general $D[G_m]$ is not guaranteed to be fully decomposible.
However, the reduction of $V_m{}^n$ by e_μ's from the $\mathcal{S}_n$ algebra leads naturally to a full decomposition of $D[G_m]$.
This is a consequence of the fact that linear transformations on $V_m{}^n$ representing $\{ g \in G_m \}$ and $\{ p \in S_n \}$ commute with each other, and each type of operator constitutes essentially the "maximal set" which has this property.

The underlying principle behind the following results is just a generalization of the familiar facts that:
(i) a Complete Set of Commuting Operators (CSCO) on a vector space share common eigenvectors.
(ii) a decomposition of reducible subspaces with respect to some subset of the commuting operators often leads naturally to diagonalization of the remaining operator(s).

We have made use of this principle to diagonalize the Hamiltonian for a general 1-D lattice by taking advantage of T_1. Similarly, as is often done in the solution to physical problems involving spherical symmetry, the Hamiltonian is diagonalized by decomposing first with respect to angular momentum operators.

- Lemma 5.1:

The rep matrices $D(G_m)$, Eq. (5.5-6), and $D(S_n)$, Eq. (5.5-13) satisfy the following symmetry relation:

(5.5-14) $$D^J{}_I = D^{J_q}{}_{I_q}$$

where $I = i_1 i_2 \ldots i_n \qquad I_q = i_{q_1} i_{q_2} \ldots i_{q_n}$

$$q = \begin{pmatrix} 1 & 2 & \ldots & n \\ q_1 & q_2 & \ldots & q_n \end{pmatrix} \in S_n$$

Linear transformations on $V_m{}^n$ satisfying this condition are said to be symmetry- preserving.

- Proof:

(5.5-6) $$D(g)^J{}_I = g^{j_1}{}_{i_1} g^{j_2}{}_{i_2} \cdots g^{j_n}{}_{i_n} \qquad \forall\ g \in G_m$$
$$= g^{j_{q_1}}{}_{i_{q_1}} g^{j_{q_2}}{}_{i_{q_2}} \cdots g^{j_{q_n}}{}_{i_{q_n}}$$
$$= D(g)^{J_q}{}_{I_q}$$

(5.5-13) $$D(p)^J{}_I = \delta_{I^J_{p^{-1}}} = \delta^J_{I\,p} \qquad \forall\ p \in S_n$$
$$= \delta_{I_q{}^J{}_{q\,p}} = D(p)^{J_q}{}_{I_q}$$

- Theorem 5.9:

The two sets of matrices $\{ D(p)\,,\ p \in S_n \}$ and $\{ D(g)\,,\ g \in G_m \}$ commute with each other.

- Proof:

Consider the action of $p\,g$ and $g\,p$ on the basis vectors in turn:

(i) $\quad p\, g \,|\, I\,\rangle = p \,|\, J\,\rangle\, D(g)^J{}_I$

$\quad\quad = \left|\, J_{p^{-1}} \right\rangle D(g)^J{}_I$

$\quad\quad = \,|\, J\,\rangle\, D(g)^{J_p}{}_I$

(ii) $\quad g\, p \,|\, I\,\rangle = g \left|\, J_{p^{-1}} \right\rangle$

$\quad\quad = \,|\, J\,\rangle\, D(g)^J{}_{I_{p^{-1}}}$

$\quad\quad = \,|\, J\,\rangle\, D(g)^{J_p}{}_I$

Hence

$$p\, g \,|\, I\,\rangle = g\, p \,|\, I\,\rangle \qquad \forall \;|\, I\,\rangle$$

Therefore

$$p\, g = g\, p$$

- Example 1: $\boldsymbol{V_2{}^2}$

Consider second rank tensors ($n = 2$) in 2-D space ($m = 2$).
The basis vectors will be denoted by

$$|\, ++\,\rangle \qquad |\, +-\,\rangle \qquad |\, -+\,\rangle \qquad |\, --\,\rangle$$

Now, $S_2 = \{\, e,\ (12)\,\}$.
Since e leads to trivial results, we need only to consider $p = (12)$ and its interplay with elements of G_2:

$$p\, g \,|\, \pm\pm\,\rangle = p \,|\, i\, k\,\rangle\, g^i{}_\pm\, g^k{}_\pm = |\, k\, i\,\rangle\, g^k{}_\pm\, g^i{}_\pm$$
$$g\, p \,|\, \pm\pm\,\rangle = g\, |\, \pm\pm\,\rangle = |\, k\, i\,\rangle\, g^k{}_\pm\, g^i{}_\pm = p\, g \,|\, \pm\pm\,\rangle$$
$$p\, g \,|\, \pm\mp\,\rangle = p \,|\, i\, k\,\rangle\, g^i{}_\pm\, g^k{}_\mp = |\, k\, i\,\rangle\, g^k{}_\mp\, g^i{}_\pm$$
$$g\, p \,|\, \pm\mp\,\rangle = g\, |\, \mp\pm\,\rangle = |\, k\, i\,\rangle\, g^k{}_\mp\, g^i{}_\pm = p\, g \,|\, \pm\mp\,\rangle$$

These equalities hold for any element $g \in G_2$.

- Irreducible Subspaces of $\boldsymbol{V_m{}^n}$

We shall now decompose $V_m{}^n$ into irreducible subspaces wrt S_n and G_m, utilizing the $e_\lambda{}^p$'s associated with various $\Theta_\lambda{}^p$ of S_n.
Let L_λ be the left ideal generated by e_λ. The main results will be:

(i) For a fixed $\alpha \in V_m{}^n$, the subspace $\{\, r \,|\, \alpha\,\rangle;\, r \in L_\lambda\,\}$ is irreducibly invariant under S_n;

(ii) For a fixed $\Theta_\lambda{}^p$, the subspace $\{\, e_\lambda{}^p \,|\, \alpha\,\rangle;\, |\, \alpha\,\rangle \in V_m{}^n\,\}$ is irreducibly invariant under G_m;

(iii) $V_m{}^n$ can be decomposed in the "factorized" basis vectors $|\, \lambda,\, \alpha,\, a\,\rangle$ where λ denotes a symmetry class specified by a Young diagram; α & a labels the various irreducible invariant subspaces under S_n & G_m, resp.

- Definition 5.6
  (Tensors of Symmetry $\boldsymbol{\Theta_\lambda{}^p}$ and Tensors of Symmetry Class $\boldsymbol{\lambda}$):

To each Young tableau $\Theta_\lambda{}^p$ we associate tensors of the symmetry $\Theta_\lambda{}^p$ consisting of

$$T_\lambda{}^p = \{\, e_\lambda{}^p \,|\, \alpha\,\rangle;\ |\, \alpha\,\rangle \in V_m{}^n\,\}.$$

For a given Young diagram characterized by λ, the set of tensors

$$T_\lambda = \{\, r\, e_\lambda \,|\, \alpha\,\rangle,\ r \in \mathcal{S}_n,\ |\, \alpha\,\rangle \in V_m{}^n\,\}$$

is said to belong to the symmetry class λ.

Theorem 5.10

For a given $|\,\alpha\,\rangle$, let $T_\lambda(\alpha) = \{\, r\, e_\lambda \,|\, \alpha\,\rangle,\ r \in \mathcal{S}_n \,\}$.

(i) $T_\lambda(\alpha)$ is an irreducible invariant subspace with respect to S_n;

(ii) if $T_\lambda(\alpha) \neq \emptyset$, then the realization of S_n on $T_\lambda(\alpha)$ coincides with the IR generated by e_λ on $\mathcal{S}_n$.

- Proof:

(i) Let $|\, x \,\rangle \in T_\lambda(\alpha)$, then by definition,

$$|\, x\,\rangle = r\, e_\lambda \,|\, \alpha\,\rangle \qquad \text{for some } r \in \mathcal{S}_n$$

hence, $\forall\ p \in S_n$,

$$p\,|\, x\,\rangle = p\, r\, e_\lambda \,|\, \alpha\,\rangle$$
$$= \ q\, e_\lambda \left|\, \alpha \right\rangle \in T_\lambda(\alpha) \qquad [\, q = p\, r \in \mathcal{S}_n \,]$$

This means $T_\lambda(\alpha)$ is invariant under S_n.

(ii) Since $T_\lambda(\alpha)$ is not empty, we know $e_\lambda \,|\, \alpha\,\rangle \neq 0$. Let $\{\, r_i\, e_\lambda \,\}$ be a basis of L_λ, then $\{\, r_i\, e_\lambda \,|\, \alpha\,\rangle\}$ form a basis of $T_\lambda(\alpha)$. Hence, if

$$p\,|\, r_i\, e_\lambda\,\rangle = |\, p\, r_i\, e_\lambda\,\rangle = |\, r_i\, e_\lambda\,\rangle\, D(\,p)^j{}_i \qquad \text{on } \mathcal{S}_n$$

then,

$$p\, r_i\, e_\lambda \,|\, \alpha\,\rangle = r_j\, e_\lambda \left|\, \alpha \right\rangle\, D(\,p)^j{}_i \qquad \text{on } T_\lambda(\alpha)$$

for all $p \in S_n$.

Hence the invariant subspace is irreducible, and the representation matrices on $T_\lambda(\alpha)$ coincide with those on $\mathcal{S}_n$. QED

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Let $\Theta_{\lambda=s} = \boxed{}\boxed{}\boxed{\cdots\cdots}\boxed{}$ then $e_s = \sum_p \dfrac{p}{n!}$ is the total symmetrizer.

Since $r\, e_s = e_s$ for all $r \in \mathcal{S}_n$, the left ideal L_s is 1-D.
Correspondingly, for any given $|\,\alpha\,\rangle \in V_m{}^n$, the irreducible subspace $T_s(\alpha)$ consists of all multiples of $e_s \,|\, \alpha\,\rangle$.
These are totally symmetric tensors, as it is straightforward to verify:

(5.5-15)
$$e_s \,|\, \alpha\,\rangle\, n! = \sum_p p\,|\, I\,\rangle\, \alpha^I$$
$$= \sum_p \left|\, I_{p^{-1}} \right\rangle \alpha^I$$
$$= |\, I\,\rangle \sum_p \alpha^{I_p}$$

hence the components are totally symmetric in the n-indices.

The realization of S_n on $T_\lambda(\alpha)$ is the 1-D identity rep because all permutations leave a totally symmetric tensor unchanged.

- Example 2:

Consider third rank tensors ($n = 3$) in two dimensions ($m = 2$). Four distinct totally symmetric tensors can be generated by starting with different elements of $V_2{}^{n=3}$.

$$e_s = \frac{1}{3!}[\, e + (12) + (13) + (23) + (123) + (321)\,]$$

(i) $\mid \alpha = 1 \rangle = \mid +++ \rangle$
$e_s \mid \alpha \rangle = \mid +++ \rangle \equiv \mid s, 1, 1 \rangle$

(ii) $\mid \alpha = 2 \rangle = \mid ++- \rangle$
$e_s \mid \alpha \rangle = \frac{1}{6} [\mid ++- \rangle + \mid ++- \rangle + \mid -++ \rangle + \mid +-+ \rangle + \mid -++ \rangle + \mid +-+ \rangle]$
$= \frac{1}{3} [\mid ++- \rangle + \mid +-+ \rangle + \mid -++ \rangle]$
$\equiv \mid s, 2, 1 \rangle$

(iii) $\mid \alpha = 3 \rangle = \mid --+ \rangle$
$e_s \mid \alpha \rangle = \frac{1}{6} [\mid --+ \rangle + \mid --+ \rangle + \mid +-- \rangle + \mid -+- \rangle + \mid +-- \rangle + \mid -+- \rangle]$
$= \frac{1}{3} [\mid --+ \rangle + \mid -+- \rangle + \mid +-- \rangle]$
$\equiv \mid s, 3, 1 \rangle$

(iv) $\mid \alpha = 4 \rangle = \mid --- \rangle$
$e_s \mid \alpha \rangle = \mid --- \rangle \equiv \mid s, 4, 1 \rangle$

In the last column, we introduced the $\mid \lambda, \alpha, a \rangle$ labelling scheme.
This classification is used extensively in the following discussions.
Each of the above totally symmetric tensors is invariant under all $p \in S_3$.
Together, they represent all totally symmetric tensors that can be constructed in ${V_2}^3$; they are tensors of the symmetry class s, where s represents the Young tableau with one single row.
We shall denote the subspace of tensors of the symmetry class s by T_s'.

■ ✸

Can we similarly generate totally anti- symmetric tensors in ${V_m}^n$? We leave as an exercise [Problem 5.6] for the reader to show that they exist only if $m \geq n$. The total anti- symmetrizer is $e_a = \sum_p (-)^p \frac{p}{n!}$. Since $p\, e_a = (-)^p e_a$, both L_a and $T_a(\alpha)$ are 1-D, and the realization of S_n on $T_a(\alpha)$ corresponds to the 1-D rep $p \to (-)^p$.

■ Example 3:

There is one and only one independent totally anti- symmetric tensor of rank n in n-D space, usually denoted by ϵ. In 2-D, its components are

$$\epsilon^{12} = -\epsilon^{21} = 1, \epsilon^{11} = \epsilon^{22} = 0$$

In 3-D, the components are $\epsilon^{i\,j\,k} = \pm 1$ according to whether $(i\; j\; k)$ is an even or odd permutation of (123); else, if any two indices are equal, then $\epsilon^{i\,j\,k} = 0$.

■ Example 4:

Consider second rank tensors ($n = 2$) in m-D ($m \geq 2$),

$$e_s \mid i\, i \rangle = \mid i\, i \rangle \qquad i = 1, 2, \ldots, m$$

$$e_s \mid i\; j \rangle = \frac{1}{2} \{ \mid i\; j \rangle + \mid j\; i \rangle \} \qquad i \neq j$$

There are $m(m-1)/2$ distinct anti- symmetric tensors, as

$$e_a \mid i\, i \rangle = 0$$

$$e_a \mid i\; j \rangle = \frac{1}{2} \{ \mid i\; j \rangle - \mid j\; i \rangle \} \qquad i \neq j$$

Let us now turn to tensors with mixed symmetry.

- Example 5:

We return to $V_2{}^3$.
Consider

$$\Theta_m = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \qquad \text{and} \qquad e_m = \frac{1}{4}[\, e + (12)\,]\,[\, e - (31)\,]$$

Since
$$e_m \mid +++ \rangle = e_m \mid --- \rangle = 0$$
only two independent irreducible invariant subspaces of tensors can be generated.

(i) By choosing $\mid \alpha \rangle = \mid ++- \rangle$, we obtain:

$$\begin{aligned} e_m \mid \alpha \rangle &= \frac{1}{4}[\, e + (12)\,]\,[\, \mid ++- \rangle - \mid -++ \rangle\,] \\ &= \frac{1}{4}[\, \mid ++- \rangle - \mid -++ \rangle + \mid ++- \rangle - \mid +-+ \rangle\,] \\ &= \frac{1}{4}[\, 2 \mid ++- \rangle - \mid -++ \rangle - \mid +-+ \rangle\,] \\ &\equiv \mid m, 1, 1 \rangle \qquad\qquad [\, \mid \lambda, \alpha, a \rangle\,] \end{aligned}$$

$$\begin{aligned} (23)\, e_m \mid \alpha \rangle &= \frac{1}{4}\,(23)[\, 2 \mid ++- \rangle - \mid -++ \rangle - \mid +-+ \rangle\,] \\ &= \frac{1}{4}[\, 2 \mid +-+ \rangle - \mid -++ \rangle - \mid ++- \rangle\,] \\ &\equiv \mid m, 1, 2 \rangle \end{aligned}$$

and, for any $r \in \mathcal{S}_3$, $r\, e_m \mid \alpha \rangle$ is a linear combination of the above two tensors. These two mixed tensors form a basis for $T_{\lambda=m}(1)$.

(ii) By choosing $\mid \alpha \rangle = \mid --+ \rangle$, we obtain:

$$\begin{aligned} e_m \mid \alpha \rangle &= \frac{1}{4}\,[\, 2 \mid --+ \rangle - \mid +-- \rangle - \mid -+- \rangle\,] \\ &\equiv \mid m, 2, 1 \rangle \end{aligned}$$

$$\begin{aligned} (23)\, e_m \mid \alpha \rangle &= \frac{1}{4}[\, 2 \mid -+- \rangle - \mid +-- \rangle - \mid --+ \rangle\,] \\ &\equiv \mid m, 2, 2 \rangle \end{aligned}$$

as the basis for another irreducible invariant subspace of tensors with mixed symmetry $T_{\lambda=m}(2)$.

The realization of the group S_3 on either $T_{\lambda=m}(1)$ or $T_{\lambda=m}(2)$ corresponds to the 2-D IR discussed in Sec. 5.2 and described earlier in Chap. 3 [cf Table 3.3].

The two tensors of mixed symmetry $\mid m, i, 1 \rangle$, $i = 1, 2$ (first ones of the two sets given above), are two linearly independent tensors of the form $e_m \mid \alpha \rangle$ with $\mid \alpha \rangle$ ranging over $V_m{}^n$. [Problem 5.8] They are tensors of the symmetry Θ_m. We call the subspace spanned by these vectors $T_m'(1)$. $T_m'(1)$ is an invariant subspace under G_2 since

$$g\, e_m \mid \alpha \rangle = e_m\, g \mid \alpha \rangle \in T_m'(1) \qquad \forall \;\mid \alpha \rangle \in V_m{}^n$$

One can also show that this invariant subspace is irreducible under G_2. [Problem 5.8]

Similarly, the two tensors $|\, m, i, 2\,\rangle$, $i = 1, 2$ (second ones of the two sets) are two linearly independent tensors of the form $e_m^{(23)}\,|\,\alpha\,\rangle$ – as can easily be verified by noting that $(23)\, e_m = e_m^{(23)}$. They are tensors of the symmetry $\Theta_m^{(23)}$, denote the subspace spanned by these tensors by $T_m'(2)$. $T_m'(2)$ is also invariant under group transformations of G_2, and it is irreducible. Together, the two sets $\{T_m'(a),\ a = 1,\ 2\}$ comprise tensors of the symmetry class m, where m denotes the Young diagram (frame) associated with the normal tableau Θ_m. For the sake of economy of indices, we shall use "α" in place of the label "i" from now on; it is understood that the range of this label is equal to the number of independent tensors that can be generated by $e_\lambda\,|\,\alpha\,\rangle$ with $|\,\alpha\,\rangle \in V_m^n$.

We note that for the 8-D tensor space V_2^3, the use of Young symmetrizers (in Examples 2 and 5) leads to the complete decomposition into irreducible tensors $|\,\lambda, \alpha, a\,\rangle$ where $\lambda\,(=\ s, m\,)$ characterizes the symmetry class (Young diagram); "α" labels the distinct (but equivalent) sets of tensors $T_\lambda(\alpha)$ invariant under S_n; and "a" labels the basis elements within each set $T_\lambda(\alpha)$, it is associated with distinct symmetries (tableaux) in the same symmetry class. We have 4 totally symmetric tensors (Example 2) and 2 sets of 2 linearly independent mixed symmetry tensors. The latter can be classified either as belonging to two invariant subspaces under S_3 $\{T_m(\alpha),\ \alpha = 1,\ 2\}$, or as belonging to two invariant subspaces under G_2 $\{T_m'(a),\ a = 1,\ 2\}$. The latter comprise of tensors of two distinct symmetries associated with Θ_m and $\Theta_m^{(23)}$.

Bearing in mind these results for V_2^3, we return to the general case.

- Theorem 5.11:

(i) Two tensor subspaces irreducibly invariant with respect to S_n and belonging to the same symmetry class either overlap completely or they are disjoint;

(ii) Two irreducible invariant tensor subspaces corresponding to two distinct symmetry classes are necessarily disjoint.

- Proof:

(i) Let $T_\lambda(\alpha)$ and $T_\lambda(\beta)$ be two invariant subspaces generated by the same irreducible symmetrizer e_λ. Either they are disjoint or they have at least one non- zero element in common. In the latter case, there are $s, s' \in \mathcal{S}_n$, such that

$$s\, e_\lambda\,|\,\alpha\,\rangle = s'\, e_\lambda\,|\,\beta\,\rangle$$

This implies,

$$r\, s\, e_\lambda\,|\,\alpha\,\rangle = r\, s'\, e_\lambda\,|\,\beta\,\rangle \qquad \forall\ r \in \mathcal{S}_n$$

When r ranges over all $\mathcal{S}_n$, so do $r\,s$ and $r\,s'$. Therefore, the left-hand side of the last equation ranges over $T_\lambda(\alpha)$ and the right-hand side ranges over $T_\lambda(\beta)$, hence the two invariant subspaces coincide.

(ii) Given any two subspaces $T_\lambda(\alpha)$ and $T_\mu(\beta)$ invariant under S_n; their intersection is also an invariant subspace. If $T_\lambda(\alpha)$ and $T_\mu(\beta)$ are irreducible, then either the intersection is the null set or it must coincide with both $T_\lambda(\alpha)$ and $T_\mu(\beta)$. If λ and μ correspond to different symmetry classes, then the second possibility is ruled out. Hence $T_\lambda(\alpha)$ and $T_\mu(\beta)$ have no elements in common if $\lambda \neq \mu$. QED

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These general results permit the complete decomposition of the tensor space V_m^n into irreducible subspace $T_\lambda(\alpha)$ invariant under S_n. As explained when working on the the example of V_2^3, we shall use α as the label for distinct subspaces corresponding to the same symmetry class λ. The decomposition can be expressed as

(5.5-16) $$V_m^n = \sum_{\lambda\oplus} \sum_{\mu\oplus} T_\lambda(\alpha)$$

The basis elements of the tensors in the various symmetry classes are denoted by $| \lambda, \alpha, a \rangle$ where a ranges from 1 to the dimension of $T_\lambda(\alpha)$. We can choose these bases in such a way that the representation matrices for S_n on $T_\lambda(\alpha)$ are identical for all α associated with the same λ , or

(5.5-17) $p | \lambda, \alpha, a \rangle = | \lambda, \alpha, b \rangle D_\lambda(p)^b{}_a$

independently of α.

The central result of this section will be that the decomposition of $V_m{}^n$ according to the symmetry classes of S_n, as described above, automatically provides a complete decomposition with respect to the general linear group G_m as well. We have already seen how this worked out in the case of $V_2{}^3$.

- Theorem 5.12:

If $g \in G_m$ and $\{ | \lambda, \alpha, a \rangle \}$ is the set of basis tensors generated according to the above procedure, then the subspaces $T_\lambda{}'(a)$ spanned by $\{ | \lambda, \alpha, a \rangle \}$ with fixed λ and a are invariant with respect to G_m, and the representations of G_m on $T_\lambda{}'(a)$ are independent of a: i.e.

(5.5-18) $g | \lambda, \alpha, a \rangle = | \lambda, \beta, a \rangle D_\lambda(g)^\beta{}_\alpha$

- Proof:

(i) Given $r e_\lambda | \alpha \rangle \in T_\lambda(\alpha)$ and $g \in G_m$, we have

$$g (r e_\lambda) | \alpha \rangle = (r e_\lambda) g | \alpha \rangle \in T_\lambda(g \alpha)$$

Hence, the operations of the linear group do not change the symmetry class of the tensor, or

$$g | \lambda, \alpha, a \rangle = | \lambda, \beta, a \rangle D_\lambda(g)^{\beta b}{}_{\alpha a}$$

(ii) We now show that $D_\lambda(g)$ is diagonal in the indices (b, a). To this end, we note, for $g \in G_m$, and $p \in S_n$,

$$g p | \lambda, \alpha, a \rangle = g | \lambda, \alpha, c \rangle D_\lambda(p)^c{}_a$$
$$= | \lambda, \beta, b \rangle D_\lambda(g)^{\beta b}{}_{\alpha c} D_\lambda(p)^c{}_a$$

and

$$p g | \lambda, \alpha, a \rangle = p | \lambda, \beta, c \rangle D_\lambda(g)^{\beta c}{}_{\alpha a}$$
$$= | \lambda, \beta, b \rangle D_\lambda(p)^b{}_c D_\lambda(g)^{\beta c}{}_{\alpha a}$$

Since $g p = p g$ (Theorem 5.9), the two product matrices on the right- hand sides can be equated to each other. For clarity, let us designate quantities in square brackets as matrices in the space of Latin indices, and suppress these indices. We obtain

(5.5-19) $\left[D_\lambda(g)^\beta{}_\alpha \right] [D_\lambda(p)] = [D_\lambda(p)] \left[D_\lambda(g)^\beta{}_\alpha \right]$

For given g, this equation holds for all $p \in S_n$. According to Schur's Lemma, the matrix $D_\lambda(g)^{\beta b}{}_{\alpha a}$ must be proportional to the unit matrix in the Latin indices. QED

- Theorem 5.13 (Irreducible Representations of G_m):

The reps of G_m on the subspace $T_\lambda{}'(a)$ of $V_m{}^n$ as described above are IRs.

- Proof:

Even though the complete proof involves some technical details [Miller], the basic idea behind it is rather easy to understand: since G_m is, so to speak, the most general group of transformations which commutes with S_n on $V_m{}^n$ on the subspace $T_\lambda{}'$ the operators $\{ D(g), g \in G_m \}$ must be "complete" – they cannot be reducible. More specifically, consider an arbitrary linear transformation A on the vector space $T_\lambda{}'(a)$. In tensor component notation,

$$x^I \longrightarrow y^I = A^I{}_J x^J$$

Because x and y belong to the same symmetry class, A must be "symmetry preserving" in the sense that,

(5.5-20) $$A^I{}_J = A^{I_p}{}_{J_p} \qquad \forall \; p \in S_n$$

We know already that the linear transformations representing $g \in G_m$ on $V_m{}^n$ are symmetry preserving [Lemma 5.1]. It can be established that, even though A does
not necessarily factorize as $D(g)$ in Eq. (5.5-6), it can nevertheless be written as a linear combination of $D(g)$. [cf. Lemma IV.7] Since this is true for all linear transformations, $D(g)$ must be irreducible. QED.

■ ✸

A concrete example on how the tensor space $V_m{}^n$ is decomposed to irreducible invariant subspaces with respect to both S_n and G_m was worked out in detail previously for $V_2{}^3$. In the context of Theorems 5.12 and 5.13, we found:

(i) associated with totally symmetric tensors ($\lambda = s$), there is an invariant subspace (with respect to G_m) $T_s{}'$ which is 4-dimensional and has basis vectors $\{ \mid s, \alpha, 1 \rangle, \; \alpha = 1, \ldots, 4 \}$ given in Example 2; and

(ii) associated with the symmetry class $\lambda = m$, there are two invariant subspaces $T_m{}'(1)$ and $T_m{}'(2)$ which give rise to equivalent 2-dimensional irreducible representations of the linear group G_2. [cf. Problem 5.8]

The irreducible representations of G_m provided by tensors of various symmetry classes as described in this section are by no means the only irreducible representations of the general linear group. The main purpose of this exposition is to illustrate the usefulness of the symmetric (or permutation) group in an important class of application – tensor analysis. In Chapter 13, we shall give a more systematic discussion of finite-dimensional representations of the classical groups which includes GL(m, C) as the most general case. We shall also utilize the tensor method to help evaluate the explicit expression for all representation matrices of the rotation group in Sec. 8.1.