Matrix

To gain more insight into the idempotent approach, we work out the details for S_3.

Regular Representation of S_3

The multiplication table of S_3 is:

S_3	e	(12)	(23)	(31)	(123)	(321)
e	e	(12)	(23)	(31)	(123)	(321)
(12)	(12)	e	(123)	(321)	(23)	(31)
(23)	(23)	(321)	e	(123)	(31)	(12)
(31)	(31)	(123)	(321)	e	(12)	(23)
(123)	(123)	(31)	(12)	(23)	(321)	e
(321)	(321)	(23)	(31)	(12)	e	(123)

Using the definition for regular representation:

$$D^R(g_m)^i{}_j = \Delta^i{}_{m\,j} = \begin{cases} 1 \text{ if} & g_i = g_m\, g_j \\ 0 & \text{otherwise} \end{cases}$$

we have

$$D^R[e] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad D^R[(12)] = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$D^R[(23)] = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad D^R[(31)] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$D^R[(123)] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad D^R[(321)] = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the basis is { g_i }:

$$e = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (12) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (23) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (31) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(123) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (321) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The dual role of g as operator & vector is easily checked out.
For example,

as group elements: $(123)\cdot(321) = e$

as operators: $D^R\,[(123)]\cdot D^R\,[(321)] = D^R\,[e]$

$$\text{ie} \quad \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&1&0 \end{pmatrix} \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 1&0&0&0&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$$

as operation on vectors: $D^R\,[(123)]\cdot(321) = e$

$$\text{ie} \quad \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&1&0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The group algebra is the set

$r = r^i\, g_i \qquad \forall\, r^i =$ complex numbers

Considered as vectors, the group algebra is a 6-dimensional vector space &

$$r = \begin{pmatrix} r^e \\ r^{12} \\ r^{23} \\ r^{31} \\ r^{123} \\ r^{321} \end{pmatrix}$$

Consider as operators, the group algebra consists of all 6×6 matrices of the form

$$D^R[r] = \begin{pmatrix} r^e & r^{12} & r^{23} & r^{31} & r^{321} & r^{123} \\ r^{12} & r^e & r^{123} & r^{321} & r^{31} & r^{23} \\ r^{23} & r^{321} & r^e & r^{123} & r^{12} & r^{31} \\ r^{31} & r^{123} & r^{321} & r^e & r^{23} & r^{12} \\ r^{123} & r^{31} & r^{12} & r^{23} & r^e & r^{321} \\ r^{321} & r^{23} & r^{31} & r^{12} & r^{123} & r^e \end{pmatrix}$$

Note that r is not invertible whenever $\det D^R[r] = 0$.

The special matrix form of the elements in the group algebra means that they can all be put into a block- diagonal form by a similarity transformation. Now,

$$\Gamma^R = A + A' + 2E$$

where A, A', E are irreducible representations of S_3.
Thus, there must exist an invertible matrix S such that:

$$S\ D^R[r]\ S^{-1} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & e & f & 0 & 0 \\ 0 & 0 & 0 & 0 & g & h \\ 0 & 0 & 0 & 0 & i & j \end{pmatrix}$$

with respect to the basis

$$u_i = S\, g_i$$

Here, u_1 & u_2 each spans an 1-dim invariant subspace, while (u_3 , u_4) & (u_5 , u_6) each spans a 2-dim invariant subspace.

Symmetrizers

In sec 5.3, we showed that the irreducible symmetrizers e_μ of S_3 can be obtained from the standard Young tableaux:

$\Theta_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$ gives $e_1 = s$

$\Theta_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$ gives $e_2 = s_2\, a_2 = e + (12) - (31) - (321)$

$\Theta_3 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$ gives $e_3 = a$

$\Theta_2^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$ gives $e_2^{(23)} = e + (31) - (12) - (123)$

In terms of vectors in basis { g_i }, we have

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \qquad e_2{}^{(23)} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

That these symmetrizers are orthogonal to each other ($e_\mu\, e_\nu \propto\ \delta_{\mu\,\nu}$) can be easily verified by direct calculations.

They are also basis vectors with respect to which $D^R(r)$ are all block diagonal for all $r \in \tilde{G}$.
Since e_2 & $e_2{}^{(23)}$ are symmetrizers of the 2-dim irreducible representation E, they need to be supplemented by their partners to make the basis complete. This is easily done by multiplying them with g_i.

For example, the partner of e_2 is

$$r_2 = (23)\, e_2 = (23) + (321) - (123) - (12) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

or

$$r_2 = D^R\,[(23)]\, e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

In terms of operators, we have

$$e_2 = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

The various products among r_2 & e_2 can be easily found using, say, mathematica:

$$e_2\, e_2 = \begin{pmatrix} 3 & 3 & 0 & -3 & -3 & 0 \\ 3 & 3 & 0 & -3 & -3 & 0 \\ 0 & -3 & 3 & 0 & 3 & -3 \\ -3 & 0 & -3 & 3 & 0 & 3 \\ 0 & -3 & 3 & 0 & 3 & -3 \\ -3 & 0 & -3 & 3 & 0 & 3 \end{pmatrix} = \ 3\, \mathrm{e}_2$$

$e_2\, r_2 = 0$

$$r_2\, e_2 = \begin{pmatrix} 0 & -3 & 3 & 0 & 3 & -3 \\ -3 & 0 & -3 & 3 & 0 & 3 \\ 3 & 3 & 0 & -3 & -3 & 0 \\ 0 & -3 & 3 & 0 & 3 & -3 \\ -3 & 0 & -3 & 3 & 0 & 3 \\ 3 & 3 & 0 & -3 & -3 & 0 \end{pmatrix} = 3\, r_2$$

$r_2\, r_2 = 0$

The partner of $e_2{}^{(23)}$ is

$$r_2{}^{(23)} = (23)\, e_2{}^{(23)} = (23) + (123) - (321) - (31) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Note that although the basis $\left\{ e_2 ,\, r_2 ,\; e_2{}^{(23)},\, r_2{}^{(23)} \right\}$ are orthogonal to e_1 & e_3, its members are not mutually orthogonal. This is expected since the 2 irreducible representations are equivalent. An orthogonal basis is easily obtained by means of the Schmidt orthogonalization method.

In terms of operators, we have

$$e_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix} \quad e_2{}^{(23)} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

■ Projection Operators

Let L_1 be an invariant subspace or left ideal of L.
Let $\{u_i\}$ be a basis of L_1.
We can then always choose a basis of L by adding to $\{u_i\}$ enough independent vectors to span L. This basis is of the form $\left\{ \{u_i\},\, \{v_j\} \right\}$ where all v 's are orthogonal to all u 's.
An arbitrary vector in L is then of the form

$$\begin{pmatrix} \mathrm{u} \\ \mathrm{v} \end{pmatrix}$$

where u is in L_1.

Every operator with respect to which L_1 is invariant must then be of the form

$$R = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \qquad \text{where} \quad A,\ B,\ C,\ 0 \ \text{are matrices of appropriate dimensions}$$

so that

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} A\,u \\ 0 \end{pmatrix} \in L_1$$

Note that R is of this form only with respect to the basis $\{\{u_i\},\ \{v_j\}\}$. It is easy to see that an arbitrary similarity transform will in general destroy it.
On the other hand, any properties of R derived under this special basis will be valid in a general basis if the said properties are invariant under a similarity transform.

Consider now the form of a projection operator P in this basis. By definition

$$P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix} \qquad \forall\ u, v$$

we see that we must be of the form

$$P = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

Obviously, det $P = 0$, so that P^{-1} doesn't exist.
If we demand

$$P^2 = P$$

we have

$$P^2 = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^2 & A\,B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

or

$$A^2 = A \ \& \qquad A\,B = B$$

The 2nd relation is automatically satisfied if $B = A\,F$ for arbitrary F.

Thus, the general solution is

$$P = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}$$

where P' is an idempotent operator ($P'^2 = P'$) and Q is arbitrary.

A special solution is $P' = I$ so that

$$P = \begin{pmatrix} I & Q \\ 0 & 0 \end{pmatrix}$$

The effects of P on an arbitrary operator $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are:

$$P\,R = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P'A + P'Q\,C & P'B + P'Q\,D \\ 0 & 0 \end{pmatrix}$$

$$R\,P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A\,P' & A\,P'Q \\ C\,P' & D\,P'Q \end{pmatrix}$$

$$P\,R\,P = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A\,P' & A\,P'Q \\ C\,P' & D\,P'Q \end{pmatrix}$$

$$= \begin{pmatrix} P'A\,P' + P'Q\,C\,P' & P\,A\,P'Q + P'Q\,D\,P'Q \\ 0 & 0 \end{pmatrix}$$

If L_1 is invariant under R, $C = 0$, so that

$$P\,R = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} P'A & P'B + P'Q\,D \\ 0 & 0 \end{pmatrix}$$

$$R\,P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A\,P' & A\,P'Q \\ 0 & D\,P'Q \end{pmatrix}$$

$$P\,R\,P = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A\,P' & A\,P'Q \\ 0 & D\,P'Q \end{pmatrix}$$

$$= \begin{pmatrix} P'A\,P' & P\,A\,P'Q + P'Q\,D\,P'Q \\ 0 & 0 \end{pmatrix}$$

Thus, in general,

$$P\,R \neq R\,P$$

If we impose the condition

$$P\,R = R\,P \qquad \forall\ R$$

P must be of the form

$$P = \alpha \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad \alpha = \text{constant}$$

The condition

$$P^2 = P$$

then restricts α to the value 1.

We emphasize that P is in this simple form only with respect to the basis $\{\{u_i\}, \{v_j\}\}$.
Under a similarity transform wrt

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{pmatrix}$$

we have

$$S\,P\,S^{-1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{pmatrix}$$

$$= \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} S_{11}\,(S^{-1})_{11} & S_{11}\,(S^{-1})_{12} \\ S_{21}\,(S^{-1})_{11} & S_{21}\,(S^{-1})_{12} \end{pmatrix}$$

If $\quad P\,R = R\,P \qquad \forall\ R$

we have

$$P^2\,R = P\,R = P\,R\,P = R\,P \qquad (\,P^2 = P\,)$$

Let

$$A\,B\,|\,x\,\rangle = C\,|\,y\,\rangle = |\,z\,\rangle$$

then

$$P\,A\,B\,|\,x\,\rangle = P\,C\,|\,y\,\rangle = P\,|\,z\,\rangle$$

$$= \ \mathrm{P\,A\,P\ \ P\,B\,P\,P}\,|\,x\,\rangle \qquad \text{if} \qquad P\,R = R\,P = P\,R\,P$$

$$= \ \mathrm{A}_P\,B_P\,|\,x_P\,\rangle$$

$$= \ \mathrm{P\,C\,P\,P}\,|\,y\,\rangle = C_P\,|\,y_P\,\rangle$$

$$= \ |\,z_P\,\rangle$$

where the subscript denotes the projected component.
Thus, the projector preserves the operator multiplication if it commutes with every operator.

Consider

$$A\,A^{-1} = E$$

we have

$$P\,A\,A^{-1} = P\,E$$

$$= \ P\,A\,P\,P\ \ A^{-1}\,P = P$$

$$= \ \mathrm{A}_P\,A^{-1}{}_P = E_P$$

Thus, P is the projected component of the identity. It also plays the role of the identity operator for projected operators A_P.

Idempotents

The basis for D^R is just the group elements $\{\,g_i\,\}$.
Let the basis of the completely decomposed representation $\mathcal{D}$ be $\{\,u_{\mu\,a}\,\}$, where μ runs through all the irreducible representations & $a = 1,\ \ldots,\ d_\alpha$ with d_α being the dimension of the representation α. By definition,

$$\mathcal{D}(r)^{\alpha\,a}{}_{\beta\,b} = \delta^{\alpha}{}_{\beta}\ \mathcal{D}^{\alpha}(r)^{a}{}_{b} \qquad \forall\ r \in \tilde{G}$$

These two basis must be related by an invertible transformation S:

$$u_{\alpha\,a} = g_i\,S^{i}{}_{\alpha\,a}$$

$$g_i = u_{\alpha\,a}\,S^{\alpha\,a}{}_{i}$$

with

$$S^{i}{}_{\alpha\,a}\,S^{\alpha\,a}{}_{j} = \delta^{i}{}_{j} \qquad\qquad S^{i}{}_{\alpha\,a}\,S^{\beta\,b}{}_{i} = \delta^{\beta}{}_{\alpha}\,\delta^{b}{}_{a}$$

Group multiplication among $\{\,u_{\mu\,a}\,\}$ is found with the help of the regular representation

$$D^R(\,g_i)^{j}{}_{k} = \Delta^{j}{}_{i\,k} = \begin{cases} 1 & \text{for} \quad \mathrm{g_j = g_i\,g_k} \\ 0 & \text{otherwise} \end{cases}$$

as

$$u_{\beta\,b}\,u_{\alpha\,a} = g_j\,S^{j}{}_{\beta\,b}\ \ g_i\,S^{i}{}_{\alpha\,a}$$

$$= \ \mathrm{g}_k\,\Delta^{k}{}_{j\,i}\,S^{j}{}_{\beta\,b}\ \ S^{i}{}_{\alpha\,a}$$

$$= \ \mathrm{g}_k\,D^R\big(\,g_j\big)^{k}{}_{i}\,S^{j}{}_{\beta\,b}\ \ S^{i}{}_{\alpha\,a}$$

$$= \ \mathrm{g}_k\,D^R\big(\,g_j\,S^{j}{}_{\beta\,b}\big)^{k}{}_{i}\ \ S^{i}{}_{\alpha\,a}$$

$$= \ \mathrm{g}_k\,D^R\big(\,u_{\beta\,b}\big)^{k}{}_{i}\ \ S^{i}{}_{\alpha\,a}$$

By definition

$$\mathcal{D}(\,r\,)^{\alpha\,a}{}_{\beta\,b} = S^{\alpha\,a}{}_{j}\ D^R(r)^{j}{}_{i}\ S^{i}{}_{\beta\,b} = \delta^{\alpha}{}_{\beta}\ \mathcal{D}^{\alpha}(r)^{a}{}_{b}$$

so that

$$D^R\big(\,u_{\beta\,b}\big)^{k}{}_{i}\ \ S^{i}{}_{\alpha\,a} = S^{k}{}_{\gamma\,c}\,\mathcal{D}\big(\,u_{\beta\,b}\big)^{\gamma\,c}{}_{\alpha\,a}$$

$$= \ \mathrm{S}^{k}{}_{\alpha\,c}\,\mathcal{D}^{(\alpha)}\big(\,u_{\beta\,b}\big)^{c}{}_{a}$$

Hence

$$u_{\beta b}\, u_{\alpha a} = g_k\, S^k{}_{\alpha c}\, \mathcal{D}^{(\alpha)}\big(u_{\beta b}\big)^c{}_a$$
$$= \; u_{\alpha c}\, \mathcal{D}^{(\alpha)}\big(u_{\beta b}\big)^c{}_a$$

which is simply the definition of irreducible representation $\mathcal{D}^{\alpha}$ of the operator $u_{\beta b}$ with respect to the irreducible basis set $\{ u_{\alpha a} ;\; a = 1, \ldots, d_\alpha \}$.

Since L^α is an invariant subspace, it is meaningful to write

$$g_i = \sum_\alpha u_{\alpha a}\, S^{(\alpha)\, a}{}_i$$
$$= \sum_\alpha g_{i\,\alpha}$$

where (α) means that α is exempted from the Einstein summation notation, &

$$g_{i\,\alpha} = u_{\alpha a}\, S^{(\alpha)\, a}{}_i$$

In particular, the identity element $E \equiv g_1$ is given by

$$E = u_{\alpha a}\, S^{\alpha\, a}{}_1 = \sum_\alpha u_{\alpha a}\, S^{(\alpha)\, a}{}_1 = \sum_\alpha g_{1\,\alpha}$$
$$g_{1\,\alpha} = u_{\alpha a}\, S^{(\alpha)\, a}{}_1$$

The idempotent e_α is defined as a vector proportional to a basis vector, say, $u_{\alpha 1}$, of L^α:

$$e_\alpha = c_\alpha\, u_{\alpha 1}$$

where c_α is a constant. Furthermore, we demand

$$E = \sum_\alpha e_\alpha$$

This means that e_α is equal to $g_{1\,\alpha}$ for a special choice of S such that

$$S^{\alpha\, a}{}_1 = \delta^a{}_1\, S^{\alpha\, 1}{}_1 \qquad \forall\ \alpha$$

Thus

$$e_\alpha = u_{\alpha 1}\;\, S^{(\alpha)\, 1}{}_1 \qquad \text{or} \qquad c_\alpha = \; S^{\alpha\, 1}{}_1$$

We also have

$$S^i{}_{\alpha a}\, S^{\alpha\, a}{}_1 = \delta^i{}_1 = S^i{}_{\alpha 1}\, S^{\alpha\, 1}{}_1$$

The representation of e_α in L^α is, by definition,

$$e_\alpha\, |\, u_{\alpha a} \rangle = \; |\; u_{\alpha b}\; \rangle\, \mathcal{D}^{(\alpha)}\, [\, e_\alpha\,]^b{}_a$$
$$= \; u_{\alpha 1}\;\, S^{(\alpha)\, 1}{}_1\; \big|\, u_{\alpha a} \big\rangle$$
$$= \; |\, u_{\alpha b} \rangle\, \mathcal{D}^{(\alpha)}(\, u_{\alpha 1})^b{}_a\, S^{(\alpha)\, 1}{}_1$$

Hence,

$$\mathcal{D}^{(\alpha)}\, [\, e_\alpha\,]^b{}_a = \delta^b{}_a$$

Next, we demand e_α to be the projection operator onto one of the basis, say, $u_{\alpha 1}$, of L^α.
Thus, with respect to the basis $\{ u_{\alpha a} ;\; a = 1, \ldots, d_\alpha \}$,

$$e_\alpha\, |\, u_{\alpha a} \rangle = \; |\, u_{\alpha b}\; \rangle\, \mathcal{D}^{(\alpha)}\, [\, e_\alpha\,]^a{}_b$$

with

$$\mathcal{D}^{(\alpha)}\, [\, e_\alpha\,]^a{}_b = \delta^a{}_1\, \mathcal{D}^{(\alpha)}\, [\, e_\alpha\,]^1{}_b$$

or

$$\mathcal{D}^{(\alpha)}\, [\, e_\alpha\,] = \begin{pmatrix} \mathcal{D}^{(\alpha)}\, [\, e_\alpha\,]^1{}_1 & \mathcal{D}^{(\alpha)}\, [\, e_\alpha\,]^1{}_1 & \ldots & \mathcal{D}^{(\alpha)}\, [\, e_\alpha\,]^1{}_{d_\alpha} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

To see if

Let us examine the definition of the projector P_α:

$$P_\alpha \ | r \rangle \equiv | \ r e_\alpha \ \rangle$$

Writing

$$r = r^i g_i = r^{\alpha a} u_{\alpha a}$$

we have

$$r e_\alpha = r^{\beta b} u_{\beta b} u_{(\alpha) a} S^{(\alpha) a}{}_1$$

To evaluate $u_{\beta b} u_{\alpha a}$, we need the regular representation

$$D^R(g_i)^j{}_k = \Delta^j{}_{ik} = \begin{cases} 1 & \text{for} \quad g_j = g_i g_k \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\begin{aligned} u_{\beta b} u_{\alpha a} &= g_j S^j{}_{\beta b} \ g_i S^i{}_{\alpha a} \\ &= g_k \Delta^k{}_{ji} S^j{}_{\beta b} \ S^i{}_{\alpha a} \\ &= g_k D^R\left(g_j\right)^k{}_i S^j{}_{\beta b} \ S^i{}_{\alpha a} \\ &= g_k D^R\left(g_j S^j{}_{\beta b}\right)^k{}_i \ S^i{}_{\alpha a} \\ &= g_k D^R\left(u_{\beta b}\right)^k{}_i \ S^i{}_{\alpha a} \end{aligned}$$

By definition

$$\mathcal{D}(r)^{\alpha a}{}_{\beta b} = S^{\alpha a}{}_j \ D^R(r)^j{}_i \ S^i{}_{\beta b} = \delta^\alpha{}_\beta \ \mathcal{D}^\alpha(r)^a{}_b$$

so that

$$\begin{aligned} D^R\left(u_{\beta b}\right)^k{}_i \ S^i{}_{\alpha a} &= S^k{}_{\gamma c} \mathcal{D}\left(u_{\beta b}\right)^{\gamma c}{}_{\alpha a} \\ &= S^k{}_{\alpha c} \mathcal{D}^{(\alpha)}\left(u_{\beta b}\right)^c{}_a \end{aligned}$$

Hence

$$\begin{aligned} u_{\beta b} u_{\alpha a} &= g_k S^k{}_{\alpha c} \mathcal{D}^{(\alpha)}\left(u_{\beta b}\right)^c{}_a \\ &= u_{\alpha c} \mathcal{D}^{(\alpha)}\left(u_{\beta b}\right)^c{}_a \end{aligned}$$

which is simply the definition of irreducible representation $\mathcal{D}^\alpha$ of the operator $u_{\beta b}$ with respect to the irreducible basis set $\{ u_{\alpha a} ; \ a = 1, \ \ldots, \ d_\alpha \}$.

Therefore

$$\begin{aligned} r e_\alpha &= r^{\beta b} \ u_{\beta b} u_{(\alpha) a} S^{(\alpha) a}{}_1 \\ &= r^{\beta b} = u_{\alpha c} \mathcal{D}^{(\alpha)}\left(u_{\beta b}\right)^c{}_a S^{(\alpha) a}{}_1 \\ &= u_{\alpha c} \mathcal{D}^{(\alpha)}\left(r^{\beta b} u_{\beta b}\right)^c{}_a S^{(\alpha) a}{}_1 \\ &= u_{\alpha c} \mathcal{D}^{(\alpha)}(r)^c{}_a S^{(\alpha) a}{}_1 \end{aligned}$$

Thus $r e_\alpha \in L^\alpha$ so that P_α is indeed a projector onto L^α.
Obviously, this relation can also be written down immediately by treating r as an operator & e_α a vector in the invariant subspace L^α.

Consider now the product $e_\alpha r e_\alpha$.

$$\begin{aligned} e_\alpha r e_\alpha &= e_\alpha u_{(\alpha) c} \mathcal{D}^{(\alpha)}(r)^c{}_a S^{(\alpha) a}{}_1 \\ &= u_{(\alpha) b} S^{(\alpha) b}{}_1 \ u_{(\alpha) c} \mathcal{D}^{(\alpha)}(r)^c{}_a S^{(\alpha) a}{}_1 \\ &= u_{(\alpha) f} \mathcal{D}^{(\alpha)}(u_{(\alpha) b})^f{}_c S^{(\alpha) b}{}_1 \ \mathcal{D}^{(\alpha)}(r)^c{}_a S^{(\alpha) a}{}_1 \\ &= u_{(\alpha) f} \mathcal{D}^{(\alpha)}(e_\alpha)^f{}_c \ \mathcal{D}^{(\alpha)}(r)^c{}_a S^{(\alpha) a}{}_1 \\ &= u_{(\alpha) f} \mathcal{D}^{(\alpha)}(e_\alpha r)^f{}_a \ S^{(\alpha) a}{}_1 \end{aligned}$$