Chapter 6
1-D Continuous Groups

Continuous groups consist of group elements labelled by one or more continuous variables, say (a_1 , a_2 , … , a_r), where each variable has a well- defined range.

This chapter explores:
1. the group of rotations in a plane SO(2),
2. the group of translations in 1-D T_1.

These groups are 1-D (i.e. the group elements only depend on one continuous parameter) and they are necessarily abelian.

A Lie group is an infinite group whose elements can be parametrized smoothly and analytically.

The definition of these properties requires introducing algebraic and geometric structures beyond group multiplication in pure group theory (which has been our only concern so far for discrete groups). The precise formulation of the general theory of Lie groups requires considerable care; it involves notions of topology and differential geometry.

All known continuous symmetry groups of interest in physics, however, are groups of matrices for which the additional algebraic and geometric structures are already familiar. These groups are usually referred to as linear Lie groups or classical Lie groups. We shall introduce the most important features and techniques of the theory of linear Lie groups by studying the important symmetry groups of space and time, starting from this chapter. Experience with these physically useful examples should provide a good foundation for studying the general theory of Lie groups. [Chevalley, Gilmore, Miller, Pontrjagin, Weyl]

Summary by Sections

1. Definition of SO(2) & its general properties.
2. Generator of the group as determined by the group structure near e..
3. IRs of SO(2) derived using eigenvectors of the generator.
   Role of global properties.
4 . Invariant integration measure on the group manifold.
   Orthonormality and completeness properties of the rep functions.
   Their relation to Fourier analysis..
5. Multi- valued reps & their relation to the topology of the group manifold.
6. T_1.
7. Generators are identified with measurable physical quantities.

All ideas introduced in this chapter will find significant generalizations in later chapters.

6.1 The Rotation Group SO(2)

Consider a system symmetric under rotations in a plane, around a fixed point O.

Let $\hat{e}_1$ and $\hat{e}_2$ be the Cartesian orthonormal basis vectors [see Fig. 6.1].
A rotation through angle ϕ by $R(\phi)$ gives

(6.1-1) $$R(\phi)\,\hat{e}_1 = \hat{e}_1 \cos\phi + \hat{e}_2 \sin\phi$$
$$R(\phi)\,\hat{e}_2 = -\hat{e}_1 \sin\phi + \hat{e}_2 \cos\phi$$

or equivalently,

(6.1-2) $$R(\phi)\,\hat{e}_i = \hat{e}_j\, R(\phi)^j{}_i$$

with the matrix $R(\phi)^j{}_i$ given by

(6.1-3) $$\mathbb{R}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

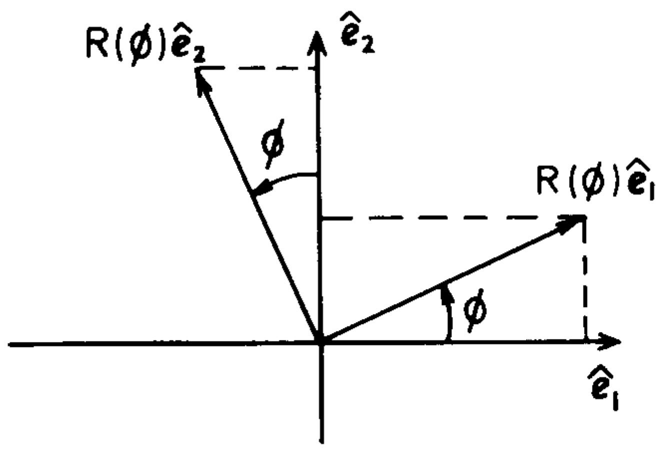

Fig. 6.1 Rotation in a plane.

Let
$$\boldsymbol{x} = \hat{e}_i\, x^i$$
Then x transforms under rotation $R(\phi)$ according to:
$$x \longrightarrow x' \equiv R(\phi)\boldsymbol{x} = R(\phi)\ \hat{e}_i\, x^i = \hat{e}_j\, R(\phi)^j{}_i\, x^i$$

Writing
$$x' = \hat{e}_i\, x'^{\,i}$$
we obtain

(6.1-4) $$x'^{\,j} = R(\phi)^j{}_i\, x^i$$

or in matrix form
$$\mathbb{x}' = \mathbb{R}(\phi)\,\mathbb{x}$$

Geometrically, it is obvious that the length of vectors remains invariant under rotations, i.e.
$$|\ \boldsymbol{x}\ |^2 = x_i\, x^i = |\ \boldsymbol{x}'\ |^2 = x_i'\, x'^{\,i}$$
or
$$\mathbb{x}^T\,\mathbb{x} = \mathbb{x}'^T\,\mathbb{x}'$$
$$= \mathbb{x}^T\,\mathbb{R}^T(\phi)\,\mathbb{R}(\phi)\,\mathbb{x}$$

Thus

(6.1-5) $$\mathbb{R}^T(\phi)\,\mathbb{R}(\phi) = \mathbb{I} \qquad \forall\ \phi$$

where $\mathbb{R}^T$ denotes the transpose of $\mathbb{R}$, and $\mathbb{I}$ is the unit matrix.

Real matrices satisfying the condition (6.1-5) are called orthogonal matrices and denoted by $O(n)$, where n is their dimension. In the present case, $n = 2$.

Eq. (6.1-5) implies that

$$\det\left[\mathbb{R}^T(\phi)\,\mathbb{R}(\phi)\right] = \det \mathbb{I}$$

$$\det\left[\mathbb{R}^T(\phi)\right]\det[\,\mathbb{R}(\phi)\,] = 1 \qquad [\,\det[\mathbb{A}\,\mathbb{B}] = \det\mathbb{A}\,\det\mathbb{B}\,]$$

$$[\,\det\,\mathbb{R}(\phi)\,]^2 = 1 \qquad [\;\det\;\mathbb{A}^T \;= \det\mathbb{A}\;]$$

so that

$$\det\,\mathbb{R}(\phi) = \pm 1$$

Now, the identity operator is given by

$$E = R(0)$$

or in matrix form

$$\mathbb{R}(0) = \mathbb{I}$$

with

$$\det\mathbb{R}(0) = \det\mathbb{I} = 1$$

Operators that evolve continuously from E must therefore satisfy

(6.1-6) $$\det\,\mathbb{R}(\phi) = 1 \qquad \forall\;\; \phi$$

They are called **proper rotations**.
Matrices satisfying (6.1-6) are said to be special.
In the present case, they are special orthogonal matrices of rank 2; and are designated as SO(2) matrices.

Operators with

$$\det\,\mathbb{R}(\phi) = -1$$

are called **improper rotations**.
They can always be expressed as a proper rotation followed by a reflection.

- Theorem 6.1:

There is a one-to-one correspondence between rotations in a plane and SO(2) matrices.

The proof of this theorem is left as an exercise. [Problem 6.1]

This correspondence is a general one, applicable to SO(n) matrices and rotations in the Euclidean space of dimension n for any n.

O(2) Group

The $O(2)$ matrices with $\phi \in [0,\, 2\,\pi)$ form a group called $O(2)$.

The law of composition is

(6.1-7) $$R(\phi_2)\,R(\phi_1) = R(\phi_2 + \phi_1)$$

with the understanding that any ϕ outside $[0,\, 2\,\pi)$ is brought back to $\phi \in [0,\, 2\,\pi)$ using

(6.1-8) $$R(\phi) = R(\phi + 2\,\pi\,n) \qquad n = \text{integer}$$

Obviously

$$R(\phi = 0) = E \qquad \text{and} \qquad R(\phi)^{-1} = R(-\phi) = R(2\,\pi - \phi)$$

Note that $O(2)$ is not a Lie group since the improper rotations are not continuously related to E.

- Theorem 6.2 (2-D Rotation Group):

The 2-D proper rotations $\{R(\phi)\}$ form a Lie group called the rotation group R_2 or SO(2) group.

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The set $\phi \in [0, 2\pi)$ corresponds to all points on the unit circle which defines the "topology" of the group parameter space (manifold) [cf Fig. 6.2].

Obviously, this parameterization is not unique.
Any monotonic function $\xi(\phi)$ of ϕ over the above domain can serve as an alternative label for the group element.
The structure of the group and its reps should not be affected by the labelling scheme.

We shall come back to the "naturalness" of the variable ϕ and the question of topology in a later section. Both issues are important for the analysis of general continuous groups.

Note that Eq. (6.1-7) implies

$$R(\phi_1)\, R(\phi_2) = R(\phi_2)\, R(\phi_1) \qquad \forall \;\; \phi_1 , \phi_2$$

Thus, the group SO(2) is abelian.

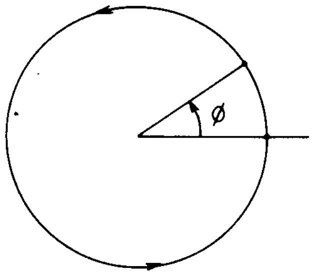

Fig. 6.2 SO(2) group manifold.

6.2 Generator of SO(2)

Consider an infinitesimal SO(2) rotation by the angle $d\phi$.

Differentiability of $R(\phi)$ in ϕ requires that $R(d\phi)$ differs from $R(0) = E$ by only a quantity of order $d\phi$ which we define by the relation

(6.2-1) $$R(d\,\phi) \equiv E - i\, d\phi\;\, J$$

where the factor $(-i\,)$ is included by convention and for later convenience.
The quantity J is independent of the rotation angle $d\phi$.

Next, consider the rotation $R(\phi + d\phi)$, which can be evaluated in two ways:

(6.2-2) $$R(\phi + d\phi) = R(\phi)\, R(d\phi) = R(\phi) - i\, d\phi\, R(\phi)\, J$$

$$R(\phi + d\phi) = R(\phi) + d\phi\, \frac{d\, R(\phi)}{d\, \phi}$$

Comparing the two equations, we obtain the differential equation,

(6.2-3) $$\frac{d\, R(\phi)}{d\, \phi} = -i\, R(\phi)\, J$$

with the boundary condition $R(0) = E$.

The solution to Eq. (6.2-3) is therefore unique, we present it in the form of a theorem.

- Theorem 6.3 (Generator of SO(2)):

All 2-D proper rotations can be expressed in terms of the operator J as:
(6.2-4) $$R(\phi) = e^{-i\,\phi\,J}$$
J is called the generator of the group.

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Thus the general group structure and its reps are determined mostly by J which, in turn, is specified by *local* behavior of the group near the identity.

This is typical of Lie groups.

Note however that certain *global* properties of the group, such as Eq. (6.1-8), cannot be deduced from (6.2-3). These global properties, mostly topological in nature, also play a role in determining the IRs of the group, as we shall see in the next section.

Let us turn from this abstract discussion to the explicit representation of $R(\phi)$ given by (6.1-3). We have, to first order in $d\phi$,

$$\mathbb{R}(d\phi) = \begin{pmatrix} 1 & -d\phi \\ d\phi & 1 \end{pmatrix}$$

Comparing with Eq. (6.2-1), we deduce
(6.2-5) $$\mathbb{J} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Thus $\mathbb{J}$ is a traceless hermitian matrix. It is easy to show that
$$\mathbb{J}^2 = \mathbb{I},\ \mathbb{J}^3 = \mathbb{J},\ \ldots \text{ etc.}$$

Therefore,
$$\begin{aligned} e^{-i\,\phi\,\mathbb{J}} &= \mathbb{I} - i\,\mathbb{J}\,\phi + \frac{1}{2!}\,\mathbb{I}\,\phi^2 - i\,\mathbb{J}\left(-\frac{\phi^3}{3!}\right) + \ldots \\ &= \mathbb{I}\cos\phi - i\,\mathbb{J}\sin\phi \\ &= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \end{aligned}$$
which reproduces Eq. (6.1-3).

6.3 IRs of SO(2)

Consider any rep of SO(2) defined on a finite dimensional vector space V.
Let $U(\phi)$ be the operator on V which corresponds to $R(\phi)$.

Then, according to Eq. (6.1-7), we must have
$$U(\phi_2)\,U(\phi_1) = U(\phi_2 + \phi_1) = U(\phi_1)\,U(\phi_2)$$
with the same understanding that $U(\phi) = U(\phi + 2\,\pi\,n)$.

For an infinitesimal transformation, we can again define an operator corresponding to the generator J in Eq. (6.2-1). In order to avoid a proliferation of symbols, we use the same letter J to denote this operator,
(6.3-1) $$U(d\phi) = E - i\,d\phi\;J$$

Repeating the arguments of the last section, we obtain
(6.3-2) $$U(\phi) = e^{-i\,\phi\,J}$$
which is now an operator equation on V. If $U(\phi)$ is to be unitary for all ϕ, J must be hermitian with real eigenvalues.

Since SO(2) is an abelian group, all its IRs are 1-D.
This means that given any $|\,\alpha\,\rangle$ in a minimal invariant subspace under SO(2) we must have:
(6.3-3) $J\,|\,\alpha\,\rangle = |\,\alpha\,\rangle\,\alpha$
(6.3-4) $U(\phi)\,|\,\alpha\,\rangle = |\,\alpha\,\rangle\,e^{-i\,\phi\,\alpha}$
where α is an eigenvalue of J.

It is easy to show that the form given by Eq. (6.3-4) automatically satisfies the group multiplication rule Eq. (6.1-7) for an arbitrary α. However, in order to satisfy the global constraint Eq. (6.1-8), a restriction must be placed on the eigenvalue α. Indeed, we must have

$$e^{i\,2\,\pi\,n\,\alpha} = 1 \qquad \forall\; n = \text{integer}$$

which implies that α is an integer. We denote this integer by m, and the corresponding representation by U^m:
(6.3-5) $J\,|\,m\,\rangle = |\,m\,\rangle\,m$
(6.3-6) $U^m(\phi)\,|\,m\,\rangle = |\,m\,\rangle\,e^{-i\,m\,\phi}$

Let us take a closer look at these reps:

(i) When $m = 0$,

$$R(\phi) \;\to\; U^0(\phi) = 1.$$

This is the identity rep;

(ii) When $m = 1$,

$$R(\phi) \;\to\; U^1(\phi) = e^{-i\,\phi}$$

This is an isomorphism between SO(2) group elements and ordinary numbers on the unit circle in the complex number plane. As $R(\phi)$ ranges over the group space, $U^1(\phi)$ covers the unit circle once, in the clockwise sense;

(iii) When $m = -1$,

$$R(\phi) \;\to\; U^{-1}(\phi) = e^{i\,\phi}$$

The situation is the same as above, except that the unit circle is covered once in the counter- clockwise direction;

(iv) When $m = \pm 2$,

$$R(\phi) \;\to\; U^{\pm 2}(\phi) = e^{\mp 2\,i\,\phi}$$

These are mappings of the group parameter space to the unit circle on the complex number plane covering the latter twice in opposite directions.

The general case follows in an obvious manner from these examples. We summarize these results in the form of a theorem.

- Theorem 6.4 (IRs of SO(2)):

The single valued IRs of SO(2) are given by $J = m$ where m is any integer, and
(6.3-7) $U^m(\phi) = e^{-i\,m\,\phi}$

Of these, only the $m = \pm 1$ ones are faithful reps.

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The defining equation for $R(\phi)$, Eq. (6.1-3), is a 2-D rep of the group. It has to be reducible. Indeed, it is equivalent to a direct sum of the $m = \pm 1$ representations. To see this. we note that due to Eq. (6.3-2), it suffices to diagonalize the matrix corresponding to the generator.

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

It is obvious that J has two eigenvalues, ± 1; and the corresponding eigenvectors are:

$$\hat{e}_{\pm} = \frac{1}{\sqrt{2}} \left(\mp \hat{e}_1 - i\, \hat{e}_2 \right) \qquad [\text{ Problem 6.2 }]$$

Thus, with respect to the new basis.

(6.3-8) $$J\, \hat{e}_{\pm} = \pm\, \hat{e}_{\pm}$$
$$R(\phi)\, \hat{e}_{\pm} = \hat{e}_{\pm}\; e^{\mp i \phi}$$

6.4 Invariant Integration Measure, Orthonormality & Completeness Relations

We now derive the orthonormality and completeness relations for the representation functions $U_m(\phi) = e^{-i m \phi}$.

Because ϕ is a continuous variable, the summation over group elements must be replaced by an integration, and the integration measure must be well defined.

Now, any function $\xi(\phi)$, monotonic in $0 \leq \phi < 2\pi$, can also serve as label.

However, for an arbitrary function f of the group elements,

$$\int d\xi\;\; f\,[\,R(\xi)\,] = \int d\phi\;\, \xi\,'(\phi)\;\, f\,[\,R(\phi)\,] \qquad [\; \xi\,' = \frac{d\xi}{d\phi}\;]$$

$$\neq \int d\phi\;\; f\,[\,R(\phi)\,]$$

Hence, "integration" of f over the group manifold is not well defined a priori.

Now, the Rearrangement Lemma lies at the heart of the proof of most important results of the representation theory. Therefore we seek to define an integration measure such that,

(6.4-1) $$\int \mathrm{d}\tau_R\;\; f\,[R\,] = \int \mathrm{d}\tau_R\;\; f\left[S^{-1}\, R\right] \qquad \forall\, S \in G$$

$$= \int \mathrm{d}\tau_{\mathrm{SR}}\;\; f\,[R\,]$$

If the group elements are labelled by the parameter ξ, then

$$\mathrm{d}\tau_R = \rho_R(\xi)\, \mathrm{d}\xi$$

where $\rho_R(\xi)$ is some appropriately defined "weight function".

- Definition 6.1 (Invariant Integration Measure):

A parameterization $R(\xi)$ in group space with an associated weight function $\rho_R(\xi)$ is said to provide an invariant integration measure if Eq. (6.4-1) holds. The validity of Eq. (6.4-1) requires

$$\mathrm{d}\tau_R = \mathrm{d}\tau_{\mathrm{SR}}$$

which imposes the condition on the weight function,

(6.4-2) $$\frac{\rho_R(\xi)}{\rho_{\mathrm{SR}}(\xi)} = \frac{d\,\xi_{\mathrm{SR}}}{d\,\xi_R}$$

This condition is automatically satisfied if we define

(6.4-3) $$\rho_R(\xi) = \frac{d\,\xi_E}{d\,\xi_R}$$

where ξ_E is the group parameter around the identity element E and $\xi_R = \xi_{\mathrm{ER}}$ is the corresponding parameter around R. [2] In evaluating the right- hand side of the above equation, R is to be regarded as fixed; the dependence of ξ_{ER} on ξ_E is determined by the group multiplication rule.

The situation is simplest when ξ_{SR} linear in ξ_R,. This is the case when $\xi = \phi$ is the rotation angle. The group multiplication rule, Eq. (6.1-7), requires

$$\phi_{ER} = \phi_E + \phi_R \qquad \rho_R = \left(\frac{d\,\phi_E}{d\,\phi_{ER}} \right)_R = 1$$

■ Theorem 6.5 (Invariant Integration Measure of **SO(2)**):

The rotation angle ϕ, Fig. 6.1, and the volume measure $d\tau_R = d\phi$, provide the proper invariant integration measure over the SO(2) group space.

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If ξ is a general parameterization of the group element, then

$$d\tau_R = \rho_R(\xi)\, d\xi = \rho_R(\phi)\, d\phi = d\phi$$

We must have, therefore,

$$\rho_R(\xi) = \frac{d\,\phi}{d\,\xi}$$

The above discussion may appear to be rather long-winded just to arrive at a relatively obvious conclusion. The motivation for including so much detail is to set up a line of reasoning which can be applied to general continuous groups in later Chapters.

Once an invariant measure is found, it is straightforward to write down the expected orthonormality and completeness relations.

Theorem 6.6: The SO(2) representation functions $U^n(\phi)$ of Theorem 6.4 satisfy the following orthonormality and completeness relations:

(6.4-4) $$\frac{1}{2\pi}\int_0^{2\pi} d\phi\ U_n^{\dagger}(\phi)\ U^m(\phi) = \delta_n^{\ m} \qquad \text{(orthogonality)}$$

$$\sum_n U^n(\phi)\, U_n^{\dagger}(\phi') = \delta(\phi - \phi') \qquad \text{(completeness)}$$

Three simple remarks of general importance are in order here:

(i) These relations are natural generalizations of Theorem 3.5 and 3.6 (for finite groups) to a continuous group; the only change is the replacement of the finite sum over group elements by the invariant integration over the continuous group parameter;

(ii) Theorem 6.6. with $U^n(\phi)$ given by Eq. (6.3-7), is equivalent to the classical Fourier Theorem for periodic functions, the continuous group parameter ϕ and the discrete representation label n are the "conjugate variables" ;

(iii) These relations are also identical to the results encountered in Chap.1, Eqs. (1.4-1) – (1.4-2), in connection with the discrete translation group T^d. Note, however, the roles of the group element label (discrete) and the representation label (continuous) are exactly reversed in comparison to the present case.

6.5 Multi-Valued Reps

For later reference, we mention here a new feature of continuous groups – the possibility of having multi- valued representations. To introduce the idea, consider the mapping

(6.5-1) $$R(\phi) \longrightarrow U_{1/2}(\phi) = e^{-\frac{i}{2}\phi}$$

This is not a unique representation of the group, as

(6.5-2) $$U_{1/2}(2\pi + \phi) = e^{-i\pi - \frac{i}{2}\phi} = -U_{1/2}(\phi)$$

whereas, we expect, on physical grounds, $R(2n + \phi) = R(\phi)$. However, since

$$U_{1/2}(4\pi + \phi) = U_{1/2}(\phi)$$

Eq.(6.5-1) does define a one-to-two mapping where each $R(\phi)$ is assigned to two complex numbers $\mp e^{-\frac{i}{2}\phi}$ differing by a factor of -1. This is a two- valued representation in the sense that the group multiplication law for SO(2) is preserved if either one of the two numbers corresponding to $R(\phi)$ can be accepted.

Clearly, we can also consider general mappings,

(6.6-3) $$R(\phi) \longrightarrow U_{n/m}(\phi) = e^{-\frac{in}{m}\phi}$$

where n and m are integers with no common factors. For any pair (n, m) this mapping defines a "m-valued representation" of SO(2) in the same sense as described above.

The following questions naturally arise:

(i) Do continuous groups always have multi-valued irreducible representation;

(ii) How do we know whether (and for what values of m do) multi-valued iepresentations exist;

(iii) When multi-valued representations exist, are they realized in physical systems? In other words, does it make sense to restrict our attention to solutions of classical and/or quantum- mechanical systems only to those corresponding to single-valued representations of the appropnate symmetry groups?

It turns out that the existence of multi- valued representations is intimately tied to "connectedness" – a global topological property – of the group parameter space. In the case of SO(2), the group parameter space (the unit circle) is "multiply-connected" [3], which implies the existence of multi- valued representations. Thus, it is possible to determine the existence and the nature of multi- valued representations from an intrinsic property of the group.

In regard to the last question posed above, so far as we know, both single- and double- valued representations, but no others, are realized in quantum mechanical systems, and only single- valued representations appear in classical solutions to physical problems. The occurrence of double- valued representations can be traced to the connectedness of the group manifolds of symmetries associated with the physical 3- and 4- dimensional spaces. This observation will become clearer after we discuss the full rotation group and the Lorentz group in the next few chapters.

6.6 T_1

Rotations in the 2-dimensional plane (by the angle ϕ) can be interpreted as translations on the unit circle (by the arc length ϕ). This fact accounts for the similarity in the form of the irreducible representation function, $U^n(\phi) = e^{-in\phi}$, in comparison to the case of discrete translation, $t_k(n) = e^{-inkb}$, discussed in Chap. 1. The "complementary" nature of these results has been noted in Sec. 6.4. We now extend the investigation to the equally important and basic continuous translation group in one dimension, which we shall refer to as T_1.

Let the coordinate axis of the one- dimensional space be labelled x. An arbitrary element of the group T_1 corresponding to translation by the distance x will be denoted by $T(x)$. Consider "states" $|\, x_0 \rangle$ [4] of a "particle" localized at the position x_0. [5] The action of $T(x)$ on $|\, x \rangle$ is:

(6.6-1) $$T(x)\,|\, x_0 \rangle \equiv |\, x + x_0 \rangle$$

It is easy to see that $T(x)$ must have the following properties:

(6.6-2a) $$T(x_1)\, T(x_2) = T(x_1 + x_2)$$

(6.6-2b) $$T(0) = E$$

(6.6-2c) $$T(x)^{-1} = T(-x)$$

These are just the properties that are required for

$$\{\, T(x)\,,\ -\infty < x < \infty \,\}$$

to form a group [cf. Eqs. (1.2-1abc)].

For an infinitesimal displacement denoted by dx, we have

(6.6-3) $$T(\mathrm{dx}) \equiv E - i\,\mathrm{dx}\,P$$

which defines the (displacement-independent) generator of translation P. Next, we write $T(x + \mathrm{dx})$ in two different ways:

(6.6-4a) $$T(x + \mathrm{dx}) = T(x) + \mathrm{dx}\,\frac{d\,T(x)}{d\,x}$$

and

(6.6-4b) $$T(x + \mathrm{dx}) = T(\mathrm{dx})\,T(x)$$

Substituting (6.6-3) in (6.6-4b), and comparing with (6.6-4a), we obtain

(6.6-5) $$\frac{d\,T(x)}{d\,x} = -i\,P\,T(x)$$

Considering the boundary condition (6.6-2b), this differential equation yields the unique solution,

(6.6-6) $$T(x) = e^{-i\,P\,x}$$

It is straightforward to see that with $T(x)$ written in this form, all the required group properties, (6.6-2a,b,c), are satisfied. This derivation is identical to that given for the rotation group SO(2). [cf. Theorem 6.3] The only difference is that the parameter x in $T(x)$ is no longer restricted to a finite range as for ϕ in $R(\phi)$.

As before, all irreducible representations of the translation group are onedimensional. For unitary representations, the generator P corresponds to a hermitian operator with real eigenvalues, which we shall denote by p. For the representation $T(x) \to U^p(x)$, We obtain:

(6.6-7) $$P \,|\, p \,\rangle = p \,|\, p \,\rangle$$

(6.6-8) $$U^p(x) \,|\, p \,\rangle = |\, p \,\rangle\, e^{-i\,p\,x}$$

It is easy to see that all the group properties, Eqs. (6.6-2a,b,c), are satisfied by this representation function for any given real number p. Therefore the value of p is totally unrestricted.

Comparing these results with those obtained in Chap. 1 for the discrete translation group T^d and in Sections (6.1–6.5) for SO(2), we remark that:

(i) The representation functions in all these cases take the exponential form [cf. Eqs. (1.3-3), (6.3-6), (6.6-8)], reflecting the common group multiplication rule [cf Eqs. (1.2-la),(6.1-7),(6.6-2a)];

(ii) For T^d, the group parameter (n in Eq. (1.3-3)) is discrete and infinite, the representation label (k) is continuous and bounded. For SO(2), the former (ϕ in Eq. (6.3-6)) is continuous and bounded, the latter (m) is discrete and infinite. Finally, for T_1 the former (x in Eq. (6.6-8)) is continuous and unbounded, so is the latter (p).

The conjugate role of the group parameter and the representation label in the sense of Fourier analysis was discussed in Sec. 6.4. The case is strengthened more by applying the orthonormality and completeness relations of representation functions to the present case of full one- dimensional translation. For this purpose we must again define an appropriate invariant measure for integration over the group elements. Just as in the case of SO(2), one needs only to pick the natural Cartesian displacement x as the integration variable. Because the range of integration is now infinite, not all integrals are strictly convergent in the classical sense. But for our purposes, it suffices to say that all previous results hold in the sense of generalized functions. We obtain:

(6.6-9) $$\int_{-\infty}^{\infty} \mathrm{dx}\ U_p{}^{\dagger}(x)\ U^{p'}(x) = N\,\delta(p - p')$$

(6.6-10) $$\int_{-\infty}^{\infty} \mathrm{dp}\ U^p(x)\,U_p{}^{\dagger}(x') = N\,\delta(x - x')$$

where N is a yet unspecified normalization constant. Since $U^p(x) = e^{-i\,p\,x}$, these equations represent a statement of the Fourier theorem for arbitrary (generalized) functions. This correspondence also gives the correct value of N, i.e. $N = 2\pi$.

6.7 Conjugate Basis Vectors

In Chap. 1 we described two types of basis vectors: $\{|\ x\ \rangle\}$, defined by Eq. (1.1-6), and $|\ u(E, k)\ \rangle$, defined by Eq. (1.3-3). The first represents "localized states" at some position x ; the second corresponds to normal modes which fill the entire lattice and have simple translational properties. Each one has its unique features and practical uses. State functions expressed in terms of these two bases are related by a Fourier expansion. Analogous procedures can be applied in the representation space of the rotation group SO(2) and the continuous translation group. We describe them in turn.

Consider a particle state localized at a position represented by polar coordinates (r, ϕ) On the 2-dimensional plane. The value of r will not be changed by any rotation; therefore we shall not be concerned about it in subsequent discussions. Intuitively,

(6.7-1) $$U(\phi)\,|\ \phi_0\ \rangle = |\ \phi + \phi_0\ \rangle$$

so that,

(6.7-2) $$|\ \phi\ \rangle = U(\phi)\ |\ 0\ \rangle \qquad 0 \le \phi < 2\pi$$

where $|\ 0\ \rangle$ represents a "standard state" aligned with a pre-chosen x-axis. How are these states related to the eigenstates of J defined by Eqs. (6.3-5) and (6.3-6)?

If we expand $|\ \phi\ \rangle$ in terms of the vectors $\{\ |\ m\ \rangle\ ;\ m = 0, \pm 1, \ldots\ \}$,

$$|\ \phi\ \rangle = \sum_m |\,m\ \rangle\langle\ m\,|\ \phi\ \rangle$$

then

$$\langle\ m\,|\ \phi\ \rangle = \langle\ m\,|\,U(\phi)\ |\ 0\ \rangle = \left\langle\ U^\dagger(\phi)\,m\,\middle|\ 0\ \right\rangle = \langle\ m\,|\ 0\ \rangle\, e^{-i\,m\,\phi}$$

States with different values of m are unrelated by rotation, and we can choose their phases (i.e. a multiplicative factor $e^{i\,\alpha\,m}$ for each m) such that

$$\langle\ m\,|\ 0\ \rangle = 1 \qquad \text{for all } m$$

thus obtaining

(6.7-3) $$|\ \phi\ \rangle = \sum_m |\,m\ \rangle\, e^{-i\,m\,\phi}$$

To invert this equation, multiply by $e^{i\,m\,\phi}$ and integrate over ϕ. We obtain

(6.7-4) $$|\ m\ \rangle = \int_0^{2\pi} \frac{d\,\phi}{2\pi}\,|\,\phi\ \rangle\, e^{i\,m\,\phi}$$

We see that by this convention, the "transfer matrix elements" $\langle\ m\,|\ \phi\ \rangle$ between the two are just the group representation functions.

An arbitrary state $|\ \psi\ \rangle$ in the vector space can be expressed in either of the two bases:

(6.7-5) $$|\ \psi\ \rangle = \sum_m |\,m\ \rangle\,\psi_m = \int_0^{2\pi} \frac{d\,\phi}{2\pi}\,|\,\phi\ \rangle\ \psi(\phi)$$

The "wave functions" ψ_m and $\psi(\phi)$ are related by

(6.7-6) $$\psi(\phi) = \langle\ \phi\,|\,\psi\ \rangle = \sum_m\ \langle\ \phi\,|\ m\ \rangle\langle\ m\,|\ \phi\ \rangle = \sum_m\ e^{i\,m\,\phi}\,\psi_m$$

and

(6.7-7) $$\psi_m = \int_0^{2\pi} \frac{d\,\phi}{2\pi}\ \psi(\phi)\, e^{-i\,m\,\phi}$$

Let us examine the action of the operator J on the states $| \, \phi \, \rangle$. From Eq. (6.7-3), we obtain

(6.7-8) $$J \, | \, \phi \, \rangle = \sum_m J \, | \, m \, \rangle \, e^{-i m \phi} = \sum_m | \, m \, \rangle \, m \, e^{-i m \phi} = i \frac{d}{d \phi} | \, \phi \, \rangle$$

For an arbitrary state, we have:

(6.7-9) $$\langle \, \phi \, | \, J \, | \, \psi \, \rangle = \langle \, J \phi \, | \, \psi \, \rangle = \frac{1}{i} \frac{d}{d \phi} \langle \, \phi \, | \, \psi \, \rangle = \frac{1}{i} \frac{d}{d \phi} \psi(\phi)$$

Readers who have had some quantum mechanics [Messiah, Schiff] will recognize that J is the angular momentum operator (measured in units of $\hbar$). The above purely group- theoretical derivation underlines the general, geometrical origin of these results.

The above discussion can be repeated for the continuous translation group. The "localized states" $| \, x \, \rangle$, Es. (6.6-1), and the "translationally covariant" states $| \, p \, \rangle$, Eq. (6.6-7) are related by

(6.7-10) $$| \, x \, \rangle = \int_{-\infty}^{\infty} \frac{d p}{2 \pi} | \, p \, \rangle \, e^{-i p x}$$

and

(6.7-11) $$| \, p \, \rangle = \int_{-\infty}^{\infty} d x \, | \, x \, \rangle \, e^{i p x}$$

where the normalization of the states is chosen, by convention, as

$$\langle \, x' \, | \, x \, \rangle = \delta(x - x') \qquad \langle \, p' \, | \, p \, \rangle = 2 \pi \, \delta(p - p')$$

The transfer matrix elements are, again, the group representation functions [Eq. (.6-8)],

(6.7-12) $$\langle \, p \, | \, x \, \rangle = e^{-i p x}$$

As before, if

(6.7-13) $$| \, \psi \, \rangle = \int | \, x \, \rangle \, \psi(x) \, d x = \int | \, p \, \rangle \, \psi(p) \, \frac{d p}{2 \pi}$$

then

(6.7-14) $$\psi(x) = \int \psi(p) \, e^{i p x} \, \frac{d p}{2 \pi}$$

(6.7-15) $$\psi(p) = \int \psi(x) \, e^{-i p x} \, d x$$

and

(6.7-16) $$\langle \, x \, | \, P \, | \, \psi \, \rangle = \langle \, P_x \, | \, \psi \, \rangle = \frac{1}{i} \frac{d}{d x} \psi(x)$$

Thus, the generator P can be identified with the linear momentum operator in quantum mechanical systems. [Messiah, Schiff]

Problems

6.1 Show that the rotation matrix $R(\phi)$, Eq. (6.1-3), is an orthogonal matrix and prove that every SO(2) matrix represents a rotation in the plane.

6.2 Show that $\hat{e}_{\pm} = \frac{1}{\sqrt{2}}\left(\mp \hat{e}_1 - i\,\hat{e}_2\right)$ are eigenvectors of J with eigenvalues ± 1 respectively [cf. Eq. (6.3-8)].

FootNotes

Notes 1: Matrices satisfying Eq. (h.1-5) but with determinant equal to -1 correspond physically to rotations combined with spatial inversion or mirror reflection. This set of matrices is not connected to the identity transformation by a continuous change of parameters. We shall include spatial inversion in our group theoretical analysis in Chap. 11.

Notes 2: We note that

$$\rho_R = \frac{d\,\xi_E}{d\,\xi_{\mathrm{ER}}} = \frac{d\,\xi_E}{d\,\xi_{\mathrm{SR}}}\,\frac{d\,\xi_{\mathrm{SR}}}{d\,\xi_R} = \rho_{\mathrm{SR}}\,\frac{d\,\xi_{\mathrm{SR}}}{d\,\xi_R}$$

Notes 3: This means that there exist closed "paths" on the unit circle which wind around it *m* times (for all Integers *m*) and which cannot be continuously deformed into each other.

Notes 4: The "state" can be interpreted in the sense of either classical mechanics or quantum mechanics. We use the state- vector convention of quantum mechanics only for the sake of clarity in notation.

Notes 5: By "particle" we simply mean an entity with no spatial extension, which can be represented by a mathematical point.