Chapter 5
Symmetric Groups

- References

The following notes are based on Chap 5, Tung & Chap 15, Inui.
Caution: the conventions used by Tung & Inui are very different (see section group product).
Basic concepts of the idempotents are described in note GroupAlgebra, which was based on App III, Tung.
Other noteworthy references are:
   This book also describes the Eigenfunction Method for finding group representations.
2. Chapter 7, Hamermeash.

5.0 Basics

- $S_n$

The $n!$ permutations of $n$ objects form a symmetric (permutation) group $S_n$ of degree $n$ & order $n!$.

The group elements are written as

$$p = \left( \begin{array}{cccc} 1 & 2 & \ldots & n \\ p_1 & p_2 & \ldots & p_n \end{array} \right) = \left( \begin{array}{c} i \\ p_i \end{array} \right)$$

Obviously, the order of the elements in the permutation symbol is immaterial, eg.:

$$\left( \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array} \right) = \left( \begin{array}{ccc} 2 & 3 & 1 \\ 1 & 3 & 2 \end{array} \right) = \left( \begin{array}{ccc} 3 & 1 & 2 \\ 3 & 2 & 1 \end{array} \right) = \ldots$$

- Group Product

There're 2 inequivalent ways to define the group product.
The active view will be adopted in this note.
In either view, 2 permutations commute if they do not involve permutations of the same index.
Section 3. Symmetric Groups

- **Active Point of View**

The **active way** used by Tung (see p.18, Tung) interprets a permutation

\[
p = \begin{pmatrix}
1 & 2 & \ldots & n \\
p_1 & p_2 & \ldots & p_n
\end{pmatrix} = (p_1 \leftrightarrow i)
\]

as taking the object originally in box \( i \) to box \( p_i \).

The product \( pq \) of 2 permutations \( p \) & \( q \) then denotes 2 consecutive actions:

1st, take objects in box \( i \) to box \( q_i \).

then, take objects in box \( i \) to box \( p_i \).

Thus, an object that is originally in box \( i \) is 1st taken to box \( q_i \), then to box \( p_{q_i} \).

Symbolically:

\[
pq = (p_i \leftrightarrow i)(q_i \leftrightarrow i)
= (p_{q_i} \leftrightarrow q_i)(q_i \leftrightarrow i)
= (p_{q_i} \leftrightarrow i)
\]

or

\[
pq = \begin{pmatrix}
1 & 2 & \ldots & n \\
p_1 & p_2 & \ldots & p_n
\end{pmatrix} \begin{pmatrix}
1 & 2 & \ldots & n \\
q_1 & q_2 & \ldots & q_n
\end{pmatrix}
= \begin{pmatrix}
q_1 & q_2 & \ldots & q_n \\
p_{q_1} & p_{q_2} & \ldots & p_{q_n}
\end{pmatrix} \begin{pmatrix}
1 & 2 & \ldots & n \\
q_1 & q_2 & \ldots & q_n
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & \ldots & n \\
p_{q_1} & p_{q_2} & \ldots & p_{q_n}
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & \ldots & n \\
(p_{q_1}) & (p_{q_2}) & \ldots & (p_{q_n})
\end{pmatrix}
\]

where \( \begin{pmatrix}
q_1 & q_2 & \ldots & q_n \\
p_{q_1} & p_{q_2} & \ldots & p_{q_n}
\end{pmatrix} \) is the rearrangement of the columns of \( \begin{pmatrix}
1 & 2 & \ldots & n \\
p_1 & p_2 & \ldots & p_n
\end{pmatrix} \) so that the 1st row \( \begin{pmatrix}
1 & 2 & \ldots & n \\
p_1 & p_2 & \ldots & p_n
\end{pmatrix} \) becomes \( \begin{pmatrix}
q_1 & q_2 & \ldots & q_n \\
p_{q_1} & p_{q_2} & \ldots & p_{q_n}
\end{pmatrix} \).

Thus \( (pq)_j = p_{q_j} \).

The inverse \( p^{-1} \) of \( p \) is:

\[
p^{-1} = \begin{pmatrix}
1 & 2 & \ldots & n \\
(p^{-1}_1) & (p^{-1}_2) & \ldots & (p^{-1}_n)
\end{pmatrix} = \begin{pmatrix}
p_1 & p_2 & \ldots & p_n \\
1 & 2 & \ldots & n
\end{pmatrix} = (i \leftrightarrow p_i) = (p_i \rightarrow i)
\]

so that

\[
p p^{-1} = (p_i \leftrightarrow i)(i \leftrightarrow p_i) = (p_i \leftrightarrow p_i) = e
\]

\[
p^{-1} p = (i \leftrightarrow p_i)(p_i \leftrightarrow i) = (i \leftrightarrow i) = e
\]

\[
p p^{-1} = \begin{pmatrix}
1 & 2 & \ldots & n \\
p_1 & p_2 & \ldots & p_n
\end{pmatrix} \begin{pmatrix}
1 & 2 & \ldots & n \\
p_1 & p_2 & \ldots & p_n
\end{pmatrix} = \begin{pmatrix}
p_1 & p_2 & \ldots & p_n \\
p_1 & p_2 & \ldots & p_n
\end{pmatrix} = e
\]

\[
p^{-1} p = \begin{pmatrix}
p_1 & p_2 & \ldots & p_n \\
1 & 2 & \ldots & n
\end{pmatrix} \begin{pmatrix}
p_1 & p_2 & \ldots & p_n \\
1 & 2 & \ldots & n
\end{pmatrix} = \begin{pmatrix}
1 & 2 & \ldots & n \\
1 & 2 & \ldots & n
\end{pmatrix} = e
\]

as expected.

Obviously, \( p_{(p^{-1})_j} = (p p^{-1})_j = e \) or \( j = (p^{-1} p)_j = (p^{-1})_p \).
Passive Point of View

The **passive way** used by Inui is (see p.16, Inui) interprets a permutation

\[ p = \begin{pmatrix} 1 & 2 & \ldots & n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix} = (i \rightarrow p_i) \]

as relabeling object \( i \) as \( p_i \).

The product \( pq \) of 2 permutations \( p \) & \( q \) then denotes 2 consecutive re-labeling:

1st, relabeling object \( i \) as \( q_i \),
then, relabeling object \( i \) as \( p_i \).

Thus, object \( i \) is finally labeled \( q_{p_i} \).

Symbolically:

\[
\begin{aligned}
p q &= (i \rightarrow p_i)(i \rightarrow q_i) \\
&= (i \rightarrow p_i)(p_i \rightarrow q_{p_i}) \\
&= (i \rightarrow q_{p_i})
\end{aligned}
\]

or

\[
\begin{aligned}
p q &= \begin{pmatrix} 1 & 2 & \ldots & n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & n \\ q_1 & q_2 & \ldots & q_n \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & \ldots & n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix} \begin{pmatrix} p_1 & p_2 & \ldots & p_n \\ q_{p_1} & q_{p_2} & \ldots & q_{p_n} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & \ldots & n \\ q_{p_1} & q_{p_2} & \ldots & q_{p_n} \end{pmatrix}
\end{aligned}
\]

where \( \begin{pmatrix} p_1 & p_2 & \ldots & p_n \\ q_{p_1} & q_{p_2} & \ldots & q_{p_n} \end{pmatrix} \) is the rearrangement of the columns of \( \begin{pmatrix} 1 & 2 & \ldots & n \\ q_1 & q_2 & \ldots & q_n \end{pmatrix} \) so that the 1st row \( (1 2 \ldots n) \)
becomes \( (p_1 p_2 \ldots p_n) \).

Thus \( (pq)_j = q_{p_j} \).

This means products in the passive view correspond to products in inverse order in the active view, & vice versa. i.e.

\[
\text{passive } pq \ldots rs \iff \text{active } sr \ldots qp
\]

Since we shall adopt the active point of view in the rest of this note, further development of the passive way will be left as exercise for those interested.

- **Example**

Let

\[
p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}
\]

The active view gives

\[
\begin{aligned}
pq &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\
qp &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}
\end{aligned}
\]
The passive view gives

\[
pq = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}
\]

\[
qp = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}
\]

**Example \( S_3 \)**

Permutations of 3 objects form the group \( S_3 \).

There are \( 3! = 6 \) group elements:

\[
e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}
\]

\[
c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}
\]

**\( S_3 \) & \( C_3^v \)**

Consider the group \( C_3^v \), which is isomorphic to \( S_3 \).

To facilitate the correspondence between elements of the 2 groups, we identify the objects under permutation to be the vertices of the triangle & the boxes to be positions in space.

Both vertices & positions are labeled 1, 2, 3 in a counterclockwise sense.

Originally, vertex labelled \( i \) is at position ( box ) \( i \).

In the **active point of view**, the permutation \( \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123) \) moves the vertex 1 at position 1 to position 2, vertex 2 at position 2 to position 3, and vertex 3 at position 3 to position 1. The result is that vertices 1, 2, 3 are now at positions 2, 3, 1, respectively. Clearly, this corresponds to the \( C_3 \) rotation of the triangle if the boxes or positions are fixed in space.

In the **passive point of view**, the permutation \( \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123) \) relabels the vertex 1 as vertex 2, vertex 2 as vertex 3, and vertex 3 as vertex 1. The result is that vertices 1, 2, 3 are now at positions 3, 1, 2, respectively. Clearly, this corresponds to the \( C_3^2 \) rotation of the triangle if the positions are fixed in space. Alternatively, we can also say that the boxes are rotated by \( C_3 \) while the triangle is held fixed.

Similarly, the elements of \( S_3 = \{ e, a, b, c, d, f \} \) can be identified with those of \( C_3^v = \{ E, C_3, C_3^2, \sigma_1, \sigma_2, \sigma_3 \} \), respectively, if we adopt the convention that in the active ( passive ) point of view, operations of \( C_3^v \) represent rotations of the triangle ( positions ).

Obviously, it is also correct to identify \( S_3 = \{ e, a, b, c, d, f \} \) with \( C_3^v = \{ E, C_3^2, C_3, \sigma_1, \sigma_2, \sigma_3 \} \), respectively, if we adopt the convention that in the active ( passive ) point of view, operations of \( C_3^v \) represent rotations of the positions ( triangle ).

This freedom of interpretation may create confusion to novices & is a major source of computational error, especially when results from different authors are quoted.

We shall henceforth adopt the active point of view for all groups, which means elements of the point groups are treated as actions on the geometric figure.

The active point of view is usually preferred by physicists while the passive one, by mathematicians.
The group multiplication table for $S_3$ can be obtained from that of $C_3$ (p.12, Inui):

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>e</th>
<th>a = C_3</th>
<th>b = C_3^2</th>
<th>c = σ_1</th>
<th>d = σ_2</th>
<th>f = σ_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>e</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>f</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>e</td>
<td>f</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>e</td>
<td>a</td>
<td>d</td>
<td>f</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>d</td>
<td>f</td>
<td>e</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>f</td>
<td>c</td>
<td>b</td>
<td>e</td>
<td>a</td>
</tr>
<tr>
<td>f</td>
<td>f</td>
<td>c</td>
<td>d</td>
<td>a</td>
<td>b</td>
<td>e</td>
</tr>
</tbody>
</table>

This table is the same as that in p.15, Tung but the roles of $a$ & $b$ are interchanged in p.17, Inui.
Both authors adopt the active view on rotation operators but Tung used the active, Inui the passive view for permutations.

### Cycles

An $n$-cycle $(i_1 \ i_2 \ ... \ \ i_n)$ is defined by

$$(i_1 \ i_2 \ ... \ \ i_n) = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ i_2 & i_3 & \cdots & i_1 \end{pmatrix}$$

### Examples

1 - cycle:

$$(i) = \begin{pmatrix} i \\ i \end{pmatrix}$$

2 - cycle, also called a transposition:

$$(i \ j) = \begin{pmatrix} i & j \\ j & i \end{pmatrix}$$

3 - cycle:

$$(i \ j \ k) = \begin{pmatrix} i & j & k \\ j & k & i \end{pmatrix}$$

The generator of a cyclic group $C_n$ is the $n$ - cycle.

Any permutation can be written as a product of cycles with no common indices, eg.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 1 & 5 & 8 & 2 & 6 \end{pmatrix} = (14)(237)(5)(68)$$

### Example $S_3$

In cycle notations, elements of $S_3$ becomes:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3) = ea = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

$$c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)(23) = (23)$$

$$d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13)$$

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)$$
For convenience, the multiplication table is reproduced in cycle notations (cf p.15, Tung):

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>$e$</th>
<th>(123)</th>
<th>(132)</th>
<th>(23)</th>
<th>(13)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>(123)</td>
<td>(132)</td>
<td>(23)</td>
<td>(13)</td>
<td>(12)</td>
</tr>
<tr>
<td>(123)</td>
<td>(123)</td>
<td>(132)</td>
<td>(23)</td>
<td>(13)</td>
<td>(12)</td>
<td></td>
</tr>
<tr>
<td>(132)</td>
<td>(132)</td>
<td>$e$</td>
<td>(12)</td>
<td>(23)</td>
<td>(13)</td>
<td></td>
</tr>
<tr>
<td>(23)</td>
<td>(23)</td>
<td>(13)</td>
<td>(12)</td>
<td>$e$</td>
<td>(123)</td>
<td></td>
</tr>
<tr>
<td>(13)</td>
<td>(12)</td>
<td>(23)</td>
<td>(132)</td>
<td>$e$</td>
<td>(123)</td>
<td></td>
</tr>
<tr>
<td>(12)</td>
<td>(12)</td>
<td>(23)</td>
<td>(13)</td>
<td>(123)</td>
<td>(132)</td>
<td>$e$</td>
</tr>
</tbody>
</table>

For comparison purposes, the following table may be more useful:

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>$C_{3v}$</th>
<th>$e$</th>
<th>$a = (123)$</th>
<th>$b = (132)$</th>
<th>$c = (23)$</th>
<th>$d = (13)$</th>
<th>$f = (12)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$a = (123)$</td>
<td>$b = (132)$</td>
<td>$c = (23)$</td>
<td>$d = (13)$</td>
<td>$f = (12)$</td>
<td>$a = (123)$</td>
</tr>
<tr>
<td>$a = (123)$</td>
<td>$a = (123)$</td>
<td>$b = (132)$</td>
<td>$c = (23)$</td>
<td>$d = (13)$</td>
<td>$f = (12)$</td>
<td>$a = (123)$</td>
<td>$b = (132)$</td>
</tr>
<tr>
<td>$b = (132)$</td>
<td>$b = (132)$</td>
<td>$e$</td>
<td>$a = (123)$</td>
<td>$d = (13)$</td>
<td>$f = (12)$</td>
<td>$a = (123)$</td>
<td>$b = (132)$</td>
</tr>
<tr>
<td>$c = (23)$</td>
<td>$c = (23)$</td>
<td>$d = (13)$</td>
<td>$f = (12)$</td>
<td>$a = (123)$</td>
<td>$b = (132)$</td>
<td>$e$</td>
<td>$a = (123)$</td>
</tr>
<tr>
<td>$d = (13)$</td>
<td>$d = (13)$</td>
<td>$f = (12)$</td>
<td>$c = (23)$</td>
<td>$b = (132)$</td>
<td>$e$</td>
<td>$a = (123)$</td>
<td>$b = (132)$</td>
</tr>
<tr>
<td>$f = (12)$</td>
<td>$f = (12)$</td>
<td>$c = (23)$</td>
<td>$d = (13)$</td>
<td>$a = (123)$</td>
<td>$b = (132)$</td>
<td>$e$</td>
<td>$a = (123)$</td>
</tr>
</tbody>
</table>

- **Cycle Structures**

  Every element of $S_n$ can be expressed as a product of cycles.

  If an element contains $k_i$ $i$-cycles ($i = 1 \ldots n$), its cycle structure is denoted by:

  \[
  (1^{k_1} 2^{k_2} \ldots i^{k_i} \ldots n^{k_n})
  \]

  with the convention that terms with $k_i = 0$ can be omitted.

  Some authors, eg. Inui, uses the $n$-tuple notation $k = (k_1 \ldots k_n)$.

  Since each element is a permutation of $n$ objects, the ‘total length’ of the cycles must be $n$. Thus

  \[
  \sum_{j=1}^{n} jk_j = n
  \]

  As an example, the permutation

  \[
  \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 1 & 5 & 8 & 2 & 6 \end{pmatrix}
  \]

  has a cycle structure $\{1^1 2^1 3^1\}$ with $k = (1 \times 2 \times 1 \times 0 \times 0 \times 0 \times 0 \times 0)$.

  \[
  \sum_{j=1}^{n} jk_j = 1 \times 1 + 2 \times 2 + 3 \times 1 + 4 \times 0 + \ldots = 8
  \]
Classes

Let

\[ p = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix} = (p_1 \leftrightarrow i) \quad q = \begin{pmatrix} 1 & 2 & \cdots & n \\ q_1 & q_2 & \cdots & q_n \end{pmatrix} = (q_i \leftrightarrow i) \]

\[ q \cdot p \cdot q^{-1} = (q_i \leftrightarrow i)(p_1 \leftrightarrow i)(i \leftrightarrow q_i) \]

\[ = (q_i \leftrightarrow i)(p_i \leftrightarrow q_i) \]
\[ = (q_{p_i} \leftrightarrow p_i)(p_i \leftrightarrow q_i) \]
\[ = (q_{p_i} \leftrightarrow q_i) \]
\[ = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ q_{p_1} & q_{p_2} & \cdots & q_{p_n} \end{pmatrix} \]
\[ = q[p] \]

where

\[ q[p] = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ q_{p_1} & q_{p_2} & \cdots & q_{p_n} \end{pmatrix} \]

is the permutation of the numbers in the array 

\[ p = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix} \]

Thus, \( q \cdot p \cdot q^{-1} \) has the same cycle structure as \( p \).

Conversely, the cycle structure can be used to characterize classes.

The number of elements in a class is therefore equal to the number of distinct elements with the same cycle structure.

Given a cycle structure \( (1^{k_1} \ 2^{k_2} \ \cdots \ i^{k_i} \ \cdots \ n^{k_n}) \), there'll be \( n! \) ways to fill it with the numbers \( 1 \ldots n \).

Obviously, permutations within each of the sets of \( k_i \) \( i \)-cycle result in the same class element. The number of such permutations is \( k_1! \cdots k_n! = \prod_j k_j! \).

Furthermore, cyclic permutations within each \( i \)-cycle also results in the same class element. For an \( i \)-cycle, there are \( i \) such permutations. Hence, the total number of such permutations is \( 1^{k_1} \cdots i^{k_i} \cdots n^{k_n} = \prod_j i^{k_j} \).

Thus, the number of elements in a class is

\[
\frac{n!}{k_1! \cdots k_n!} \frac{1^{k_1} \cdots i^{k_i} \cdots n^{k_n}}{\prod_j k_j! \ i^{k_j}} = \frac{n!}{\prod_j k_j! \ i^{k_j}}
\]
**Example**

Let 

\[ p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1 \times 2) \quad q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (12 \times 3) \]

By direct multiplication, we have:

\[ q p q^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix} \]

Alternatively,

\[ q p q^{-1} = q \left[ p \right] = q \left[ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right] = \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2 & q_1 & q_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix} = (2 \times 3) \]
Notations & Properties of Cycle Structures

1. In writing a permutation in terms of products of cycles, the 1–cycle is usually omitted, eg.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 3 & 7 & 1 & 5 & 8 & 2 & 6
\end{pmatrix} = (14)(237)(68)
\]

2. The order of the cycles in the product is immaterial since they do not have common indices.

3. Any cyclic permutations of the indices of a cycle are equivalent, eg.

\[(ijk) = (jki) = (kji) = \ldots\]

4. Any transposition can be written as product of adjacent transpositions according to the following recursive formula:

\[(i, i+1) = (i+1, i+2)(i, i+1)(i+1, i+2)\]

eg.

For the group \(S_4\):

\[
\begin{align*}
(1 \times 3) &= (2 \times 3)(1 \times 2)(2 \times 3) \\
(2 \times 4) &= (3 \times 4)(2 \times 3)(3 \times 4) \\
(1 \times 4) &= (2 \times 4)(1 \times 2)(2 \times 4)
\end{align*}
\]

\[= (3 \times 4)(2 \times 3)(3 \times 4)(1 \times 2)(3 \times 4)(2 \times 3)(3 \times 4)\]

5. Some useful relations:

\[
(abcd) = (ab)(bc)(cd) = (abc)(cd) = (abc)(bc)(d) = (ab)(bc)(c)d\]

\[(abcde)^{-1} = (edcba)\]

6. The parity \(\delta_P\) of a permutation is defined by

\[\delta_P = (-1)^N\]

where \(N\) is the number of transpositions in the permutation.

The permutation is even (odd) if \(\delta_P = 1 (-1)\).

8. An \(n\)–cycle is a product of \(n-1\) transpositions. (see 5.)

Thus, an \(n\)–cycle is an even (odd) permutation if \(n = \text{odd (even)}\).

9. Consider the sum of 2 integers \(m \& n:\)

\[n + m = \begin{cases} 
\text{even} & \text{if n, m are both odd or both even} \\
\text{odd} & \text{if 1 of n, m is even, the other odd}
\end{cases}\]

Hence, the product of 2 permutations \(p \& q\) is

\[pq = \begin{cases} 
\text{even} & \text{if p, q are both odd or both even} \\
\text{odd} & \text{if 1 of p, q is even, the other odd}
\end{cases}\]

10. The inverse of a cycle is just the cycle written in reverse order.

ie. \(p = (ij \ldots lm)\) \(\implies p^{-1} = (ml \ldots ji)\)

This can be proved by writing the cycle in terms of a product of transpositions.

Writing an arbitrary permutation as products of cycles, its inverse is obtained by writing each cycle in reversed order.

Thus, a permutation & its inverse have the same parity & cycle structure.

11. Similarity transforms \(pqp^{-1}\) do not change cycle structures.

Elements with the same cycle structure must belong to the same class & vice versa.

A similarity transform will be even (odd) if \(q\) is even (odd) since the \(pp^{-1}\) part is always even.
Example $S_3$

In cycle structure notations, elements of $S_3$ becomes:

\[ e = (1^3) \quad a = (1^0 \ 2^0 \ 3^1) = (3^1) \quad b = (1^0 \ 2^0 \ 3^1) = (3^1) \]
\[ c = (1^1 \ 2^1 \ 3^0) = (1^1 \ 2^1) \quad d = (1^1 \ 2^1 \ 3^0) = (1^1 \ 2^1) \ f = (1^1 \ 2^1 \ 3^0) = (1^1 \ 2^1) \]

There're therefore 3 classes with cycle structures

\[ [e] = (1^3) \quad [a, \ b] = (3^1) \quad [c, \ d, \ f] = (1^1 \ 2^1) \]

or, in the $k$ notation:

\[ [e] = (3 \times 0 \times 0) \quad [a, \ b] = (0 \times 0 \times 1) \quad [c, \ d, \ f] = (1 \times 1 \times 0) \]

The # of elements in these classes can be calculated using the formula

\[ \frac{n!}{k_1! \ldots k_n! \ 1^{i_1} \ldots i^{k_i}} \]

Thus

\[ (1^3) = (1^1 \ 2^0 \ 3^0) \quad \rightarrow \quad 3! / (3! \times 0! \times 0! 1^1 2^0 3^0) = 1 \]
\[ (3^1) = (1^0 \ 2^0 \ 3^1) \quad \rightarrow \quad 3! / (0! \times 0! \times 1! 1^0 2^0 3^1) = 2 \]
\[ (1^1 \ 2^1) = (1^1 \ 2^1 \ 3^0) \quad \rightarrow \quad 3! / (1! \times 1! \times 0! 1^1 2^1 3^0) = 3 \]

as expected.

Since $S_3$ is isomorphic to $C_3$, its character table is:

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>$(1^3)$</th>
<th>$2 \ (3^1)$</th>
<th>$3 \ (1^1 \ 2^1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

### 5.1 1-D Representations

Alternating Group $A_n$

Since the product of 2 even permutations is even, the set of all even permutations of $n$ objects is a group called the **alternating group $A_n$**.

Since an even permutation remains even under a similarity transform, $A_n$ is an invariant subgroup of $S_n$ with $S_n/A_n = S_2$ where

\[
\begin{array}{c|cc}
S_2 & (1^2) & (2^1) \\
\chi^1 & 1 & 1 \\
\chi^2 & 1 & -1 \\
\end{array}
\]

Thus, there are always two 1-D representations for $S_n$.

The 1st is obtained by assigning 1 to all elements & is called the identity representation.

The other is obtained by assigning even odd permutations.

This also implies that the number of even & odd permutations in $S_n$ are equal.
Definition:

\[ s = \sum_p p \]

is called the **symmetrizer**

\[ a = \sum_p (-)^p p \]

is called the **anti-symmetrizer**

The left coset decomposition of \( S_n \) is

\[ S_n = \sum_{i=1}^{k} R_i S_2 \]

\[ k = \frac{n_{S_0}}{n_{S_2}} = \frac{n!}{2} \]

**Theorem:** \( s \) & \( a \) are essentially idempotent & primitive

**Proof:**

Rearrangement theorem \( \rightarrow \)

\[ q s = q \sum_p p = \sum_p p' = s q \quad \forall \ q \in S_n \]

\[ \therefore \quad \sum_p p s = \sum_p s = n! s \quad \rightarrow \quad s \text{ is essentially idempotent.} \]

\[ s q s = s s = n! s \quad \rightarrow \quad s \text{ is primitive} \]

Similarly:

\[ q a = q \sum_p (-)^p p = \sum_p (-)^p p' = (-)^q a = a q \]

\[ \therefore \quad \sum_p (-)^p p a = \sum_p (-)^p a = n! a \quad \rightarrow \quad a \text{ is essentially idempotent.} \]

\[ a q a = a (-)^q a = (-)^q n! a \quad \rightarrow \quad a \text{ is primitive} \]

Since there are equal number of even & odd permutations in \( S_n \),

\[ s a = \sum_p p a = \sum_p (-)^p a = 0 \]

Hence:

\[ s q a = s (-)^q a = s a = 0 \quad \rightarrow \quad s \text{ & } a \text{ are inequivalent.} \]

---

### 5.2 Partitions & Young Diagrams

**Partition of \( n \)**

A **partition** \( \lambda \) of an integer \( n \) is a set \( \lambda = \{ \lambda_1 \ldots \lambda_r \} \) such that

1. \( \lambda_i \) = positive integers.
2. \( \sum_{i=1}^{r} \lambda_i = n \)
3. \( \lambda_i \geq \lambda_{i+1} \) (decending order)

Some basic definitions:

- **a.** \( \lambda = \mu \quad \iff \quad \lambda_i = \mu_i \quad \forall \ i \)
- **b.** \( \lambda > \mu \) if 1st non-zero number in the sequence \( \{ \lambda_i - \mu_i \} \) is positive.
  \( \lambda < \mu \) if negative.
Young Diagram

The Young diagram of a partition \( \lambda = (\lambda_1 \ldots \lambda_r) \) is an arrangement of \( n \) squares into \( r \) rows such that the \( i \) th row has \( \lambda_i \) squares.

**Example**

**Partition of 3**

Possible partitions are \([3], [2, 1], [1, 1, 1] = [1^3]\).

The corresponding Young diagrams are:

\[
\begin{align*}
([3]) & \quad \rightarrow \begin{array}{c}
\square
\end{array} \\
([2, 1]) & \quad \rightarrow \begin{array}{c}
\square
\end{array} \\
([1, 1, 1]) & \quad \rightarrow \begin{array}{c}
\square
\end{array}
\end{align*}
\]

**Theorem:** Young Diagrams & IRs of \( S_n \)

\# of distinct Young diagrams of partition of \( n \) = \# of classes of \( S_n \)

\[= \# \text{ of IR's of } S_n.\]

Given a Young diagram:
- Each column represents a cycle.
- \# of vertical blocks in each column is equal to the length of the cycle.

**Proof:**

Every class of \( S_n \) is characterized by its cycle structure \( (1^{k_1} \ldots i^{k_i} \ldots n^{k_n}) \) where \( k_j \) is the number of \( j \)-cycles. Since the 'total length' of the cycles must be \( n \), we have

\[
n = \sum_j j k_j = k_1 + 2 k_2 + \ldots + n k_n = (k_1 + k_2 + \ldots + k_n) + (k_2 + \ldots + k_n) + \ldots + (k_n + k_n) + k_n = \sum_{j=1}^{n} \sum_{i=j}^{n} k_j = \sum_{j=1}^{n} \lambda_j
\]

where

\[
\lambda_j = \sum_{i=j}^{n} k_i \quad j = 1 \ldots n
\]

Since \( k_j \) are all non-negative integers, \( \lambda_j \leq \lambda_{j+1} \quad \forall \ j = 2 \ldots n.\)
Thus, \( \{ \lambda_j \} \) is a partition of \( n.\)
Specifically,
\[ \lambda_j = \lambda_1 + \lambda_2 + \ldots + \lambda_n \]
= \# of cycles in each permutation.
= \# of blocks in the 1st row of the Young diagram.
# of blocks in the 1st column = \# of non-zero \( \lambda_j \)’s
= length of the longest cycle

The reason for the last statement is as follows:
Let \( m \) be the length of the longest cycle.
This implies \( \lambda_m \) is the last non-zero number in the partition.
\[ \lambda_m = \sum_{i=m}^{n} k_i = k_m \]

Obviously,
\[ \lambda_j \neq 0 \quad \forall \ j \leq m \]
\[ \lambda_j = 0 \quad \forall \ j > m \]

Thus, the \# of blocks in the 1st and longest column is just \( m \).

By the same token, we see that
\# of blocks in the 2nd column = length of the 2nd longest cycle
or, in general,
\# of blocks in each column = length of a cycle

To summarize, given a cycle structure, the corresponding Young diagram is constructed by writing each \( j \)-cycle present as a column of \( j \) blocks. The columns are placed adjacent with longer ones to the left of shorter ones.

For example, the class \( (1^3 \ 3^2 \ 4^1) \) with \( k = (3 \times 0 \times 2 \times 1) \) corresponds to the following Young diagram with \( \lambda = \{6, 3, 3, 1, 0, \ldots, 0\} \).

```
  6
  |
  |
  |
  3
  3
  |
  |
  1

\( (1^3 \ 3^2 \ 4^1) \)
```

Thus, there is a 1–1 correspondence between the partitions (Young diagrams) of \( n \) to the classes of \( S_n \).
The number of distinct Young diagrams is therefore equal to the number of classes of \( S_n \).

- **Example:** \( S_3 \)

The cycle structures of the classes of \( S_3 \) are
\[ (e) = (1^3) \quad (a, b) = (3^1) \quad (c, \ d, \ f) = (1^1 \ 2^1) \]

With \( k = (k_1 \ldots k_n) \), we have
\[ k = (3, 0, 0) \quad \text{for} \quad (e) \]
\[ k = (0, 0, 1) \quad \text{for} \quad (a, b) \]
\[ k = (1, 1, 0) \quad \text{for} \quad (c, \ d, \ f) \]

Thus, the partitions \( \lambda = \left\{ \lambda_j = \sum_{i=j}^{n} k_i \right\} \) are
\[ \lambda = (3, 0, 0) \quad \text{for} \quad (e) \]
\[ \lambda = (1, 1, 1) \quad \text{for} \quad (a, b) \]
\[ \lambda = (2, 1, 0) \quad \text{for} \quad (c, \ d, \ f) \]
The corresponding Young diagrams are:

- For \((e) = (1^3)\):
  
- For \((\{a, b\}) = (3^1)\):
  
- For \((\{c, d, f\}) = (1^1 2^1)\):

Note that given \(n\), its partitions are easily constructed. So are the Young diagrams.

The classes & their cycle structures of \(S_n\) can then be read off directly from the Young diagrams.

### Young Tableau

When the squares of a Young diagram are filled with the numbers \(\{1, 2, \ldots, n\}\), it becomes a **Young tableau** \(\Theta_\lambda\).

Since each Young diagram represents a class of \(S_n\), an element (permutation) in the class is just the cyclic permutations of the numbers within each column in the Young tableau. In this sense, each Young tableau is an element of a class in \(S_n\) if we treat tableaux that differ by mere cyclic permutations in any column as equivalent.

Using \(S_3\) as an example,

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>or</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

represents \((1 \times 3) = d \in (1^1 2^1)\)

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>or</th>
<th>2</th>
<th>3</th>
<th>or</th>
<th>3</th>
<th>2</th>
<th>or</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

represents \((1 \times 2 \times 3) = b \in (3^1)\)

Obviously, permutations of numbers in a row or non-cyclic permutations within a column will in general result in a different member of the class.

When the numbers are filled sequentially from top left to right bottom, the resultant Young tableau is called **normal** & denoted by \(\Theta_\lambda\).

If the magnitudes of the numbers in every row increase from left to right, & those in every column increase from top to bottom, the tableau is called **standard**.

We state without proof that the number of distinct standard tableaux of a Young diagram is equal to the dimension of the IR generated by that diagram.

Any Young tableau \(\Theta_\lambda\) can be obtained from \(\Theta_\lambda\) by an appropriate permutation \(p\).

This is denoted by \(\theta_\lambda = \Theta^p_\lambda = p \Theta_\lambda\),

Obviously \(q \Theta^p_\lambda = q p \Theta_\lambda = \Theta_q^{\lambda p}\).
Example  
Partition of 3

<table>
<thead>
<tr>
<th>Normal</th>
<th>Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3</td>
<td>1 3 2</td>
</tr>
</tbody>
</table>

5.3 Symmetrizers & Anti-Symmetrizers of Young Tableau

**Horizontal Permutations** \( \{ h^p_\lambda \} \)

The horizontal permutations \( \{ h^p_\lambda \} \) of a Young tableau \( \Theta^p_\lambda \) are permutations which leaves invariant the sets of numbers in each row of \( \Theta^p_\lambda \).

**Vertical Permutations** \( \{ v^p_\lambda \} \)

The vertical permutations \( \{ v^p_\lambda \} \) of a Young tableau \( \Theta^p_\lambda \) are permutations which leaves invariant the sets of numbers in each column of \( \Theta^p_\lambda \).

**Symmetrizers** \( s^p_\lambda \)

The symmetrizer \( s^p_\lambda \) of a Young tableau \( \Theta^p_\lambda \) is the sum of all horizontal permutations \( h^p_\lambda \):

\[
s^p_\lambda = \sum h^p_\lambda
\]

**Anti-Symmetrizers** \( a^p_\lambda \)

The anti-symmetrizer \( a^p_\lambda \) of a Young tableau \( \Theta^p_\lambda \) is the signed sum of all vertical permutations \( v^p_\lambda \):

\[
a^p_\lambda = \sum (-)^v v^p_\lambda
\]

**Irreducible / Young Symmetrizers** \( e^p_\lambda \)

The irreducible symmetrizer \( e^p_\lambda \) of a Young tableau \( \Theta^p_\lambda \) is the sum of the products of the horizontal & vertical symmetrizers.

\[
e^p_\lambda = s^p_\lambda a^p_\lambda = \sum (-)^v h^p_\lambda v^p_\lambda
\]
Example $S_3$

In cycle notations, elements of $S_3$ are:

$$e = e \hspace{1cm} a = (123) \hspace{1cm} b = (132)$$
$$c = (23) \hspace{1cm} d = (13) \hspace{1cm} f = (12)$$

The classes of $S_3$ have cycle structures:

$$\{e\} = (1^3) \hspace{1cm} \{a, b\} = (3^1) \hspace{1cm} \{c, d, f\} = (1^1 2^1)$$

The corresponding partitions are found using the formula

$$\lambda_i = \sum_{j=i}^n k_j \hspace{1cm} k_j = \# \text{ of } j \text{- cycle}$$

For class $\{e\}$:

$$k = \{k_1, k_2, k_3\} = [3, 0, 0]$$
$$\lambda = \{\lambda_1, \lambda_2, \lambda_3\} = [3 + 0 + 0, 0 + 0, 0] = [3, 0, 0] = [3]$$

For class $\{a, b\}$:

$$k = \{k_1, k_2, k_3\} = [0, 0, 1]$$
$$\lambda = \{\lambda_1, \lambda_2, \lambda_3\} = [0 + 0 + 1, 0 + 1, 1] = [1, 1, 1]$$

For class $\{c, d, f\}$:

$$k = \{k_1, k_2, k_3\} = [1, 1, 0]$$
$$\lambda = \{\lambda_1, \lambda_2, \lambda_3\} = [1 + 1 + 0, 1 + 0, 0] = [2, 1, 0] = [2, 1]$$

The normal Young tableau are:

$$\Theta_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} :$$

$$h_3: \{e, a, b, c, d, f\}$$
$$v_3: e$$
$$s_1 = \sum_p p = e + a + b + c + d + f \equiv S$$
$$a_1 = e$$
$$e_1 = s_1 a_1 = S$$

$$\Theta_2 = \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} :$$

$$h_3: \{e, f = (12)\}$$
$$v_3: \{e, d = (13)\}$$
$$s_2 = e + f = e + (12)$$
$$a_2 = e - d = e - (13)$$
$$e_2 = s_2 a_2 = (e + f) (e - d) = e - d + f - f d = e - d + f - b$$
$$\quad = e - (13) + (12) - (132)$$
\[ \Theta_3 \]

\[
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\]

\[
h_3 : e \\
v_3 : \{ e, a, b, c, d, f \} \\
s_3 = e \\
a_3 = \sum_p (-1)^p p = e - a - b + c + d + f = A \\
e_3 = A = e - a - b + c + d + f
\]

The only \textbf{standard Young tableau} is:

\[ \Theta_2' = \Theta_2^{23} = \begin{array}{c} 1 \ 3 \\ 2 \end{array} \]

\[
h_2' : \{ e, d = (13) \} \\
v_2' : \{ e, f = (12) \} \\
s_2 = e + d = e + (13) \\
a_2 = e - f = e - (12) \\
e_2 = s_2 a_2 = (e + d) (e - f) = e + d - f - d f = e + d - f - a \\
= e + (13) - (12) - (123)
\]

\section*{5.4 IR of \( S_n \)}

\begin{itemize}
\item \( \{ h_3 \} \) & \( \{ v_3 \} \) are subgroups of \( S_n \)
\end{itemize}

See Lemma IV.2 , Tung

\( e \) must belong to \( \{ h_3 \} \).

\( h_3^{-1} \) must be a horizontal permutations & hence in \( \{ h_3 \} \).

Products of two horizontal permutations must be another horizontal permutations & hence in \( \{ h_3 \} \).

Therefore, \( \{ h_3 \} \) is a group.

Since \( h_3 \) are permutations of \( n \) numbers, they must be elements of \( S_n \).

Thus \( \{ h_3 \} \) is a subgroup of \( S_n \).

Same argument is applicable to \( \{ v_3 \} \).

\begin{itemize}
\item \( s_i \) & \( a_i \) are essentially idempotent
\end{itemize}

See Lemma IV.2 & p.67–8, Tung.
Using the rearrangement theorem on the subgroups \( \{ h_1 \} & \{ v_1 \} \), we have
\[
s_3 h_3 = h_3' h_3 = h_3'' = s_3 \quad \text{where} \quad h_3'' = h_3' h_3
\]
Similarly,
\[
h_3', s_3 = s_3.
\]
\[
a_3 v_3 = \sum_v (-)^{v_3} v_3' v_3 = \sum_v (-)^{v_3} v_3' v_3'' = (-)^{v_3} a_3 \quad \text{where} \quad v_3'' = v_3' v_3
\]
Similarly,
\[
v_3 a_3 = (-)^{v_3} a_3
\]
\[
s_3 s_3 = \sum_h h_3 \sum_h h_3' = \sum_h h_3'' = \sum_h s_3 = n_3 s_3
\]
\[
a_3 a_3 = \sum_v (-)^{v_3} v_3 \sum_v (-)^{v_3} v_3' = \sum_v (-)^{v_3} v_3'' = \sum_v a_3 = m_3 a_3
\]
where \( n_3 = \# \) of horizontal permutations in \( \theta_3 \) & \( m_3 = \# \) of vertical permutations in \( \theta_3 \).

Now, the diagram \( \theta_3 \) consists of rows of \( \lambda_j \) blocks. Thus the \( \# \) of horizontal permutations \( n_3 \) is simply
\[
n_3 = \lambda_1 \ldots \lambda_n!
\]
The diagram \( \theta_3 \) consists of \( k_j \) columns of \( j \) blocks. Thus the \( \# \) of vertical permutations \( m_3 \) is
\[
m_3 = (1!)^{k_1} \ldots (j!)^{k_j} \ldots (n!)^{k_n}
\]

To summarize,
\[
s_3 h_3 = h_3 s_3 = s_3 \quad s_3 s_3 = n_3 s_3 \quad n_3 = \lambda_1 \ldots \lambda_n!
\]
\[
a_3 v_3 = v_3 a_3 = (-)^{v_3} a_3 \quad a_3 a_3 = m_3 a_3 \quad m_3 = (1!)^{k_1} \ldots (n!)^{k_n}
\]
Hence, \( s_3 \) & \( a_3 \) are essentially idempotent.

\[ x_3^p = px_3 p^{-1} \quad x = h, v, s, a, e \]

See Lemma IV.1, Tung.

Consider a tableau \( \theta_j \) & its permutation \( \theta_j^p = p \theta_j \).

An operation \( x_3 \) on \( \theta_j \) gives a tableau \( \theta_j^p = x_3 \theta_j \).

The problem is to find the corresponding operation \( x_3^p \) on \( \theta_j^p \) so that \( \theta_j^p = x_3^p \theta_j^p \) is the permutation of \( \theta_j^p \), i.e., \( \theta_j^p = p \theta_j^p \).

Thus
\[
p \theta_j^p = x_3 \theta_j = p x_3 p^{-1} p \theta_j = p x_3 p^{-1} \theta_j^p
\]
Setting this to
\[
\theta_j^p = x_3^p \theta_j^p
\]
We have
\[
x_3^p = px_3 p^{-1} .
\]
Example

Let \( p = (123) \)

\[
\begin{pmatrix}
1 & 2 \\
3 & \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix}
2 & 3 \\
1 & \end{pmatrix}
\]

Let \( x_3 \equiv h_3 = (12) \)

\[
\begin{pmatrix}
2 & 1 \\
3 & \end{pmatrix}
\]

\[
h_3^p = p \cdot h_3 \cdot p^{-1} = p \begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 3 & 2 \end{pmatrix} = \begin{pmatrix}
2 & 3 & 1 \\
3 & 2 & 1 \\
1 & 1 & 1 \end{pmatrix} = (23)
\]

\[
h_3^p \cdot \theta_3^p = \begin{pmatrix}
3 & 2 \\
1 & \end{pmatrix} = p \cdot h_3 \cdot \theta_3 = \begin{pmatrix}
3 & 2 \\
1 & \end{pmatrix}
\]

\[ p \neq h_3 \cdot v_3 \quad \iff \quad \exists 2 \text{ nos in 1 row of } \theta_3 = \text{ those in 1 column of } \theta_3^p. \]

See Lemma IV.3, Tung

Proof \( \iff \)

Assume \( p = h_3 \cdot v_3 \).

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 3 & 2 \end{pmatrix}
\]

where \( v_{3,3} = h_3 \cdot v_3 \cdot h_3^{-1} \) is the vertical permutation on tableau \( \theta_{3,3}^p \) that corresponds to \( v_3 \) on \( \theta_3 \).

Consider 2 numbers \( a \ & \ b \) in the same row of \( \theta_3 \).

They will remain in the same row in \( \theta_{3,3}^p \) since \( h_3 \) is a permutation of numbers in each row of \( \theta_3 \).

Thus, \( a \ & \ b \) must be in different columns of \( \theta_{3,3}^p \).

The operation \( v_{3,3}^p \) is a permutation of numbers in each column of \( \theta_{3,3}^p \).

Hence, \( a \ & \ b \) must remain in different columns of \( \theta_{3,3}^p \).

To summarize,

\[
p = h_3 \cdot v_3 \quad \Rightarrow \quad 2 \text{ numbers in the same row of } \theta_3 \text{ must be in different columns of } \theta_{3,3}^p.
\]

Taking the negation,

\[
p \neq h_3 \cdot v_3 \quad \iff \quad \text{At least 2 numbers in the same row of } \theta_3 \text{ appear in same columns of } \theta_{3,3}^p.
\]

Proof \( \Rightarrow \)

Let every number in the same row of \( \theta_3 \) be in different columns of \( \theta_{3,3}^p \).

Tableau \( \theta_{3,3}^p \) can be obtained from \( \theta_3 \) in the following manner.
Starting with the numbers in the 1st column of \( \theta_1^p \), they must belong to different rows in \( \theta_h \).

Hence, they can be brought to the 1st column by a horizontal permutation on \( \theta_h \).

The same procedure can be applied to all columns of \( \theta_1^p \).

Subsequently, we have a tableau \( \theta_{h,k}^p = h_k \theta_l \) each of whose columns contains the same numbers of the corresponding columns in \( \theta_h^p \).

Obviously, \( \theta_{h,k}^p \) can be brought to \( \theta_h^p \) by pure vertical permutations, ie.,

\[
\theta_h^p = v_{h,k}^p \theta_{h,k}^p = h_k \theta_k \theta_h \theta_k^{-1} h_k \theta_k \theta_h \theta_k = p \theta_h
\]

\[\Rightarrow p = h_k \theta_k \]

To summarize:

Every number in the same row of \( \theta_h \) be in different columns of \( \theta_h^p \) \( \Rightarrow p = h_k \theta_k \)

Taking the negation:

At least 2 numbers in the same row of \( \theta_h \) appear in same columns of \( \theta_h^p \) \( \Leftrightarrow p \neq h_k \theta_k \)

\[
- \quad p \neq h_k \theta_k \quad \Rightarrow \quad \exists \; h_k \; \& \; \tilde{v}_k \; \therefore \; p = h_k \; p \; \tilde{v}_k
\]

\[p \neq h_k \theta_k \quad \Rightarrow \quad \text{At least 2 numbers in the same row of } \theta_h \text{ appear in same columns of } \theta_h^p\]

Let the transposition of these 2 numbers be \( t \).

\[\Rightarrow t \in \{ h_k \} \; \& \; t \in \{ v_{h,k}^p \} \]

Thus we can write

\[t = \tilde{h}_k = v_{h,k}^p \]

where \( \tilde{h}_k \) is the transition in \( \{ h_k \} \) that equals to \( t \) & similarly for \( v_{h,k}^p \).

Using \( v_{h,k}^p = p \tilde{v}_k p^{-1} \) where \( \tilde{v}_k \in \{ v_k \} \)

we have

\[
\tilde{h}_k \; p \; \tilde{v}_k = \tilde{h}_k \; p \; p^{-1} \; v_{h,k}^p \; p = \tilde{h}_k \; v_{h,k}^p \; p = t^2 \; p = p
\]

since any transition is its own inverse, ie. \( t^2 = e \).

\[
- h_k \; r \; v_k = (-)^{v_k} \; r \; \forall \; h_k \; , \; v_k
\]

\[\Rightarrow \quad r \propto e_k \]

Let \( r = \sum_{p \in G} \alpha_p \; p \)

\[G = S_n \]

\[h_k \; r \; v_k = (-)^{v_k} \; r \quad \Longrightarrow \quad \sum_{p \in G} \alpha_p \; h_k \; p \; v_k = (-)^{v_k} \sum_{p \in G} \alpha_p \; p \]

Now,

\[
\sum_{p \in G} \alpha_p \; h_k \; p \; v_k = \sum_{q \in G} (\alpha_{h_k^{-1} \cdot q \cdot v_k^{-1}}) \; q \quad \text{where } q = h_k \; p \; v_k \; \text{so that } p = h_k^{-1} \; q \; v_k^{-1}
\]

\[= (-)^{v_k} \sum_{q \in G} \alpha_q \; q \]

\[\Rightarrow \quad \alpha_{h_k^{-1} \cdot q \cdot v_k^{-1}} = (-)^{v_k} \; \alpha_q \quad \forall \; h_k \; \& \; v_k\]
If $q \neq h_\lambda v_\lambda$,

$\exists$ transpositions $\tilde{h}_\lambda$ & $\tilde{v}_\lambda$ $\ni$ $q = \tilde{h}_\lambda q \tilde{v}_\lambda$

$\alpha_{h_\lambda^{-1} q \tilde{v}_\lambda} = a_q = (-)^{\lambda} a_q = -a_q$ where $(-)^\lambda = -1$ for transpositions.

Thus $a_q = 0$.

If $q = h_\lambda v_\lambda$,

$\alpha_{h_\lambda^{-1} q v_\lambda^{-1}} = a_e = (-)^{\lambda} a_q$

ie. $a_q = (-)^{\lambda} a_e = (-)^{\lambda} \xi$ where $\xi = a_e$ is a constant independent of $q$.

Thus

$r = \sum_{\rho \in G} \alpha_{\rho} \bar{p} = \xi \sum_{h_\lambda v_\lambda} (-)^{\lambda} h_\lambda v_\lambda = \xi e_\lambda$

$\blacklozenge \ x_\mu^p x_\nu^q \propto \delta_{\rho \nu} \quad x = s, a, e$

See Lemma IV.6, Tung

Without loss of generality, we can assume $\mu > \nu$, ie. the 1st non-zero $\mu_1 - \nu_1$ is positive.

Since tableau $\theta_\mu^q$ has rows of lengths $\{\mu_1 \ldots \mu_n\}$ & similarly for $\theta_\nu^p$, the longer rows in both tableau are equal in length until the $i$ th row corresponding to the 1st non-zero $\mu_i - \nu_i$. The $i$ th row of $\theta_\mu^q$ will be longer than any row in $\theta_\nu^p$ with label $j \geq i$.

Consider 1st the normal tableaux $\Theta_\mu$ & $\Theta_\nu$. The $i$ th in $\Theta_\mu$ will contain all the numbers in the $i$ th row of $\Theta_\nu$ plus at least 1 extra number, say $a$. Now, $a$ must appear in a row of $\Theta_\nu$ with label $j \geq i$.

In other words, there must be a number $b$ in the $i$ th row of $\Theta_\mu$ such that it is in the same column with $a$ in $\Theta_\nu$.

Since $\theta_\mu^q = q \Theta_\mu$, $\theta_\nu^q = q \Theta_\nu$, we see that the numbers $q_\mu$ & $q_\nu$ will be in the same row of $\theta_\mu^q$ & the same column in $\theta_\nu^q$. So do any $h_\mu \theta_\mu^q = \theta_\mu^{h_\mu q}$ & $\nu_\nu \theta_\nu^q = \theta_\nu^{\nu \nu q}$.

Now, given $\theta_\mu^q = q \Theta_\mu$, $\theta_\nu^p = p \Theta_\nu$, there exists $h_\mu$, $\nu_\nu$, $r$ such that $q = h_\mu r$ & $p = \nu_\nu r$.

Hence, $\theta_\mu^q = h_\mu \theta_\mu^q r_\nu^p \& \theta_\nu^p = \nu_\nu \theta_\nu^q$, so that there is at least 2 numbers appearing in both in a row of $\theta_\mu^q$ & a column of $\theta_\nu^p$.

Now, let the transposition of these 2 numbers be $t = \tilde{h}_\mu = \tilde{v}_\nu$.

$\rightarrow \quad t s_\mu^p s_\nu^q t = s_\mu^p$

$\quad t a_\nu^p = a_\nu^p t = (-)^{\nu} a_\nu^p = -a_\nu^p$

Hence

$a_\mu^p s_\nu^q = a_\mu^p t s_\nu^q = -a_\mu^p s_\nu^q = 0$

$s_\nu^p a_\mu^p = s_\nu^p t a_\mu^p = -s_\nu^p a_\mu^p = 0$

$e_\mu^p e_\nu^q = s_\mu^p a_\mu^p s_\nu^p a_\nu^p \neq 0$

$\blacklozenge \ s_1 \ r \ a_1 \ = \ \xi \ r \ e_1$

See Theorem 5.3, Tung

Let $u = s_1 \ r \ a_1$

$h_\lambda u v_\lambda = h_\lambda s_3 \ r \ a_1 v_\lambda = s_1 \ r \ a_3 \ (-)^{\lambda} = u (-)^\lambda \ \forall h_\lambda, v_\lambda$

$\rightarrow \quad u = s_1 \ r \ a_1 = \xi \ e_1$
\( e_\lambda^2 = \eta e_\lambda \)

See Theorem 5.3, Tung
\[ e_\lambda^2 = s_\lambda a_\lambda s_\lambda a_\lambda = \xi_{a_\lambda s_\lambda} e_\lambda = \eta e_\lambda \]

Thus, \( e_\lambda \) is essentially idempotent.

- \( e_\lambda \) are primitive idempotents

See Theorem 5.4, Tung
\[ e_\lambda r e_\lambda = s_\lambda a_\lambda r s_\lambda a_\lambda = \xi_{a_\lambda r s_\lambda} e_\lambda \]
Hence, \( e_\lambda \) are primitive idempotents.

- IRs generated by \( e_\lambda \) & \( e_\mu \) are equivalent

IRs generated by 2 primitive idempotents \( e_1 \) & \( e_2 \) are equivalent iff \( e_1 r e_2 \neq 0 \) for some \( r \in G \).

Now, using \( e_\mu^p = p e_\lambda p^{-1} \), we have
\[ e_\lambda^{-1} e_\mu^p = e_\lambda^{-1} p e_\lambda^{-1} p = e_\lambda^{-1} e_\mu^p = e_\lambda^{-1} \eta e_\lambda^{-1} \neq 0 \]

- IRs generated by \( e_\mu \) & \( e_\nu \) are inequivalent if \( \mu \neq \nu \)

\[ \forall \; p \in G \quad e_\mu p e_\nu = e_\mu p e_\nu p^{-1} p = e_\mu e_\nu^p p = 0 \quad \text{if} \quad \mu \neq \nu \]
\[ \forall \; r \in G \quad r = \sum p \quad e_\mu r e_\nu = 0 \]
Hence, the IRs generated by \( e_\mu \) & \( e_\nu \) are inequivalent.

- \( e_\lambda \) of all normal Young tableaux generate all IRs of the group

See Theorem 5.7, Tung

Since
1. \# of Young diagrams = \# of normal Young tableaux = \# of classes = \# of inequivalent IRs.
2. There's one \( e_\lambda \) for each normal tableau \( \Theta_\lambda \).
3. IRs generated by \( e_\lambda \) & \( e_\mu \) are inequivalent if \( \lambda \neq \mu \).

Hence, the set of all \( e_\lambda \)'s generates all inequivalent IRs of \( S_n \).

- Example: \( S_3 \)

As shown before, the partitions of 3 are \( \lambda = \{ 3 \}, \{ 2, 1 \}, \{ 1, 1, 1 \} \)
For $\lambda = [3]$, there is only one standard tableau $\Theta_1$. 

Hence, the IR is $1 - D$. 
The Young symmetrizer is 
$$e_1 = S$$ 

Since $e_1 g = e_1 \forall g \in S_3$, the IR is the identity representation, as expected.

For $\lambda = [2, 1]$, there are 2 standard tableaux $\Theta_2$ & $\Theta_2^{(23)}$. 
The IRs are $2 - D$. 
For $\Theta_2$, the Young symmetrizer is 
$$e_2 = e - (13) + (12) - (132)$$

- $e_i$ of all standard tableaux generate the complete IR decomposition of the Regular Representation

The # of standard tableaux for each Young diagram is equal to the dimension of the IR generated from it. Since $\Gamma^{\text{reg}} = \sum_\mu n_\mu \Gamma^\mu$, the theorem is proved.

- $S_4$

The partitions of 4 are 
$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

The corresponding Young diagrams & cycle structures are are:

$\lambda = [4]$ 

\begin{array}{c|c|c|c} 
 & & & \\
\end{array} 

(1^4) 

k = (4 \times 0 \times 0 \times 0) 

# of elements: $4! / 1^4 \times 4! = 1$ 

Elements: $e$

$\lambda = [3 \times 1]$ 

\begin{array}{c|c|c} 
 & & \\
\end{array} 

(1^2 2^1)k = (2 \times 1 \times 0 \times 0) 

# of elements: $4! / 1^2 \times 2 \times 2! = 6$ 

Elements: (1 $\times$ 2), (1 $\times$ 3), (1 $\times$ 4), (2 $\times$ 3), (2 $\times$ 4), (3 $\times$ 4)

$\lambda = [2^2]$ 

\begin{array}{c|c} 
 & \\
\end{array} 

(2^2) 

k = (0 \times 2 \times 0 \times 0) 

# of elements: $4! / 2^2 \times 2! = 3$ 

Elements: (1 $\times$ 2) (3 $\times$ 4), (1 $\times$ 3) (2 $\times$ 4), (1 $\times$ 4) (2 $\times$ 3)

$\lambda = [2 \times 1^2]$ 

\begin{array}{c|c} 
 & \\
\end{array} 

(1^1 3^1)k = (1 \times 0 \times 1 \times 0) 

# of elements: $4! / 3 = 8$ 

Elements: (1 $\times$ 2 $\times$ 3), (1 $\times$ 2 $\times$ 4), (1 $\times$ 3 $\times$ 2), (1 $\times$ 3 $\times$ 4), (1 $\times$ 4 $\times$ 2), (1 $\times$ 4 $\times$ 3), (2 $\times$ 3 $\times$ 4), (4 $\times$ 3 $\times$ 2)
\[ \lambda = \{ 1^4 \} \quad (4^1) \quad k = (0 \times 0 \times 0 \times 1) \]

# of elements: 4! / 4 = 6
Elements: (1 \times 2 \times 3 \times 4), (1 \times 2 \times 4 \times 3), (1 \times 3 \times 2 \times 4), (1 \times 3 \times 4 \times 2), (1 \times 4 \times 2 \times 3), (1 \times 4 \times 3 \times 2)

Thus, there are 5 classes, namely, \{ 1^4 \}, \{ 1^2 \ 2^1 \}, \{ 2^2 \}, \{ 1^1 \ 3^1 \}, \& \{ 4^1 \}.

\[ \lambda = \{ 4 \} \]

There's only 1 standard tableau:

\[
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array}
\]

\[ [h_4] = S_n \quad s_4 = \sum g \]

\[ [v_4] = e \quad a_4 = e \]

\[ e_4 = s_4 \]

Since \( e_4 g = e_4 \), this gives the identity representation.

\[ \lambda = \{ 3 \times 1 \} \]

There are 3 standard tableaux so that the IRs are 3 - D.

\[
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
4 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 \\
\hline
\end{array}
\]

For the normal tableau:

\[ [h_3] = e, \quad (1 \times 2 \times 3), \quad (1 \times 3 \times 2), \quad (1 \times 2), \quad (1 \times 3), \quad (2 \times 3) \]

\[ s_3 = e + (1 \times 2 \times 3) + (1 \times 3 \times 2) + (1 \times 2) + (1 \times 3) + (2 \times 3) \]

\[ [v_3] = e, \quad (1 \times 4) \]

\[ a_3 = e - (1 \times 4) \]

\[ e_3 = [e + (1 \times 2 \times 3) + (1 \times 3 \times 2) + (1 \times 2) + (1 \times 3) + (2 \times 3)] e - (1 \times 4) \]

\[ = e + (1 \times 2 \times 3) + (1 \times 3 \times 2) + (1 \times 2) + (1 \times 3) + (2 \times 3) \]

\[ - (1 \times 4) - (1 \times 4 \times 2 \times 3) - (1 \times 4 \times 3 \times 2) - (1 \times 4 \times 2) - (1 \times 4 \times 3) - (2 \times 3) (1 \times 4) \]

### 5.5 Symmetry Classes of Tensors

Let \( V_m \) be a \( m \)-D vector space.

The set \( \{ g \} \) of all invertible linear transformations on \( V_m \) forms a group called the **general linear group** \( \text{GL}(m, \mathbb{C}) \).

In this chapter, we call it simply \( G_m \).

Given any basis \( \{ | i \rangle \}, \quad i = 1, 2, \ldots, m \) on \( V_m \), a natural matrix rep of \( G_m \) is:

\[ (5.5-1) \quad g | i \rangle = | j \rangle g^j_i \]

where \( g^j_i \) are elements of an invertible \( m \times m \) matrix (i.e. \( \det g \neq 0 \)).
Definition 5.5 (Tensor Space):

The direct product space $V_m \times V_m \times \ldots \times V_m$, involving $n$ factors of $V$ shall be referred to as the **tensor space** and denoted by $V_m^n$.

* 
Given a basis $\{| i \rangle \}$ on $V_m$, a natural basis for $V_m^n$ is obtained in the form:

$$
| i_1 i_2 \ldots i_n \rangle = | i_1 \rangle | i_2 \rangle \ldots | i_n \rangle
$$

Tung's notation

$$
| I \rangle
$$

My notation

An arbitrary element $x$ of the tensor space $V_m^n$ has the decomposition,

$$
| x \rangle = | i_1 i_2 \ldots i_n \rangle x^{i_1 i_2 \ldots i_n}
$$

where $x^I = x^{i_1 i_2 \ldots i_n}$ are the **tensor components** of $x$.

Elements of $G_m$ (defined on $V_m$) induce the following linear transformations on $V_m^n$

$$
g | I \rangle = | J \rangle D(g)_I^J
$$

where

$$
D(g)_I^J = g^{i_1}_{j_1} g^{i_2}_{j_2} \ldots g^{i_n}_{j_n}
$$

$\forall \ g \in G_m$

It can easily be verified that $\{D(g)\}$ forms a $(n \cdot m)$-D rep of $G_m$, and that for any $\ | x \rangle \in V_m^n$,

$$
g | x \rangle = | x_g \rangle = | J \rangle x_g^J
$$

where

$$
x_g^J = D(g)_I^J x^I
$$

Independently, $S_n$ also has a natural realization on $V_m^n$.

In particular, consider the mapping

$$
p \in S_n \; \rightarrow \; p = \text{linear transformation on } V_m^n
$$

defined by,

$$
p | x \rangle = | x_p \rangle
$$

where $\ | x \rangle \in V_m^n$ and

$$
x_p^I = x^I
$$

with

$$
I = i_1 i_2 \ldots i_n
$$

$$
I_p = i_{p_1} i_{p_2} \ldots i_{p_n}
$$

In terms of the basis vectors $\{| I \rangle\}$, the action of $p$ goes as

$$
p | I \rangle = | I_{p}^{-1} \rangle
$$

where

$$
I_{p}^{-1} = i_{p_1}^{-1} i_{p_2}^{-1} \ldots i_{p_n}^{-1}
$$

Therefore, if we write

$$
p | I \rangle = | J \rangle D(p)_I^J
$$

then

$$
D(p)_I^J = \delta_{I_{p}^{-1}}^J = \delta^J_{I_p}
$$

$$
= \delta_{i_{p_1}^{-1} i_1} \ldots \delta_{i_{p_n}^{-1} i_n} = \delta_{i_1} \ldots \delta_{i_n}
$$
The last equality involves permuting the $n$ $\delta$-factors by $p$. The reader should verify that Eq. (5.5-9), or equivalently (5.5-11), does provide a rep for $S_n$. [Problem 5.5]

Both $D[G_m]$ and $D[S_n]$ are in general reducible.

As $S_n$ is a finite group, $D[S_n]$ can be decomposed into IRs. This will be achieved through the $e_{\mu}'s$.

$G_m$ is an infinite group. A general $D[G_m]$ is not guaranteed to be fully decomposable. However, the reduction of $V_m^n$ by $e_{\mu}'s$ from the $S_n$ algebra leads naturally to a full decomposition of $D[G_m]$. This is a consequence of the fact that linear transformations on $V_m^n$ representing $g \in G_m$ and $p \in S_n$ commute with each other, and each type of operator constitutes essentially the "maximal set" which has this property.

The underlying principle behind the following results is just a generalization of the familiar facts that:

(i) a Complete Set of Commuting Operators (CSCO) on a vector space share common eigenvectors.

(ii) a decomposition of reducible subspaces with respect to some subset of the commuting operators often leads naturally to diagonalization of the remaining operator(s).

We have made use of this principle to diagonalize the Hamiltonian for a general 1-D lattice by taking advantage of $T_1$. Similarly, as is often done in the solution to physical problems involving spherical symmetry, the Hamiltonian is diagonalized by decomposing first with respect to angular momentum operators.

**Lemma 5.1:**

The rep matrices $D(G_m)$, Eq. (5.5-6), and $D(S_n)$, Eq. (5.5-13) satisfy the following symmetry relation:

\[ D_I' = D_{I_\mu} \]

where

\[ I = i_1 i_2 \ldots i_n \quad I_\mu = i_{q_1} i_{q_2} \ldots i_{q_n} \]

\[ q = \begin{pmatrix} 1 & 2 & \ldots & n \\ q_1 & q_2 & \ldots & q_n \end{pmatrix} \in S_n \]

Linear transformations on $V_m^n$ satisfying this condition are said to be **symmetry-preserving**.

**Proof:**

\[ D(g)_I' = g^{i_1}_{i_{q_1}} g^{i_2}_{i_{q_2}} \ldots g^{i_n}_{i_{q_n}} \quad \forall \ g \in G_m \]

\[ = g^{i_{q_1}}_{i_1} g^{i_{q_2}}_{i_2} \ldots g^{i_{q_n}}_{i_n} \]

\[ = D(g)_{I_\mu} \]

\[ D(p)_I' = \delta_{I_\mu^{i_1}} = \delta_{J_I p} \quad \forall \ p \in S_n \]

\[ = \delta_{I_\mu^{i_1}} = D(p)_{I_\mu} \]

**Theorem 5.9:**

The two sets of matrices $\{ D(p) \cdot p \in S_n \}$ and $\{ D(g) \cdot g \in G_m \}$ commute with each other.

**Proof:**

Consider the action of $p g$ and $g p$ on the basis vectors in turn:
\[ p g | I \rangle = p | J \rangle D(g)^I_j \]
\[ = | J_{p^{-1}} \rangle D(g)^I_j \]
\[ = | J \rangle D(g)^I_j \]

\[
\begin{align*}
(i) & \quad g p | I \rangle = g | J \rangle \\
& \quad = | J_{p^{-1}} \rangle D(g)^I_j \\
& \quad = | J \rangle D(g)^I_j \\
\end{align*}
\]
Hence
\[
pg | I \rangle = gp | I \rangle \quad \forall | I \rangle
\]
Therefore
\[
p g = g p
\]

**Example 1: \( V_2^2 \)**

Consider second rank tensors \( n = 2 \) in 2-D space \( m = 2 \).
The basis vectors will be denoted by
\[
| ++ \rangle \quad | - + \rangle \quad | - - \rangle \quad | ++ \rangle
\]
Now, \( S_2 = \{ e, (12) \} \).
Since \( e \) leads to trivial results, we need only to consider \( p = (12) \) and its interplay with elements of \( G_2 \):
\[
\begin{align*}
pg | \pm \pm \rangle &= p | i k \rangle g^i_k g^k_\pm = | k i \rangle g^k_\pm g^i_\pm \\
gp | \pm \pm \rangle &= g | \pm \pm \rangle = | k i \rangle g^k_\pm g^i_\pm = pg | \pm \pm \rangle \\
pg | \pm \mp \rangle &= p | i k \rangle g^i_k g^k_\mp = | k i \rangle g^k_\mp g^i_\mp \\
gp | \pm \mp \rangle &= g | \pm \mp \rangle = | k i \rangle g^k_\mp g^i_\mp = pg | \pm \mp \rangle
\end{align*}
\]
These equalities hold for any element \( g \in G_2 \).

**Irreducible Subspaces of \( V_m^n \)**

We shall now decompose \( V_m^n \) into irreducible subspaces wrt \( S_n \) and \( G_m \), utilizing the \( e_\lambda^\rho \)'s associated with various \( \Theta_\lambda^\rho \) of \( S_n \).
Let \( L_\lambda \) be the left ideal generated by \( e_\lambda \). The main results will be:

(i) For a fixed \( \alpha \in V_m^n \), the subspace \( \{ r | \alpha \rangle ; r \in L_\lambda \} \) is irreducibly invariant under \( S_n \);

(ii) For a fixed \( \Theta_\lambda^\rho \), the subspace \( \{ e_\lambda^\rho | \alpha \rangle ; | \alpha \rangle \in V_m^n \} \) is irreducibly invariant under \( G_m \);

(iii) \( V_m^n \) can be decomposed in the "factorized" basis vectors \( | \lambda, \alpha, \alpha \rangle \) where \( \lambda \) denotes a symmetry class specified by a Young diagram; \( \alpha \) & \( \alpha \) labels the various irreducible invariant subspaces under \( S_n \) & \( G_m \), resp.

**Definition 5.6**

(Tensors of Symmetry \( \Theta_\lambda^\rho \) and Tensors of Symmetry Class \( \lambda \))

To each Young tableau \( \Theta_\lambda^\rho \) we associate tensors of the symmetry \( \Theta_\lambda^\rho \) consisting of
\[
T_\lambda^\rho = \{ e_\lambda^\rho | \alpha \rangle ; | \alpha \rangle \in V_m^n \}.
\]
For a given Young diagram characterized by \( \lambda \), the set of tensors
\[
T_\lambda = \{ r e_\lambda | \alpha \rangle , r \in S_n , | \alpha \rangle \in V_m^n \}
\]
is said to belong to the symmetry class \( \lambda \).
Theorem 5.10
For a given \( | \alpha \rangle \), let \( T_\lambda(\alpha) = \{ r e_\lambda | \alpha \rangle, \ r \in S_n \} \).

(i) \( T_\lambda(\alpha) \) is an irreducible invariant subspace with respect to \( S_n \);

(ii) if \( T_\lambda(\alpha) \neq \emptyset \), then the realization of \( S_n \) on \( T_\lambda(\alpha) \) coincides with the IR generated by \( e_\lambda \) on \( S_n \).

Proof:

(i) Let \( | x \rangle \in T_\lambda(\alpha) \), then by definition,
where
\[
| x \rangle = r e_\lambda | \alpha \rangle
\]
for some \( r \in S_n \)
hence, \( \forall \ p \in S_n, \)
\[
p | x \rangle = p r e_\lambda | \alpha \rangle = q e_\lambda | \alpha \rangle \in T_\lambda(\alpha)
\]
\( [ q = p \ r \in S_n ] \)

This means \( T_\lambda(\alpha) \) is invariant under \( S_n \).

(ii) Since \( T_\lambda(\alpha) \) is not empty, we know \( e_\lambda | \alpha \rangle \neq 0 \). Let \( \{ r_i e_\lambda \} \) be a basis of \( L_\lambda \), then \( \{ r_i e_\lambda \ | \alpha \} \) form a basis of \( T_\lambda(\alpha) \).
Hence, if
\[
p | r_i e_\lambda \rangle = | p r_i e_\lambda \rangle = | r_i e_\lambda \rangle D( p )^i_j \text{ on } S_n
\]
then,
\[
p r_i e_\lambda | \alpha \rangle = r_i e_\lambda | \alpha \rangle D( p )^i_j \text{ on } T_\lambda(\alpha)
\]
for all \( p \in S_n \).
Hence the invariant subspace is irreducible, and the representation matrices on \( T_\lambda(\alpha) \) coincide with those on \( S_n \). QED

\[ \ast \ast \]

Let \( \Theta_{\lambda=\alpha} = \ldots \ldots \ldots \) then \( e_\lambda = \sum_p \frac{p}{n!} \) is the total symmetrizer.

Since \( r e_\lambda = e_\lambda \) for all \( r \in S_n \), the left ideal \( L_\lambda \) is 1-D.
Correspondingly, for any given \( | \alpha \rangle \in V_m^n \), the irreducible subspace \( T_\lambda(\alpha) \) consists of all multiples of \( e_\lambda | \alpha \rangle \).
These are totally symmetric tensors, as it is straightforward to verify:

\[
(5.5-15) \quad e_\lambda | \alpha \rangle n! = \sum_p p | I \rangle \alpha^i = \sum_p | I_{p-i} \rangle \alpha^i = | I \rangle \sum_p \alpha^i
\]
hence the components are totally symmetric in the \( n \)-indices.
The realization of \( S_n \) on \( T_\lambda(\alpha) \) is the 1-D identity rep because all permutations leave a totally symmetric tensor unchanged.

Example 2:

Consider third rank tensors \((n = 3)\) in two dimensions \((m = 2)\). Four distinct totally symmetric tensors can be generated by starting with different elements of \( V_2^{n=3} \).

\[
e_3 = \frac{1}{3!} [ e + (12) + (13) + (23) + (123) + (321) ]
\]
(i) \( | \alpha = 1 \rangle = | +++ \rangle \)
\( e_s | \alpha \rangle = | +++ \rangle \equiv | s, 1, 1 \rangle \)

(ii) \( | \alpha = 2 \rangle = | ++- \rangle \)
\( e_s | \alpha \rangle = \frac{1}{6} \left[ | ++- \rangle + | +-- \rangle + | -++ \rangle + | -+- \rangle + | +++ \rangle + | ++- \rangle \right] 
= \frac{1}{3} [ | ++- \rangle + | -++ \rangle + | +-- \rangle ] 
\equiv | s, 2, 1 \rangle \)

(iii) \( | \alpha = 3 \rangle = | -+- \rangle \)
\( e_s | \alpha \rangle = \frac{1}{6} \left[ | -+- \rangle + | +-- \rangle + | -++ \rangle + | +-- \rangle + | ++- \rangle + | -+- \rangle \right] 
= \frac{1}{3} [ | -+- \rangle + | +-- \rangle + | +-- \rangle ] 
\equiv | s, 3, 1 \rangle \)

(iv) \( | \alpha = 4 \rangle = | -++ \rangle \)
\( e_s | \alpha \rangle = | -++ \rangle \equiv | s, 4, 1 \rangle \)

In the last column, we introduced the \( | \lambda, \alpha, a \rangle \) labelling scheme. This classification is used extensively in the following discussions. Each of the above totally symmetric tensors is invariant under all \( p \in S_3 \).

Together, they represent all totally symmetric tensors that can be constructed in \( V_2^3 \); they are tensors of the symmetry class \( s \), where \( s \) represents the Young tableau with one single row.

We shall denote the subspace of tensors of the symmetry class \( s \) by \( T_s \).

\[ \] Can we similarly generate totally anti- symmetric tensors in \( V_m^n \)? We leave as an exercise [ Problem 5.6 ] for the reader to show that they exist only if \( m \geq n \). The total anti- symmetric is \( e_a = \sum_p (-p) \frac{p}{n!} \). Since \( p e_a = (-p) e_a \), both \( L_a \) and \( T_a(a) \) are 1-D, and the realization of \( S_n \) on \( T_a(a) \) corresponds to the 1-D rep \( p \rightarrow (-p) \).

**Example 3:**

There is one and only one independent totally anti- symmetric tensor of rank \( n \) in \( n\)-D space, usually denoted by \( e \). In 2-D, its components are
\( e^{12} = -e^{21} = 1, e^{11} = e^{22} = 0 \)

In 3-D, the components are \( e^{ijk} = \pm 1 \) according to whether \( (ijk) \) is an even or odd permutation of \( (123) \); else, if any two indices are equal, then \( e^{ijk} = 0 \).

**Example 4:**

Consider second rank tensors (\( n = 2 \)) in \( m\)-D (\( m \geq 2 \)),
\[ e_s | ii \rangle = | ii \rangle \quad i = 1, 2, \ldots, m \]
\[ e_s | ij \rangle = \frac{1}{2} ( | ij \rangle + | ji \rangle ) \quad i \neq j \]

There are \( m(m-1)/2 \) distinct anti- symmetric tensors, as
\[ e_a | ii \rangle = 0 \]
\[ e_a | ij \rangle = \frac{1}{2} ( | ij \rangle - | ji \rangle ) \quad i \neq j \]
Let us now turn to tensors with mixed symmetry.

- **Example 5:**

  We return to $V_2^3$.

  Consider

  \[
  \Theta_m = \begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix} \quad \text{and} \quad e_m = \frac{1}{4} \left( e + (12) \right) \left( e - (31) \right)
  \]

  Since

  \[ e_m \mid +++ \rangle = e_m \mid --- \rangle = 0 \]

  only two independent irreducible invariant subspaces of tensors can be generated.

  (i) By choosing $\mid \alpha \rangle = \mid +++ \rangle$, we obtain:

  \[
  e_m \mid \alpha \rangle = \frac{1}{4} \left( e + (12) \right) \left( \mid +++ \rangle - \mid --- \rangle - \mid +-- \rangle \right)
  \]

  \[
  = \frac{1}{4} \left( \mid +++ \rangle - \mid +-- \rangle + \mid +-- \rangle - \mid +-- \rangle \right)
  \]

  \[
  = \frac{1}{4} \left( 2 \mid ++- \rangle - \mid --+ \rangle - \mid +-- \rangle \right)
  \]

  \[
  = \mid m, 1, 1 \rangle \quad \text{[} \mid \lambda, \alpha, \alpha \rangle \text{]}
  \]

  \[
  (23) \ e_m \mid \alpha \rangle = - \left( 23 \right) \left( \mid ++- \rangle - \mid --+ \rangle - \mid +-- \rangle \right)
  \]

  \[
  = \frac{1}{4} \left( 2 \mid ++- \rangle - \mid --+ \rangle - \mid +-- \rangle \right)
  \]

  \[
  = \mid m, 1, 2 \rangle
  \]

  and, for any $r \in S_3$, $r e_m \mid \alpha \rangle$ is a linear combination of the above two tensors. These two mixed tensors form a basis for $T_{\lambda \nu \omega}(1)$.

  (ii) By choosing $\mid \alpha \rangle = \mid --- \rangle$, we obtain:

  \[
  e_m \mid \alpha \rangle = \frac{1}{4} \left( 2 \mid --+ \rangle - \mid +-- \rangle - \mid +-- \rangle \right)
  \]

  \[
  = \mid m, 2, 1 \rangle
  \]

  \[
  (23) \ e_m \mid \alpha \rangle = \frac{1}{4} \left( 2 \mid --+ \rangle - \mid +-- \rangle - \mid +-- \rangle \right)
  \]

  \[
  = \mid m, 2, 2 \rangle
  \]

  as the basis for another irreducible invariant subspace of tensors with mixed symmetry $T_{\lambda \nu \omega}(2)$.

  The realization of the group $S_3$ on either $T_{\lambda \nu \omega}(1)$ or $T_{\lambda \nu \omega}(2)$ corresponds to the 2-D IR discussed in Sec. 5.2 and described earlier in Chap. 3 [ cf Table 3.3 ].

  The two tensors of mixed symmetry $\mid m, i, 1 \rangle$, $i = 1, 2$ (first ones of the two sets given above), are two linearly independent tensors of the form $e_m \mid \alpha \rangle$ with $\mid \alpha \rangle$ ranging over $V_m$ [ Problem 5.8 ] They are tensors of the symmetry $\Theta_m$. We call the subspace spanned by these vectors $T_{m \nu \omega}(1)$. $T_{m \nu \omega}(1)$ is an invariant subspace under $G_2$ since

  \[
  g e_m \mid \alpha \rangle = e_m g \mid \alpha \rangle \in T_{m \nu \omega}(1) \quad \forall \quad \mid \alpha \rangle \in V_m
  \]

  One can also show that this invariant subspace is irreducible under $G_2$. [ Problem 5.8 ]
Similarly, the two tensors \( | m, i, 2 \rangle \), \( i = 1, 2 \) (second ones of the two sets) are two linearly independent tensors of the form \( e_{m}^{(23)} | \alpha \rangle \) as can easily be verified by noting that \( (23) e_{m} = e_{m}^{(23)} \). They are tensors of the symmetry \( \Theta_{m}^{(23)} \), denote the subspace spanned by these tensors by \( T_{m}^{(2)} \). \( T_{m}^{(2)} \) is also invariant under group transformations of \( G_{2} \), and it is irreducible. Together, the two sets \( \{ T_{m}^{(2)} | a \rangle, a = 1, 2 \) comprise tensors of the symmetry class \( m \), where \( m \) denotes the Young diagram (frame) associated with the normal tableau \( \Theta_{m} \). For the sake of economy of indices, we shall use \( \langle \alpha \rangle \) in place of the label \( \langle i \rangle \) from now on; it is understood that the range of this label is equal to the number of independent tensors that can be generated by \( e_{m}^{l} | \alpha \rangle \) with \( \langle \alpha \rangle \in V_{n}^{6} \).

We note that for the 8-D tensor space \( V_{2}^{8} \), the use of Young symmetrizers (in Examples 2 and 5) leads to the complete decomposition into irreducible tensors \( | \lambda, \alpha, a \rangle \) where \( \lambda (= s, m) \) characterizes the symmetry class (Young diagram); \( \langle \alpha \rangle \) labels the distinct (but equivalent) sets of tensors \( T_{\alpha} \) invariant under \( S_{n} \); and \( \langle \alpha \rangle \) labels the basis elements within each set \( T_{\alpha} \), it is associated with distinct symmetries (tableaux) in the same symmetry class. We have 4 totally symmetric tensors (Example 2) and 2 sets of 2 linearly independent mixed symmetry tensors. The latter can be classified either as belonging to two invariant subspaces under \( S_{3} \) \( \{ T_{m}^{(a)} | a \rangle, a = 1, 2 \) \), or as belonging to two invariant subspaces under \( G_{2} \) \( \{ T_{m}^{(a)} | a \rangle, a = 1, 2 \) \). The latter comprise of tensors of two distinct symmetries associated with \( \Theta_{m} \) and \( \Theta_{m}^{(23)} \).

Bearing in mind these results for \( V_{2}^{8} \), we return to the general case.

**Theorem 5.11:**

(i) Two tensor subspaces irreducibly invariant with respect to \( S_{n} \) and belonging to the same symmetry class either overlap completely or they are disjoint;

(ii) Two irreducible invariant tensor subspaces corresponding to two distinct symmetry classes are necessarily disjoint.

**Proof:**

(i) Let \( T_{\lambda}^{(a)} \) and \( T_{\lambda}^{(b)} \) be two invariant subspaces generated by the same irreducible symmetrizer \( e_{\lambda} \). Either they are disjoint or they have at least one non-zero element in common. In the latter case, there are \( s, s' \in S_{n} \), such that

\[
se_{\lambda} | \alpha \rangle = s'e_{\lambda} | \beta \rangle
\]

This implies,

\[
rs e_{\lambda} | \alpha \rangle = rs' e_{\lambda} | \beta \rangle \quad \forall r \in S_{n}
\]

When \( r \) ranges over all \( S_{n} \), so do \( rs \) and \( rs' \). Therefore, the left-hand side of the last equation ranges over \( T_{\lambda}^{(a)} \) and the right-hand side ranges over \( T_{\lambda}^{(b)} \), hence the two invariant subspaces coincide.

(ii) Given any two subspaces \( T_{\lambda}^{(a)} \) and \( T_{\mu}^{(b)} \) invariant under \( S_{n} \); their intersection is also an invariant subspace. If \( T_{\lambda}^{(a)} \) and \( T_{\mu}^{(b)} \) are irreducible, then either the intersection is the null set or it must coincide with both \( T_{\lambda}^{(a)} \) and \( T_{\mu}^{(b)} \). If \( \lambda \) and \( \mu \) correspond to different symmetry classes, then the second possibility is ruled out. Hence \( T_{\lambda}^{(a)} \) and \( T_{\mu}^{(b)} \) have no elements in common if \( \lambda \neq \mu \). QED

These general results permit the complete decomposition of the tensor space \( V_{n}^{n} \) into irreducible subspace \( T_{\lambda}^{(a)} \) invariant under \( S_{n} \). As explained when working on the the example of \( V_{2}^{8} \), we shall use \( \alpha \) as the label for distinct subspaces corresponding to the same symmetry class \( \lambda \). The decomposition can be expressed as

\[
V_{n}^{n} = \sum_{\lambda \in \Theta} \sum_{\mu \in \Theta} T_{\lambda}^{(a)}
\]
The basis elements of the tensors in the various symmetry classes are denoted by \(| \lambda, \alpha, a \rangle\) where \(a\) ranges from 1 to the dimension of \(T_\lambda(a)\). We can choose these bases in such a way that the representation matrices for \(S_n\) on \(T_\lambda(a)\) are identical for all \(a\) associated with the same \(\lambda\), or

\[(5.5-17) \quad p \mid \lambda, \alpha, a \rangle = \mid \lambda, \alpha, b \rangle D_\lambda(p)_a^b\]

independently of \(a\).

The central result of this section will be that the decomposition of \(V_m^n\) according to the symmetry classes of \(S_n\), as described above, automatically provides a complete decomposition with respect to the general linear group \(G_m\) as well. We have already seen how this worked out in the case of \(V_2^3\).

**Theorem 5.12:**

If \(g \in G_m\) and \(\{ \mid \lambda, \alpha, a \rangle \}\) is the set of basis tensors generated according to the above procedure, then the subspaces \(T_\lambda'(a)\) spanned by \(\{ \mid \lambda, \alpha, a \rangle \}\) with fixed \(\lambda\) and \(a\) are invariant with respect to \(G_m\), and the representations of \(G_m\) on \(T_\lambda'(a)\) are independent of \(a\); i.e.,

\[(5.5-18) \quad g \mid \lambda, \alpha, a \rangle = \mid \lambda, \beta, a \rangle D_\lambda(g)_{a}^{\beta}a\]

**Proof:**

(i) Given \(r e_\lambda \mid \alpha \rangle \in T_\lambda(a)\) and \(g \in G_m\), we have

\[g (r e_\lambda \mid \alpha \rangle = (r e_\lambda) g \mid \alpha \rangle \in T_\lambda(g \alpha)\]

Hence, the operations of the linear group do not change the symmetry class of the tensor, or

\[g \mid \lambda, \alpha, a \rangle = \mid \lambda, \beta, a \rangle D_\lambda(g)_{a}^{\beta}a\]

(ii) We now show that \(D_\lambda(\ g\) is diagonal in the indices \((b, a)\). To this end, we note, for \(g \in G_m\), and \(p \in S_n\),

\[p g \mid \lambda, \alpha, a \rangle = p \mid \lambda, \alpha, c \rangle D_\lambda(p)_a^c \mid \lambda, \alpha, a \rangle = \mid \lambda, \beta, b \rangle D_\lambda(g)_{a}^{\beta} D_\lambda(p)_a^c\]

and

\[p g \mid \lambda, \alpha, a \rangle = p \mid \lambda, \beta, c \rangle D_\lambda(g)_{a}^{\beta} \mid \lambda, \alpha, a \rangle = \mid \lambda, \beta, b \rangle D_\lambda(p)_a^c D_\lambda(g)_{a}^{\beta}a\]

Since \(g p = p g\) (Theorem 5.9), the two product matrices on the right-hand sides can be equated to each other. For clarity, let us designate quantities in square brackets as matrices in the space of Latin indices, and suppress these indices. We obtain

\[(5.5-19) \quad \left[D_\lambda(g)_{a}^{\beta}\right] \left[D_\lambda(p)\right] = \left[D_\lambda(p)\right] = \left[D_\lambda(g)_{a}^{\beta}\right]\]

For given \(g\), this equation holds for all \(p \in S_n\). According to Schur’s Lemma, the matrix \(D_\lambda(g)_{a}^{\beta}a\) must be proportional to the unit matrix in the Latin indices. QED

**Theorem 5.13 (Irreducible Representations of \(G_m\))**:

The reps of \(G_m\) on the subspace \(T_\lambda'(a)\) of \(V_m^n\) as described above are IRs.

**Proof:**

Even though the complete proof involves some technical details [Miller], the basic idea behind it is rather easy to understand: since \(G_m\) is, so to speak, the most general group of transformations which commutes with \(S_n\) on \(V_m^n\) on the subspace \(T_\lambda'\) the operators \(\left[D(\ g)\right], \ g \in G_m\) must be “complete” —they cannot be reducible. More specifically, consider an arbitrary linear transformation \(A\) on the vector space \(T_\lambda'(a)\). In tensor component notation,

\[x' \longrightarrow y' = A'_{\ ij} x'\]
Because \( x \) and \( y \) belong to the same symmetry class, \( A \) must be "symmetry preserving" in the sense that,

\[
A^I_j = A^{I'}_{j_p} \quad \forall \ p \in S_n
\]

We know already that the linear transformations representing \( g \in G_m \) on \( V_m^n \) are symmetry preserving [Lemma 5.1]. It can be established that, even though \( A \) does not necessarily factorize as \( D( g ) \) in Eq. (5.5-6), it can nevertheless be written as a linear combination of \( D( g ) \). [cf. Lemma IV.7] Since this is true for all linear transformations, \( D( g ) \) must be irreducible. QED.

** A concrete example on how the tensor space \( V_m^n \) is decomposed to irreducible invariant subspaces with respect to both \( S_n \) and \( G_m \) was worked out in detail previously for \( V_2^3 \). In the context of Theorems 5.12 and 5.13, we found:

(i) associated with totally symmetric tensors \( (\lambda = s) \), there is an invariant subspace (with respect to \( G_m \)) \( T_s' \), which is 4-dimensional and has basis vectors \( \{ s, \alpha, 1 \} \), \( \alpha = 1, \ldots, 4 \} \) given in Example 2; and

(ii) associated with the symmetry class \( \lambda = m \), there are two invariant subspaces \( T_m'(1) \) and \( T_m'(2) \) which give rise to equivalent 2-dimensional irreducible representations of the linear group \( G_2 \). [cf. Problem 5.8]

The irreducible representations of \( G_m \) provided by tensors of various symmetry classes as described in this section are by no means the only irreducible representations of the general linear group. The main purpose of this exposition is to illustrate the usefulness of the symmetric (or permutation) group in an important class of application —tensor analysis. In Chapter 13, we shall give a more systematic discussion of finite-dimensional representations of the classical groups which includes \( GL(m, C) \) as the most general case. We shall also utilize the tensor method to help evaluate the explicit expression for all representation matrices of the rotation group in Sec. 8.1.