▪ **Projection Operators**

Let L_1 be an invariant subspace or left ideal of L.
Let $\{u_i\}$ be a basis of L_1.
We can then always choose a basis of L by adding to $\{u_i\}$ enough independent vectors to span L. This basis is of the form $\{\{u_i\}, \{v_j\}\}$ where all v ' s are orthogonal to all u ' s.
An arbitrary vector in L is then of the form

$$\begin{pmatrix} \mathrm{u} \\ \mathrm{v} \end{pmatrix}$$

where u is in L_1.

Every operator with respect to which L_1 is invariant must then be of the form

$$\mathrm{R} = \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{C} \end{pmatrix} \qquad \text{where} \quad A,\ B,\ C,\ 0 \ \text{ are matrices of appropriate dimensions}$$

so that

$$\begin{pmatrix} \mathrm{A} & \mathrm{B} \\ 0 & \mathrm{C} \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{A\,u} \\ 0 \end{pmatrix} \in \mathrm{L}_1$$

Note that R is of this form only with respect to the basis $\{\{u_i\}, \{v_j\}\}$. It is easy to see that an arbitrary similarity transform will in general destroy it.
On the other hand, any properties of R derived under this special basis will be valid in a general basis if the said properties are invariant under a similarity transform.

Consider now the form of a projection operator P in this basis. By definition

$$P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix} \qquad \forall\ u, v$$

we see that we must be of the form

$$P = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

Obviously, det $P = 0$, so that P^{-1} doesn't exist.
If we demand

$$P^2 = P \qquad (\text{ idempotent })$$

we have

$$P^2 = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^2 & A\,B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

or

$$A^2 = A \;\& \qquad A\,B = B$$

The 2nd relation is automatically satisfied if $B = A\,F$ for arbitrary F.

Thus, the general solution is

$$P = \begin{pmatrix} P' & P'\,Q \\ 0 & 0 \end{pmatrix}$$

where P' is an idempotent operator ($P'^2 = P'$) and Q is arbitrary.

A special solution is $P' = I$ so that

$$P = \begin{pmatrix} I & Q \\ 0 & 0 \end{pmatrix}$$

The effects of P on an arbitrary operator $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are:

$$P\,R = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P'A + P'Q\,C & P'B + P'Q\,D \\ 0 & 0 \end{pmatrix}$$

$$R\,P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A\,P' & A\,P'Q \\ C\,P' & D\,P'Q \end{pmatrix}$$

$$P\,R\,P = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A\,P' & A\,P'Q \\ C\,P' & D\,P'Q \end{pmatrix}$$

$$= \begin{pmatrix} P'A\,P' + P'Q\,C\,P' & P\,A\,P'Q + P'Q\,D\,P'Q \\ 0 & 0 \end{pmatrix}$$

Thus, $P\,R$ & $P\,R\,P$ is again a projector (not necessarily idempotent) but $R\,P$ is not.

If L_1 is invariant under R, $C = 0$, so that

$$P\,R = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} P'A & P'B + P'Q\,D \\ 0 & 0 \end{pmatrix}$$

$$R\,P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A\,P' & A\,P'Q \\ 0 & D\,P'Q \end{pmatrix}$$

$$P\,R\,P = \begin{pmatrix} P' & P'Q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A\,P' & A\,P'Q \\ 0 & D\,P'Q \end{pmatrix}$$

$$= \begin{pmatrix} P'A\,P' & P\,A\,P'Q + P'Q\,D\,P'Q \\ 0 & 0 \end{pmatrix}$$

Thus, in general,

$$P\,R \neq R\,P$$

If we impose the condition

$$P\,R = R\,P \qquad \forall\; R$$

P must be of the form

$$P = \alpha \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad \alpha = \text{constant}$$

The condition

$$P^2 = P$$

then restricts α to the value 1.

We emphasize that P is in this simple form only with respect to the basis $\{\{u_i\}, \{v_j\}\}$.
Under a similarity transform wrt

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{pmatrix}$$

we have

$$S\,P\,S^{-1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{pmatrix}$$

$$= \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} S_{11}\,(S^{-1})_{11} & S_{11}\,(S^{-1})_{12} \\ S_{21}\,(S^{-1})_{11} & S_{21}\,(S^{-1})_{12} \end{pmatrix}$$

If $\quad P\,R = R\,P \qquad \forall\, R$
we have

$$P^2\,R = P\,R = P\,R\,P = R\,P \qquad (P^2 = P)$$

Let

$$A\,B\,|\,x\,\rangle = C\,|\,y\,\rangle = |\,z\,\rangle$$

then

$$P\,A\,B\,|\,x\,\rangle = P\,C\,|\,y\,\rangle = P\,|\,z\,\rangle$$

$$= \mathrm{P\,A\,P\;\;P\,B\,P\,P}\,|\,x\,\rangle \qquad \text{if} \qquad P\,R = R\,P = P\,R\,P$$

$$= \mathrm{A}_P\,B_P\,|\,x_P\,\rangle$$

$$= \mathrm{P\,C\,P\,P}\,|\,y\,\rangle = C_P\,|\,y_P\,\rangle$$

$$= |\,z_P\,\rangle$$

where the subscript denotes the projected component.
Thus, the projector preserves the operator multiplication if it commutes with every operator.

Consider

$$A\,A^{-1} = E$$

we have

$$P\,A\,A^{-1} = P\,E$$

$$= \mathrm{P\,A\,P\,P}\;\;A^{-1}\,P = P$$

$$= \mathrm{A}_P\,A^{-1}{}_P = E_P$$

Thus, P is the projected component of the identity. It also plays the role of the identity operator for projected operators A_P.