Appendix III
Group Algebra & Reduction of Regular Representation

III.1. Group Algebra

■ Definition

Given a group $\{G, \ \cdot\}$, the corresponding group algebra is $\{\mathcal{G}, \ +, \ \cdot, \ \mathbb{C}\}$.

Note: the multiplication in the algebra is the group multiplication. It is associative.

■ Properties

$\{\mathcal{G}, \ +, \ \cdot\}$ is a field.
$\{\mathcal{G}, \ +, \ \mathbb{C}\}$ is a vector space with natural basis the group elements $\{ g_i \mid i = 1 \ldots n_G \}$.

It is possible to make $\{\mathcal{G}, +, \mathbb{C}\}$ an inner product space with

$$\left(g_i \, , \ g_j \right) \equiv \delta_{i\,j} \qquad \text{or} \qquad \left\langle g_i \,\middle|\, g_j \right\rangle = \langle \, i \, | \, j \, \rangle \equiv \delta_{i\,j}$$

■ Basis / Regular Representation

As already mentioned, $\{\mathcal{G}, \ +, \ \mathbb{C}\}$ is a vector space with natural basis G. Therefore

$$r = \sum_i r_i \, g_i \qquad \forall \ r \in \mathcal{G}, \ r_i \in \mathbb{C}$$

Using the Einstein notation:

$$r = r^i \, g_i \qquad r^i = r_i$$

Multiplication between $r, \ q \in \mathcal{G}$ is

$$r \cdot q = \sum_i r_i \, g_i \cdot \sum_j q_j \, g_j$$
$$= \sum_{i\,j} r_i \, q_j \ g_i \cdot g_j$$

It is convenient to write

$$g_i \cdot g_j = \sum_k g_k \, \Delta^k_{i\,j}$$

where $\Delta^k_{i\,j} = \begin{cases} 1 & \text{for } g_k = g_i \cdot g_j \\ 0 & \text{otherwise} \end{cases}$

Hence

$$r \cdot q = \sum_{i\,j\,k} r_i \, q_j \, \Delta^k_{i\,j} \, g_k$$
$$= r^i \, q^j \, \Delta^k_{i\,j} \, g_k$$

Let $\quad p = r \cdot q = \sum_i p_i \, g_i = p^i \, g_i$

$\longrightarrow \quad p_i = \sum_{j\,k} r_j \, q_k \, \Delta^i_{j\,k}$

or $\quad p^i = \Delta^i_{j\,k} \, r^j \, q^k$

Hereafter, we shall use only the Einstein notation & drop the ' $\cdot$ ' symbol.

Elements of $\mathcal{G}$ plays a double role.

1. They are vectors of the vector space $\mathcal{G}$.
2. They are operators on the vector space $\mathcal{G}$.

The role of operator is induced by the group multiplication.

The dual role of elements of $\mathcal{G}$ is best displayed in the bra-ket notation:

Treating r, q, p as vectors, we have:

$$| r \rangle = r^i \, | i \rangle \qquad | q \rangle = q^i \, | i \rangle \qquad | p \rangle = p^i \, | i \rangle$$

However, In the multiplication $r q = p$, we must treat r as an operator with transform q to p. That is:

$$r \, | q \rangle = | p \rangle = p^i \, | i \rangle = \Delta^i_{j k} \, r^j q^k \, | i \rangle$$

Using $| q \rangle = q^k \, | k \rangle$, we have:

$$r \, | q \rangle = q^k \, r \, | k \rangle$$

Comparing the two expressions, we see that

$$r \, | k \rangle = | i \rangle \, \Delta^i_{j k} \, r^j$$

Thus, the matrix representation of the operator r in the basis $\{ | i \rangle \}$ is

$$D_{i k}(r) = \Delta^i_{j k} \, r^j$$

In particular, the representation of a group element g_m is obtained by setting

$$r = g_m \qquad r^j = \delta^j_m$$

so that

$$D_{i k}(g_m) = \Delta^i_{m k} = \begin{cases} 1 & \text{for} \quad g_i = g_m \, g_k \text{ or } g_m = g_i \, g_k^{-1} \\ 0 & \text{otherwise} \end{cases}$$

This is called the **regular representation** & usually denoted by D^R or D^{reg}.

■ Caution:

Some authors , eg., Inui, use the notation

$$\Delta^k_{i j} = \begin{cases} 1 & \text{for} \quad \mathrm{g_k = g_i \cdot g_j{}^{-1}} \\ 0 & \text{otherwise} \end{cases}$$

so that

$$D^R_{i k}(g_m) = \Delta^m_{i k}$$

■ Definition: Representation of $\mathcal{G}$

A **representation** D of the algebra $\mathcal{G}$ is a mapping which preserves the algebraic structure.

An **irreducible representation** (IR) of $\mathcal{G}$ is a representation that contains no non-trivial invariant subspaces.

■ Theorem

$D(\mathcal{G})$ is a representation of algebra $\mathcal{G}$ $\Longleftrightarrow$ $D(G)$ is a representation of group G.

$D(\mathcal{G})$ is an IR of algebra $\mathcal{G}$ $\Longleftrightarrow$ $D(G)$ is an IR of group G.

III.2. Left Ideals, Projection Operators

Left Ideals

The IR decomposition of D^R is

$$D^R = \sum_{\mu=1}^{n_c} n_\mu D^\mu$$

which means the vector space $\mathcal{G}$ can be decomposed into a direct sum of invariant subspaces L_a^μ :

$$\mathcal{G} = \sum_{\mu=1}^{n_c} L^\mu = \sum_{\mu=1}^{n_c} \sum_{a=1}^{n_\mu} L_a^\mu$$

so that

$$p \,|\, r \rangle = |\, p\, r \rangle \in L^\mu \qquad \forall\ p \in \mathcal{G},\ \ r \in L^\mu$$

$$p \,|\, r \rangle = |\, p\, r \rangle \in L_a^\mu \qquad \forall\ p \in \mathcal{G},\ \ r \in L_a^\mu$$

which means L^μ & L_a^μ are **left ideals** of $\mathcal{G}$.
Ideals that contains no smaller ideals are called **minimal**.
Minimal left ideals L_a^μ are therefore invariant subspaces of $\mathcal{G}$.

Projection Operators

A projection operator P_a^μ onto the left ideals L_a^μ is defined by

$$P_a^\mu \,|\, r \rangle = \left| \, r_a^\mu \right\rangle \in L_a^\mu \qquad \forall\, r \in \mathcal{G}$$

with

$$P_a^\mu P_b^\nu = \delta^{\mu\nu} \delta_{ab} P_a^\mu \qquad \text{[Idempotency / Orthogonality]}$$

$$\sum_{\mu a} P_a^\mu = 1 \qquad \text{[Completeness]}$$

Thus an arbitrary element r can be decomposed into a direct sum of its components in each minimal left ideal L_a^μ :

$$|\, r \rangle = \sum_{\mu a} P_a^\mu \,|\, r \rangle = \sum_{\mu a} \left| \, r_a^\mu \right\rangle$$

where $\left| \, r_a^\mu \right\rangle = P_a^\mu \,|\, r \rangle$

Treated as operators, the same decomposition also applies:

$$r = \sum_{\mu a} r_a^\mu = \sum_{\mu a} P_a^\mu \ r$$

Consider now

$$r\, P_a^\mu \ |\, s \rangle = \sum_{\nu b} r_b^\nu \left| s_a^\mu \right\rangle = \left| \, r_a^\mu s_a^\mu \right\rangle$$

$$P_a^\mu \ r \,|\, s \rangle = P_a^\mu \ |\, r\, s \rangle = \left| \, r_a^\mu s_a^\mu \right\rangle$$

Since s is arbitrary, we have:

$$r\, P_a^\mu = P_a^\mu \ r$$

III.3. Idempotents

Consider the decomposition of the identity element e :

$$e = \sum_{\mu} e^{\mu} = \sum_{\mu\, a} e_a^{\mu} \qquad e^{\mu} \in L^{\mu} e_a^{\mu} \in L_a^{\mu}$$

$$e\, e = \sum_{\mu\, \nu} e^{\mu}\, e^{\nu}$$

but $\quad e\, e = e = \sum_{\mu} e^{\mu}$

so that $e^{\mu}\, e^{\nu} = e^{\mu}\, \delta^{\mu\,\nu}$

$\therefore \quad e^{\mu}$ is an idempotent.

■ Definition: Idempotents

Elements of a set $\{ r^{\mu} \}$ which satisfies the condition

$$r^{\mu}\, r^{\nu} = r^{\mu}\, \delta^{\mu\,\nu}$$

are called **idempotents**.

■ Definition: Essentially Idempotent

Elements of a set $\{ r^{\mu} \}$ which satisfies the condition

$$r^{\mu}\, r^{\nu} = r^{\mu}\, \delta^{\mu\,\nu}\, \alpha^{\mu} \qquad \alpha^{\mu} \in \mathbb{C}$$

are called **essentially idempotent**.

■ Definition: Primitive Idempotents

An idempotent which generates a minimal left ideal is called a **primitive idempotent**.

■ Theorem: $\quad P_a^{\mu} | r \rangle = \left| r\, e_a^{\mu} \right\rangle$

P^{μ} defined by

$$P^{\mu} | r \rangle = | r\, e^{\mu} \rangle \qquad \forall\ r \in \mathcal{G}$$

is a projection operator onto L^{μ}.

P_a^{μ} defined by

$$P_a^{\mu} | r \rangle = \left| r\, e_a^{\mu} \right\rangle \qquad \forall\ r \in \mathcal{G}$$

is a projection operator onto L_a^{μ}.

■ Proof

First we show that P^{μ} is a linear operator on the vector space $L = \mathcal{G}$.

Let α , $\beta \in \mathbb{C}$,

$$\begin{aligned} P^{\mu} | \alpha\ r + \beta\, s \rangle &= | (\alpha\ r + \beta\, s)\, e^{\mu} \rangle \\ &= | \alpha\ r\, e^{\mu} + \beta\, s\, e^{\mu} \rangle \\ &= \alpha\ | r\, e^{\mu} \rangle + \beta\, | s\, e^{\mu} \rangle \\ &= \alpha\ P^{\mu} | r \rangle + \beta\, P^{\mu} | s \rangle \qquad \text{QED.} \end{aligned}$$

Next, we need to show that P^{μ} satisfies the four characteristics of a projection operator.

1. $P^\mu \,|\, r \rangle = |\, r\, e^\mu \rangle \in L^\mu \qquad \forall\, r \in \mathcal{G}$

Proof:

$$r = r\, e = r \sum_\mu e^\mu = \sum_\mu r\, e^\mu$$

Since e^μ is a left ideal, $r\, e^\mu \in L^\mu$. QED.

The decomposition of r is therefore

$$r = \sum_\mu r^\mu \qquad r^\mu = r\, e^\mu$$

2. $P^\mu P^\nu = \delta^{\mu\nu} P^\mu$

Proof:

$$P^\mu P^\nu \,|\, r \rangle = P^\mu \,|\, r\, e^\nu \rangle = |\, r\, e^\nu e^\mu \rangle = |\, r\, e^\mu \rangle\, \delta^{\mu\nu} = P^\mu \,|\, r \rangle\, \delta^{\mu\nu}$$

$\longrightarrow \quad P^\mu P^\nu = P^\mu \delta^{\mu\nu}$

3. $\sum_\mu P^\mu = 1$

Proof:

$$\sum_\mu P^\mu \,|\, r \rangle = \sum_\mu |\, r\, e^\mu \rangle = \left|\, r \sum_\mu e^\mu \right\rangle = |\, \mathrm{r\, e} \rangle = |\, r \rangle$$

$\longrightarrow \quad \sum_\mu P^\mu = 1$

4. $r\, P^\mu = P^\mu\, r$

Proof:

$r\, P^\mu \,|\, s \rangle = r \,|\, s\, e^\mu \rangle = |\, r\, s\, e^\mu \rangle \;\; \forall\; r,\; s \in \mathcal{G}$

$P^\mu\; r \,|\, s \rangle = P^\mu \,|\, r\, s \rangle = |\, r\, s\, e^\mu \rangle$

$\longrightarrow \quad r\, P^\mu = P^\mu\; r$

Proof for the P_a^μ case is analogous.

■ Theorem: Primitive Idempotent Criterion

An idempotent e^μ is primitive

$\Longleftrightarrow \quad e^\mu\, r\, e^\mu = \alpha_r\, e^\mu \qquad \alpha_r \in \mathbb{C}$

■ Proof $\longrightarrow$

e^μ is a primitive idempotent

$\longrightarrow \quad L^\mu = \{\, r\, e^\mu \mid r \in \mathcal{G} \,\}$ is minimal.

$\longrightarrow$ Realization of $\mathcal{G}$ is irreducible.

For each $r \in \mathcal{G}$, define operator R by

$$R \,|\, q \rangle \equiv |\, q\, e^\mu\, r\, e^\mu \rangle \qquad \forall\, q \in \mathcal{G}$$

Now:

$$R \,|\, q \rangle = |\, (q\, e^\mu\, r)\, e^\mu \rangle = P^\mu \,|\, q\, e^\mu\, r \rangle \in L^\mu$$

$\longrightarrow \quad R$ is a projection onto L^μ.

Also: $\forall\, s \in \mathcal{G}$,

$$R\, s \,|\, q \rangle = R \,|\, s\, q \rangle = |\, s\, q\, e^{\mu}\, r\, e^{\mu} \rangle$$

$$s\, R \,|\, q \rangle = s \,|\; q\, e^{\mu}\, r\, e^{\mu} \rangle = |\, s\, q\, e^{\mu}\, r\, e^{\mu} \rangle$$

$\longrightarrow \qquad R\, s = s\, R \qquad \forall\, s \in \mathcal{G}$

Schur's lemma $\longrightarrow$

$$\mathcal{D}^{\mu}(R) = \alpha_r\, I^{\mu}$$

where $\mathcal{D}^{\mu}(R)$ is the matrix representation of R and I^{μ} is the $n_{\mu} \times n_{\mu}$ unit matrix with n_{μ} = dimension of L^{μ}.

Let $\{|\, i \rangle\}$ be an orthonormal basis for L^{μ}, this means

$$\langle\, i \,|\, R \,|\, j \rangle = \alpha_r \left\langle\, i \,|\, e^{\mu} \,|\, j \right\rangle = \alpha_r\, \delta_{i\, j}$$

where e^{μ} is the unit operator on L^{μ} so that:

$$\left\langle\, i \,|\, e^{\mu} \,|\, j \right\rangle = \delta_{i\, j} = \langle\, i \,|\, e^{\mu}\; j \rangle = \langle\, i \,|\, j \rangle$$

By definition:

$$\langle\, i \,|\, R \,|\, j \rangle = \langle\, i \,|\, j\, e^{\mu}\, r\, e^{\mu} \rangle$$

Comparing the two expressions for $\langle\, i \,|\, R \,|\, j \rangle$

$\longrightarrow \qquad j\, e^{\mu}\, r\, e^{\mu} = \alpha_r\, e^{\mu}\; j = \alpha_r\; j\, e^{\mu} \qquad\qquad \forall\;\, j \qquad$ (j is an operator here)

$\therefore \qquad e^{\mu}\, r\, e^{\mu} = \alpha_r\, e^{\mu}$

- **Proof** $\longleftarrow$

Here, we assume

$$e^{\mu}\, r\, e^{\mu} = \alpha_r\, e^{\mu} \qquad\qquad \forall\, r \in \mathcal{G}$$

& try to prove that e^{μ} is primitive.

If e^{μ} were not primitive, we should be able to write

$$e^{\mu} = e' + e''$$

where

$$e' = e_1 \oplus 0 \qquad\qquad e'' = 0 \oplus e_2$$

so that e' & e'' are idempotents for the invariant subspaces 1 & 2 of L^{μ}.

Since e^{μ} is the unit operator on L^{μ},

$$e^{\mu}\, e' = e' = e'\, e^{\mu}$$

$\longrightarrow$

$$e^{\mu}\, e'\, e^{\mu} = e'\, e^{\mu} = e'$$

Hence

$$e^{\mu}\, e'\, e^{\mu} = \alpha_{e'}\, e^{\mu} = e'$$

Since e' is assumed to be an idempotent,

$$e'\, e' = e'$$

$\longrightarrow \qquad \alpha_{e'}\, e^{\mu}\, \alpha_{e'}\, e^{\mu} = \alpha_{e'}\, e^{\mu}$

or $\qquad \alpha_{e'}{}^{2}\, e^{\mu} = \alpha_{e'}\, e^{\mu} (\, e^{\mu}\, e^{\mu} = e^{\mu}\,)$

Hence

$$\alpha_{e'} = 0 \text{ or } \qquad \alpha_{e'} = 1$$

Since $\quad e' = \alpha_{e'}\, e^{\mu}$,

$$\alpha_{e'} = 0 \longrightarrow \qquad e' = 0 \quad \text{so that } \; e^{\mu} = e''$$

$$\alpha_{e'} = 1 \longrightarrow \qquad e' = e^{\mu} \;\; \text{so that } \; e'' = 0$$

In both cases, e^{μ} is not decomposable as assumed. QED.

- **Theorem: IR Criterion**

Two primitive idempotents e^{μ} & e^{ν} generate equivalent IR's

$\Longleftrightarrow \qquad \exists\, r \in \mathcal{G} \ni \qquad e^{\mu}\;\, r\;\, e^{\nu} \neq 0$

Proof $\longrightarrow$

Let the IR's generated by e^μ & e^ν be D^μ & D^ν , resp.
By definition, D^μ & D^ν are equivalent
$\longrightarrow \quad \exists\, S \quad \ni \quad D^\mu(r) = S^{-1}\, D^\nu(r)\, S \qquad \forall\, r \in \mathcal{G}$
S here can be interpreted as a transformation from the basis that spans L^ν to another that spans L^μ.
On the other hand, S can also be treated as the representation of an operator σ on $L = \sum_\mu L^\mu$ which brings each vector in L^μ to another in L^ν. Hence:

$$r = \sigma^{-1}\, r\, \sigma \qquad \forall\, r \in \mathcal{G}$$

Let:

$$|\, s\,\rangle \equiv \sigma\, |\, e^\mu\,\rangle \in L^\nu$$

Using $e^\mu\, e^\mu = e^\mu$, we have:

$$|\, s\,\rangle = \sigma\, |\, e^\mu\,\rangle = \sigma\, |\, e^\mu\, e^\mu\,\rangle = \sigma\, e^\mu\, |\, e^\mu\,\rangle = e^\mu\, \sigma\, |\, e^\mu\,\rangle = e^\mu\, |\, s\,\rangle = |\, e^\mu\, s\,\rangle$$

$\longrightarrow \quad s = e^\mu\, s$

Also $\quad |\, s\,\rangle \in L^\nu \quad \longrightarrow \quad s = s\, e^\nu$ since e^ν is the unit operator in L^ν.
Combining the 2 expressions:

$$e^\mu\, s = s\, e^\nu \quad \longrightarrow \quad s = e^\mu\, s\, e^\nu \neq 0 \qquad \text{QED}$$

- **Proof** $\longleftarrow$

$\exists\; r \in \mathcal{G} \quad \ni \quad e^\mu\, r\, e^\nu = s \neq 0$
Define a transformation $S : L^\mu \longrightarrow L^\nu$ by

$$S\, |\, q^\mu\,\rangle \equiv |\, q^\mu\, s\,\rangle = |\, q^\mu\, e^\mu\, r\, e^\nu\,\rangle \in L^\nu \qquad \forall\; q^\mu \in L^\mu$$

Consider now $p \in \mathcal{G}$:

$$S\; p\, |\, q^\mu\,\rangle = S\, |\, p\, q^\mu\,\rangle = S\, |\, p^\mu\, q^\mu\,\rangle = |\, p^\mu\, q^\mu\, s\,\rangle = |\, p\, q^\mu\, e^\mu\, r\, e^\nu\,\rangle$$
$$= \; p\, |\, q^\mu\, e^\mu\, r\, e^\nu\,\rangle = p\, S\, |\, q^\mu\,\rangle \in L^\nu$$

$\longrightarrow \quad \mathcal{S}\, D^\mu(p) = D^\nu(p)\, \mathcal{S} \qquad \forall\; p \in \mathcal{G}$
where $\mathcal{S}$ is the matrix form of the transformation S.
$\therefore \quad D^\mu$ & D^ν are equivalent.

Summary

Left ideal decomposition

$$L = \sum_\mu L^\mu = \sum_{\mu\, a} L^\mu_a$$

Left Ideals

$$L^\mu = \{\, r\, e^\mu \mid r \in \mathcal{G}\,\}$$
$$L^\mu_a = \left\{\, r\, e^\mu_a \,\middle|\, r \in \mathcal{G}\,\right\} \qquad \text{(minimal left ideal)}$$

Projection Operators

The right translation by e^μ_a gives the projection P^μ_a.

$$P^\mu_a\, |\, r\,\rangle = \left|\, r\, e^\mu_a\,\right\rangle \qquad \forall\, r \in \mathcal{G}$$

- **Idempotents**

$$e_a^{\mu}\, e_b^{\nu} = \delta^{\mu\,\nu}\, \delta_{a\,b}\, e_a^{\mu}$$
$$\sum_{\mu\,a} e_a^{\mu} = 1$$

Minimal idempotents that generate IR's:

$$e_a^{\mu}\; r\, e_b^{\nu} = \delta_{a\,b}\, \delta^{\mu\,\nu}\; \alpha_r\, e_a^{\mu} \qquad \forall\, r \in \mathcal{G}$$

Complete Reduction of Regular Representations

The IR decomposition can be found if all inequivalent primitive idempotents are known.

Any primitive idempotent can be written as a linear combination of the group elements:

$$e_i = \sum_j c_{i\,j}\, g_j$$

where c is a $n_i\, n_C \times n_i\, n_C$ matrix , $n_C =$ # of classes= # of IRs , n_i = dimension of the i th IR. Note that $(\mu\,,\,a\,)$ is treated as a single index i.

The conditions

$$e_i\, r\, e_j = \delta_{i\,j}\, \lambda_r\, e_i$$

can be used to determine c if a particular e_i is known.

Any group has an **identity representation** Γ_1 obtained by setting all elements to 1.

The rearrangement theorem :

$$h \sum_{g\in G} g = \sum_{g\in G} g \qquad \forall\; h \in G$$

suggests the corresponding primitive idempotent e_1 to be:

$$e_1 = \alpha \sum_{g\in G} g$$

where α is the normalization which makes $e_1\, e_1 = e_1$.

Using the rearrangement theorem, we see that

$$e_1\, e_1 = \alpha^2\, n_G \sum_{g\in G} g = \alpha\, n_G\, e_1$$

$$\longrightarrow \qquad \alpha = \frac{1}{n_G}$$

$$e_1 = \frac{1}{n_G} \sum_{g\in G} g$$

Thus

$$e_1\, e_1 = e_1$$
$$g\, e_1 = e_1 = e_1\, g \qquad \forall\, g \in G$$
$$e_1\, g\, e_1 = e_1\, e_1 = e_1$$

$\Longrightarrow$

$$\forall\; r \in \mathcal{G}, \qquad \text{ie.,} \qquad r = \sum_i r^i\, g_i$$

$$e_1\, r\, e_1 = \sum_i r^i\, e_1\, g_i\, e_1 = \sum_i r^i\, e_1 = \lambda_r\, e_1 \qquad \text{where} \quad \lambda_r = \sum_i r^i$$

which establishes e_1 as a primitive idempotent as claimed.

Other primitive idempotents can then be constructed with the help of the idempotent criteria. This is a tedious process that works practically only for groups of low order. See Hamermesh for a systematic scheme for the group S_n.

Example: C_3

$C_3 = \{ e, a, b = a^2 = a^{-1} \}$

Since it is abelian, $n_C = n_G = 3$ & $n_i = 1$.

$$e_1 = \frac{1}{3}(e + a + b)$$

Let $e_2 = x e + y a + z b$
where x, y, z are constants to be determined.

$e_1 e_2 = 0$
$\longrightarrow \quad 0 = (e + a + b) (x e + y a + z b)$
$\quad = (x + y + z) (e + a + b)$
$\therefore \quad x + y + z = 0 \qquad$ (a)

$e_2 e_2 = e_2$
$\longrightarrow \quad x e + y a + z b = (x e + y a + z b) (x e + y a + z b)$
Using
$a^2 = b \qquad b^2 = a \qquad a b = b a = e$
$(p + q + r)^2 = p^2 + q^2 + r^2 + 2 (p q + q r + r p)$
we have
$x e + y a + z b = \left(x^2 + 2 y z \right) e + \left(z^2 + 2 x y \right) a + \left(y^2 + 2 x z \right) b$
$\therefore \quad x = x^2 + 2 y z \qquad$ (1)
$y = z^2 + 2 x y \qquad$ (2)
$z = y^2 + 2 x z \qquad$ (3)
Since the sum of these 3 eqs is just
$x + y + z = (x + y + z)^2$
$\longrightarrow \quad x + y + z = 0 \qquad$ or $\qquad x + y + z = 1$
Thus, eqs 1–3 can be compatible with (a) but some of their solutions which satisfy $x + y + z = 1$ must be discarded.
Furthermore, the trivial solution $x = y = z = 0$ is also discarded for obvious reasons.

To avoid spurious solutions, we shall seek solutions of eqs 1–3 & then impose (a).
Rearrangement of eqs 1–3 give:
$x (1 - x) = 2 y z \qquad$ (1a)
$y (1 - 2 x) = z^2 \qquad$ (2a)
$z (1 - 2 x) = y^2 \qquad$ (3a)

2a / 3a $\longrightarrow \quad \frac{y}{z} = \frac{z^2}{y^2}$
$y^3 - z^3 = (y - z) \left(y^2 + y z + z^2 \right) = 0$
$\therefore \quad y = z \qquad$ or $\qquad y^2 + y z + z^2 = 0$

- ***y = z***

Eqs 1a–3a becomes:
$x (1 - x) = 2 y^2 \qquad$ (1b)
$y (1 - 2 x) = y^2 \qquad$ (2b)
$y (1 - 2 x) = y^2 \qquad$ (3b)
2b $\longrightarrow \quad y = 0 \qquad$ or $\qquad y = 1 - 2 x$

For $y = z = 0$

1b $\longrightarrow$ $x = 0$ or $x = 1$

which give $x + y + z = 0$ or $x + y + z = 1$

Both solutions are not acceptible as discussed before.

For $y = z = 1 - 2x$

1b $\longrightarrow$ $x(1 - x) = 2\left(1 - 4x + 4x^2\right)$

$9x^2 - 9x + 2 = 0$

$(3x - 2)(3x - 1) = 0$

$\therefore$ $x = \frac{2}{3}$ or $x = \frac{1}{3}$

which give

$y = z = -\frac{1}{3}$ or $y = z = \frac{1}{3}$

$x + y + z = 0$ or $x + y + z = 1$

Thus, the acceptible solution is

$$x = \frac{2}{3} \qquad y = z = -\frac{1}{3} \qquad \text{(b)}$$

- **$y^2 + yz + z^2 = 0$**

This gives

$$y = \frac{1}{2}\left(-z \pm \sqrt{z^2 - 4z^2}\right) = \frac{z}{2}\left(-1 \pm \sqrt{3}\, i\right)$$

Eqs 1a–3a becomes:

$$x(1-x) = z^2\left(-1 \pm \sqrt{3}\, i\right) \qquad (1c)$$

$$\frac{1}{2}\left(-1 \pm \sqrt{3}\, i\right)(1-2x) = z \ (2c)$$

$$1 - 2x = \frac{z}{4}\left(-1 \pm \sqrt{3}\, i\right)^2 \qquad (3c)$$

2c / 3c $\longrightarrow$

$$\frac{1}{2}\left(-1 \pm \sqrt{3}\, i\right) = \frac{4}{\left(-1 \pm \sqrt{3}\, i\right)^2} = \frac{4}{-2 \mp 2\sqrt{3}\, i} = -\frac{2}{1 \pm \sqrt{3}\, i} = -\frac{1}{2}\left(1 \mp \sqrt{3}\, i\right)$$

so that 2c & 3c are equivalent as expected.

2c $\rightarrow$ 1c gives

$$x(1-x) = \frac{1}{4}\left(-1 \pm \sqrt{3}\, i\right)^3 (1-2x)^2 = 2(1-2x)^2$$

This eq was already solved in the $y = z$ case. Thus

$$x = \frac{2}{3} \qquad \text{or} \qquad x = \frac{1}{3}$$

2c $\longrightarrow$

$$z = -\frac{1}{6}\left(-1 \pm \sqrt{3}\, i\right) \text{ or } \qquad z = \frac{1}{6}\left(-1 \pm \sqrt{3}\, i\right)$$

$$y = -\frac{1}{12}\left(-1 \pm \sqrt{3}\, i\right)^2 \qquad \text{or} \qquad y = \frac{1}{12}\left(-1 \pm \sqrt{3}\, i\right)^2$$

$$= \frac{1}{6}\left(1 \pm \sqrt{3}\, i\right) \qquad\qquad = -\frac{1}{6}\left(1 \pm \sqrt{3}\, i\right)$$

so that

$$x + y + z = 1 \qquad \text{or} \qquad x + y + z = 0$$

Thus, the acceptible solutions are:

$$x = \frac{1}{3} \qquad y = -\frac{1}{6}\left(1 \pm \sqrt{3}\, i\right) \qquad z = \frac{1}{6}\left(-1 \pm \sqrt{3}\, i\right)$$

or

$$x = \frac{1}{3} \qquad y = \frac{1}{3}\, e^{\mp \frac{2\pi}{3} i} \qquad z = \frac{1}{3}\, e^{\pm \frac{2\pi}{3} i} \qquad (c)$$

- **Solution**

To summarize, we've found 3 idempotents:

(b) $\longrightarrow$ $\quad e' = \frac{1}{3}(2e - a - b)$

(c) $\longrightarrow e_{\pm} = \frac{1}{3}\left(e + e^{\mp \frac{2\pi}{3} i}\, a + e^{\pm \frac{2\pi}{3} i}\, b\right)$

Since there're only 3 IR's, only 2 of them are primitive idempotents.

Note: our $e_{\pm}$ is equal to the $e_{\mp}$ used in p.312 of Tung.

The criterion for an idempotent to be primitive is

$$e_i\, r\, e_i = \lambda_r\, e_i \qquad \forall\ r \in \mathcal{G} \qquad \text{with } \lambda_r = \text{const}$$

Each of the idempotents must now be checked against the criterion. This is tedious but straightforward. We shall skip this & show merely that $e\,'$ is not primitive:

$$\begin{aligned} e\,'\,a\,e\,' &= \frac{1}{9}\,(2\,e - a - b)\,(2\,a - b - e) \\ &= \frac{1}{9}\,(4\,a - 2\,b - 2\,e - 2\,b + e + a - 2\,e + a + b) \\ &= \frac{1}{9}\,(-3\,e + 6\,a - 3\,b) \\ &= -\frac{1}{3}\,(e - 2\,a + b) \\ &= a\,e' \end{aligned}$$

Since the IRs are $1 - D$, they are simply the coefficients of $g\,e_\pm$.

Thus

$$\begin{aligned} e\,e_\pm &= e_\pm \\ a\,e_\pm &= \frac{1}{3}\left(a + e^{\mp\frac{2\pi}{3}i}\,b + e^{\pm\frac{2\pi}{3}i}\,e\right) \\ &= \frac{1}{3}\left(e + e^{\mp\frac{2\pi}{3}i}\,a + e^{\pm\frac{2\pi}{3}i}\,b\right)e^{\pm\frac{2\pi}{3}i} \\ &= e_\pm\,e^{\pm\frac{2\pi}{3}i} \\ b\,e_\pm &= \frac{1}{3}\left(b + e^{\mp\frac{2\pi}{3}i}\,e + e^{\pm\frac{2\pi}{3}i}\,a\right) = e_\pm\,e^{\mp\frac{2\pi}{3}i} \end{aligned}$$

$\longrightarrow$

C_3	e	a	a^{-1}
Γ_1	1	1	1
Γ_+	1	$e^{\frac{2\pi}{3}i}$	$e^{-\frac{2\pi}{3}i}$
Γ_-	1	$e^{-\frac{2\pi}{3}i}$	$e^{\frac{2\pi}{3}i}$