Matrix

To gain more insight into the idempotent approach, we work out the details for $S_3$.

- **Regular Representation of $S_3$**

The multiplication table of $S_3$ is:

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>(12)</th>
<th>(23)</th>
<th>(31)</th>
<th>(123)</th>
<th>(321)</th>
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<tr>
<td>e</td>
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<td>(321)</td>
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<td>(23)</td>
<td>(31)</td>
<td>(12)</td>
<td>e</td>
<td>(123)</td>
</tr>
</tbody>
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Using the definition for regular representation:

$$D_R^g(i) = \Delta^i_m(j) = \begin{cases} 1 & \text{if } g_i = g_m g_j \\ 0 & \text{otherwise} \end{cases}$$

we have

$$D_R^e = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_R^{(12)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D_R^{(23)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad D_R^{(31)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$D_R^{(123)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_R^{(321)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
where the basis is \( \{ g_i \} \):

\[
\begin{align*}
  e &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
  (12) &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
  (23) &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
  (31) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
  (123) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
  (321) &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
\]

The dual role of \( g \) as operator \& vector is easily checked out. For example,

as group elements: \((123) \cdot (321) = e\)

as operators: \(D^g[(123)] \cdot D^g[(321)] = D^g[e]\)

as operation on vectors: \(D^g[(123)] \cdot (321) = e\)

\[
\begin{align*}
  D^g &= \begin{pmatrix}
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0
  \end{pmatrix} \cdot \begin{pmatrix}
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0
  \end{pmatrix} = \begin{pmatrix}
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1
  \end{pmatrix}
\end{align*}
\]

as vectors, the group algebra is a 6-dimensional vector space &
Consider as operators, the group algebra consists of all $6 \times 6$ matrices of the form

$$D^R [r] = \begin{pmatrix}
  r^e & r^{12} & r^{23} & r^{31} & r^{123} & r^{123} \\
  r^{12} & r^e & r^{123} & r^{321} & r^{31} & r^{23} \\
  r^{23} & r^{321} & r^e & r^{123} & r^{12} & r^{31} \\
  r^{31} & r^{123} & r^{321} & r^e & r^{23} & r^{12} \\
  r^{123} & r^{31} & r^{12} & r^{23} & r^e & r^{321} \\
  r^{321} & r^{23} & r^{31} & r^{12} & r^{123} & r^e
\end{pmatrix}$$

Note that $r$ is not invertible whenever $\det D^R [r] = 0$.

The special matrix form of the elements in the group algebra means that they can all be put into a block-diagonal form by a similarity transformation. Now,

$$\Gamma^R = A + A' + 2E$$

where $A, A', E$ are irreducible representations of $S_3$.

Thus, there must exist an invertible matrix $S$ such that:

$$S \ D^R [r] \ S^{-1} = \begin{pmatrix}
  a & 0 & 0 & 0 & 0 & 0 \\
  0 & b & 0 & 0 & 0 & 0 \\
  0 & 0 & c & d & 0 & 0 \\
  0 & 0 & e & f & 0 & 0 \\
  0 & 0 & 0 & 0 & g & h \\
  0 & 0 & 0 & 0 & i & j
\end{pmatrix}$$

with respect to the basis

$$u_i = S g_i$$

Here, $u_1 & u_2$ each spans an 1-dim invariant subspace, while $(u_3, u_4) \& (u_5, u_6)$ each spans a 2-dim invariant subspace.

### Symmetrizers

In sec 5.3, we showed that the irreducible symmetrizers $e_{\mu}$ of $S_3$ can be obtained from the standard Young tableaux:

$\Theta_1 = \begin{array}{c}
  1 \\
  2 \\
  3
\end{array}$ gives $e_1 = s$

$\Theta_2 = \begin{array}{c}
  1 \\
  2 \\
  3
\end{array}$ gives $e_2 = s_2 a_2 = e + (12) - (31) - (321)$

$\Theta_3 = \begin{array}{c}
  1 \\
  2 \\
  3
\end{array}$ gives $e_3 = a$

$\Theta^{(23)}_2 = \begin{array}{c}
  1 \\
  2 \\
  3
\end{array}$ gives $e_2^{(23)} = e + (31) - (12) - (123)$
In terms of vectors in basis \( \{ g_i \} \), we have

\[
\begin{align*}
e_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
e_2 &= \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\
e_3 &= \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \\
e_2^{(23)} &= \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}
\end{align*}
\]

That these symmetrizers are orthogonal to each other (\( e_\mu e_\nu \propto \delta_{\mu \nu} \)) can be easily verified by direct calculations. They are also basis vectors with respect to which \( D^R(r) \) are all block diagonal for all \( r \in \hat{G} \).

Since \( e_2 \) & \( e_2^{(23)} \) are symmetrizers of the 2-dim irreducible representation \( E \), they need to be supplemented by their partners to make the basis complete. This is easily done by multiplying them with \( g_i \).

For example, the partner of \( e_2 \) is

\[
r_2 = (23) e_2 = (23) + (321) - (123) - (12) = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}
\]

or

\[
r_2 = D^R[(23)] e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}
\]

In terms of operators, we have

\[
\begin{align*}
e_2 &= \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{pmatrix} & \quad r_2 &= \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}
\end{align*}
\]

The various products among \( r_2 \) & \( e_2 \) can be easily found using, say, mathematica:

\[
e_2 e_2 = \begin{pmatrix} 3 & 3 & 0 & -3 & -3 & 0 \\ 3 & 3 & 0 & -3 & -3 & 0 \\ 0 & -3 & 3 & 0 & 3 & -3 \\ -3 & 0 & -3 & 3 & 0 & 3 \\ 0 & -3 & 3 & 0 & 3 & -3 \\ -3 & 0 & -3 & 3 & 0 & 3 \end{pmatrix} = 3 e_2
\]
\[ e_2^2 = 0 \]
\[ r_2 e_2 = \begin{pmatrix} 0 & -3 & 3 & 0 & 3 & -3 \\ -3 & 0 & -3 & 3 & 0 & 3 \\ 3 & 3 & 0 & -3 & 3 & 0 \\ 0 & -3 & 3 & 0 & 3 & -3 \\ -3 & 0 & -3 & 3 & 0 & 3 \\ 3 & 3 & 0 & -3 & 3 & 0 \end{pmatrix} = 3 r_2 \]

\[ r_2 e_2 = 0 \]

The partner of \( e_2^{(23)} \) is

\[ r_2^{(23)} = (23) e_2^{(23)} = (23) + (123) - (321) - (31) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \]

Note that although the basis \( \{ e_2, r_2, e_2^{(23)}, r_2^{(23)} \} \) are orthogonal to \( e_1 \) & \( e_3 \), its members are not mutually orthogonal. This is expected since the 2 irreducible representations are equivalent. An orthogonal basis is easily obtained by means of the Schmidt orthogonalization method.

In terms of operators, we have

\[ e_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \]
\[ e_2 = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ -1 & 0 & -1 & 1 & 0 \end{pmatrix} \]
\[ e_3 = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 \end{pmatrix} \]
\[ e_2^{(23)} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix} \]

**Projection Operators**

Let \( L_1 \) be an invariant subspace or left ideal of \( L \).

Let \( \{ u_i \} \) be a basis of \( L_1 \).

We can then always choose a basis of \( L \) by adding to \( \{ u_i \} \) enough independent vectors to span \( L \). This basis is of the form \( \{ u_i, v_j \} \) where all \( v \)'s are orthogonal to all \( u \)'s.

An arbitrary vector in \( L \) is then of the form

\[ \begin{pmatrix} u \\ v \end{pmatrix} \]

where \( u \) is in \( L_1 \).
Every operator with respect to which \( L_1 \) is invariant must then be of the form

\[
R = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}
\]

where \( A, B, C \) are matrices of appropriate dimensions

so that

\[
\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} A u \\ 0 \end{pmatrix} \in L_1
\]

Note that \( R \) is of this form only with respect to the basis \( \{ u_1 \}, \{ v_j \} \). It is easy to see that an arbitrary similarity transform will in general destroy it.

On the other hand, any properties of \( R \) derived under this special basis will be valid in a general basis if the said properties are invariant under a similarity transform.

Consider now the form of a projection operator \( P \) in this basis. By definition

\[
P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix} \quad \forall \ u, v
\]

we see that we must be of the form

\[
P = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}
\]

Obviously, \( \det P = 0 \), so that \( P^{-1} \) doesn't exist.

If we demand

\[ P^2 = P \]

we have

\[
P^2 = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}
\]

or

\[ A^2 = A \quad \text{and} \quad AB = B \]

The 2nd relation is automatically satisfied if \( B = AF \) for arbitrary \( F \).

Thus, the general solution is

\[
P = \begin{pmatrix} P^+ & P^+ Q \\ 0 & 0 \end{pmatrix}
\]

where \( P^+ \) is an idempotent operator ( \( P^+ \cdot 2 = P^+ \)) and \( Q \) is arbitrary.

A special solution is \( P^+ = I \) so that

\[
P = \begin{pmatrix} I & Q \\ 0 & 0 \end{pmatrix}
\]

The effects of \( P \) on an arbitrary operator \( R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) are:

\[
PR = \begin{pmatrix} P^+ & P^+ Q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P^+ A + P^+ Q C & P^+ B + P^+ Q D \\ 0 & 0 \end{pmatrix}
\]

\[
RP = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} P^+ & P^+ Q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A P^+ & A P^+ Q \\ C P^+ & D P^+ Q \end{pmatrix}
\]

\[
P R P = \begin{pmatrix} P^+ & P^+ Q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A P^+ & A P^+ Q \\ C P^+ & D P^+ Q \end{pmatrix}
\]

\[
= \begin{pmatrix} P^+ A P^+ + P^+ Q C P^+ & P^+ A P^+ Q + P^+ Q D P^+ Q \\ 0 & 0 \end{pmatrix}
\]
If $L_1$ is invariant under $R$, $C = 0$, so that

$$PR = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} P'A & P'B + P'QD \\ 0 & 0 \end{pmatrix}$$

$$RP = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} AP' & AP'Q \\ 0 & DP'Q \end{pmatrix}$$

$$PRP = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} AP' & AP'Q \\ 0 & DP'Q \end{pmatrix} = \begin{pmatrix} P'AP' & PAP'Q + P'QD\ P'Q \\ 0 & 0 \end{pmatrix}$$

Thus, in general,

$$PR \neq RP$$

If we impose the condition

$$PR = RP \quad \forall R$$

$P$ must be of the form

$$P = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \alpha = \text{constant}$$

The condition

$$P^2 = P$$

then restricts $\alpha$ to the value 1.

We emphasize that $P$ is in this simple form only with respect to the basis $\{ u_i \}, \{ v_j \}$.

Under a similarity transform wrt

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \quad S^{-1} = \begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{pmatrix}$$

we have

$$SPS^{-1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{pmatrix}$$

$$= \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} S_{11} (S^{-1})_{11} & S_{11} (S^{-1})_{12} \\ S_{21} (S^{-1})_{11} & S_{21} (S^{-1})_{12} \end{pmatrix}$$

If $PR = RP \quad \forall R$

we have

$$P^2 R = PR = PRP = RP \quad (P^2 = P)$$
Let
\[ AB | x \rangle = C | y \rangle = | z \rangle \]
then
\[ PAB | x \rangle = PC | y \rangle = P| z \rangle \]
\[ = P \, A \, P \, P \, B \, P \, P \, | x \rangle \quad \text{if} \quad PR = RP = PRP \]
\[ = A_P \, B_P \, | x_P \rangle \]
\[ = PC \, P \, | y \rangle = C_P \, | y_P \rangle \]
\[ = | z_P \rangle \]
where the subscript denotes the projected component.

Thus, the projector preserves the operator multiplication if it commutes with every operator.

Consider
\[ AA^{-1} = E \]
we have
\[ PAA^{-1} = PE \]
\[ = P \, A \, P \, A^{-1} \, P = P \]
\[ = A_P \, A^{-1}_P = E_P \]

Thus, \( P \) is the projected component of the identity. It also plays the role of the identity operator for projected operators \( A_P \).

### Idempotents

The basis for \( D^R \) is just the group elements \( \{ g_i \} \).

Let the basis of the completely decomposed representation \( D \) be \( \{ u_{\mu a} \} \), where \( \mu \) runs through all the irreducible representations & \( a = 1, \ldots, d_a \) with \( d_a \) being the dimension of the representation \( a \). By definition,
\[ D(r)^{a \alpha}_{\beta \beta} = \delta^{a \alpha}_{\beta \beta} \quad \forall r \in \hat{G} \]

These two basis must be related by an invertible transformation \( S \):
\[ u_{a a} = g_i S^i_{a a} \]
\[ g_i = u_{a a} S^a_{i j} \]

with
\[ S^i_{a a} S^a_{j b} = \delta^i_j \quad \quad S^i_{a a} S^b_{i \alpha} = \delta^{\alpha}_{a} \delta^{b}_{a} \]

Group multiplication among \( \{ u_{\mu a} \} \) is found with the help of the regular representation
\[ D^R(g_i)_{\beta \beta} = \Delta_{i k} = \begin{cases} 1 & \text{for } g_j = g_i \, g_k \\ 0 & \text{otherwise} \end{cases} \]
as
\[ u_{\beta b} u_{a a} = g_j S^j_{\beta b} \, g_i S^i_{a a} \]
\[ = g_k \, \Delta^k_{ij} \, S^j_{\beta b} \, S^i_{a a} \]
\[ = g_k \, D^R(g_i)^{j}_{\beta b} S^i_{a a} \]
\[ = g_k \, D^R(g_j)^{i}_{\beta b} S^i_{a a} \]
\[ = g_k D^R(u_{bb})^{ij}_{\beta} S^i_{a a} \]

By definition
\[ D(r)^{a \alpha}_{\beta \beta} = S^{a \alpha}_{j} \, D^R(r)^{j}_{\beta b} \, S^i_{a a} = \delta^{a \alpha}_{\beta \beta} \quad \forall r \in \hat{G} \]

so that
\[ D^R(u_{bb})^{ij}_{\beta} S^i_{a a} = S^k_{a \gamma} D^R(u_{bb})^{\gamma \alpha}_{a a} \]
\[ = S^k_{a \gamma} D^R(u_{bb})^{\gamma \alpha}_{a a} \]
Hence

\[ u_{\beta \gamma} u_{\alpha \alpha} = g_{\alpha} S^k_{\alpha \alpha} \mathcal{D}^{(0)} \left( u_{\beta \gamma} \right)_k \]

\[ = u_{\alpha \alpha} S^k_{\alpha \alpha} \mathcal{D}^{(0)} \left( u_{\beta \gamma} \right)_k \]

which is simply the definition of irreducible representation \( \mathcal{D}^\alpha \) of the operator \( u_{\beta \gamma} \) with respect to the irreducible basis set \( \{ u_{\alpha \alpha} : \alpha = 1, \ldots, d_\alpha \} \).

Since \( L^\alpha \) is an invariant subspace, it is meaningful to write

\[ g_{\alpha \alpha} = \sum_\alpha u_{\alpha \alpha} S^{(\alpha) \alpha}_{\alpha \alpha} \]

\[ = \sum_\alpha g_{\alpha \alpha} \]

where \( (\alpha) \) means that \( \alpha \) is exempted from the Einstein summation notation, &

\[ g_{\alpha \alpha} = u_{\alpha \alpha} S^{(\alpha) \alpha}_{\alpha \alpha} \]

In particular, the identity element \( E = g_1 \) is given by

\[ E = u_{\alpha \alpha} S^{\alpha \alpha} = \sum_\alpha u_{\alpha \alpha} S^{(\alpha) \alpha}_{\alpha \alpha} = \sum_\alpha g_{\alpha \alpha} \]

\[ g_{\alpha \alpha} = u_{\alpha \alpha} S^{(\alpha) \alpha}_{\alpha \alpha} \]

The idempotent \( e_\alpha \) is defined as a vector proportional to a basis vector, say, \( u_{\alpha 1} \), of \( L^\alpha \):

\[ e_\alpha = e_\alpha u_{\alpha 1} \]

where \( e_\alpha \) is a constant. Furthermore, we demand

\[ E = \sum_\alpha e_\alpha \]

This means that \( e_\alpha \) is equal to \( g_{\alpha 1} \) for a special choice of \( S \) such that

\[ S^{\alpha \alpha} = \delta^{\alpha 1} S^{\alpha 1} \quad \forall \alpha \]

Thus

\[ e_\alpha = u_{\alpha 1} S^{(\alpha) 1} \quad \text{or} \quad c_\alpha = S^{\alpha 1} \]

We also have

\[ S_{\alpha \alpha} S^{\alpha \alpha} = \delta_{\alpha 1} = S_{\alpha 1} S^{\alpha 1} \]

The representation of \( e_\alpha \) in \( L^\alpha \) is, by definition,

\[ e_\alpha \left| u_{\alpha \alpha} \right\rangle = \left| u_{\alpha \alpha} \right\rangle \mathcal{D}^{(0)} \left( e_\alpha \right)_a \]

\[ = u_{\alpha 1} S^{(\alpha) 1} \left| u_{\alpha \alpha} \right\rangle \]

\[ = \mathcal{D}^{(0)} \left( u_{\alpha 1} \right)_b S^{(\alpha) 1} \]

Hence,

\[ \mathcal{D}^{(0)} \left( e_\alpha \right)_b = \delta^{b \alpha} \]

Next, we demand \( e_\alpha \) to be the projection operator onto one of the basis, say, \( u_{\alpha 1} \), of \( L^\alpha \).

Thus, with respect to the basis \( \{ u_{\alpha \alpha} : \alpha = 1, \ldots, d_\alpha \} \),

\[ e_\alpha \left| u_{\alpha \alpha} \right\rangle = \left| u_{\alpha \alpha} \right\rangle \mathcal{D}^{(0)} \left( e_\alpha \right)_a \]

with

\[ \mathcal{D}^{(0)} \left( e_\alpha \right)_b = \delta^{\alpha 1} \mathcal{D}^{(0)} \left( e_1 \right)_b \]

or

\[ \mathcal{D}^{(0)} \left( e_\alpha \right) = \begin{pmatrix} \mathcal{D}^{(0)} \left( e_\alpha \right)_1 & \mathcal{D}^{(0)} \left( e_\alpha \right)_1 & \ldots & \mathcal{D}^{(0)} \left( e_\alpha \right)_1 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \]
To see if

Let us examine the definition of the projector \( P_a: \)

\[
P_a \{ r \} \equiv \{ r e_a \}
\]

Writing

\[
r = r^i g_i = r^a u_{a a}
\]

we have

\[
e_a = r^b u_{b b} u_{(a) a} S^{(a) a_1}
\]

To evaluate \( u_{b b} u_{a a} \), we need the regular representation

\[
D^R( g) f = \delta_{f k} = \begin{cases} 1 & \text{for } g_j = \delta_{j k} \\
0 & \text{otherwise}
\end{cases}
\]

so that

\[
u_{b b} u_{a a} = g_j S_{j b} g_i S_{i a a} = g_k \Delta_{ji} S_{j b} S_{i a a} = g_k D^R( g) S_{j b} S_{i a a} = g_k D^R( g) S_{j b} S_{i a a} = g_k D^R( u_{b b}) f S_{i a a}
\]

By definition

\[
D( r)^a b = S^{a j} D^R( r) f S_{j b} = \delta^{a b} D^R( r)^a b
\]

so that

\[
D^R( u_{b b}) f S_{i a a} = S^{k c} D( u_{b b} c) \gamma^c a_a = S^{k c} D^R( u_{b b} c) a_a
\]

Hence

\[
u_{b b} u_{a a} = g_k S_{k c} D^R( u_{b b} c) a_a = u_k D^R( u_{b b} c) a_a
\]

which is simply the definition of irreducible representation \( D^R \) of the operator \( u_{b b} \) with respect to the irreducible basis set \( \{ u_{a a}; a = 1, \ldots, d_a \} \).

Therefore

\[
e_a = r^b u_{b b} u_{(a) a} S^{(a) a_1}
\]

\[
= r^b = u_{a c} D^R( u_{b b} c) a S^{(a) a_1} = u_{a c} D^R( r^b u_{b b}) c a S^{(a) a_1} = u_{a c} D^R( r)^c a S^{(a) a_1}
\]

Thus \( r e_a \in L^a \) so that \( P_a \) is indeed a projector onto \( L^a \).

Obviously, this relation can also be written down immediately by treating \( r \) as an operator & \( e_a \) a vector in the invariant subspace \( L^a \).

Consider now the product \( e_a r e_a \).

\[
e_a r e_a = u_{(a) b} D^R( r)^c d S^{(a) a_1}
\]

\[
= u_{(a) b} S^{(a) b} u_{(a) c} D^R( r)^c d S^{(a) a_1} = u_{(a) f} D^R( u_{(a) b}) f S^{(a) b} D^R( r)^c d S^{(a) a_1}
\]

\[
= u_{(a) f} D^R( r)^c d S^{(a) a_1}
\]

\[
= u_{(a) f} D^R( e_a r)^c d S^{(a) a_1}
\]