Tensors

Definition: GL(m, ℂ)

Let V_m be a complex linear space of dimension m.
The **General Linear Group** GL(m, ℂ) is the group of all **invertible linear transformations** on V_m.

Definition: Tensor Space V_m^n

The tensor space V_m^n is the direct product of n linear spaces V_m.

$$V_m^n \equiv V_m \times V_m \times \ldots \times V_m$$

Elements of V_m^n are called **tensors** of rank n.

Note

The above is actually the definition of **contravariant** tensors.
Covariant & **mixed** tensors are obtained by replacing all or some of the V_m 's by their **dual spaces** W_m.

In the theory of **tensor analysis**, tensors are defined in terms of the transformation properties of their components. Thus, coordinate systems play a central part of the theory at the very beginning.

The definition given here emphasizes the geometric nature of tensors which are independent of coordinate systems. It is the approach adopted in the theory of **differential geometry**.

Natural Basis

Given a basis $\{ | i \rangle \}$ for V_m, the **natural basis** for V_m^n is

$$| i_1 \ldots i_n \rangle = | i_1 \rangle \times \ldots \times | i_n \rangle$$

$$\longrightarrow \qquad | x \rangle = \sum_{i_1 \ldots i_n} x^{i_1 \cdots i_n} | i_1 \ldots i_n \rangle \equiv x^{i_1 \cdots i_n} | i_1 \ldots i_n \rangle \qquad \forall x \in V_m^n$$

Setting

$$I = \{ i_1 \ldots i_n \}$$

we can write:

$$| i_1 \ldots i_n \rangle = | I \rangle$$

$$| x \rangle = \sum_I x^I | I \rangle$$

The x^I 's are called the **tensor components** of x.

Operators

For any operator g on V_m defined by

$$g \mid i \rangle = \mid j \rangle \, g^j{}_i$$

the corresponding operator G on V_m^n as

$$\begin{aligned} G \mid I \rangle &\equiv g \mid i_1 \rangle \times \ldots \times g \mid i_n \rangle \\ &= \mid j_1 \rangle \times \ldots \times \mid j_n \rangle \, g^{j_1}{}_{i1} \cdots \; g^{j_n}{}_{i_n} \\ &= \sum_J \mid J \rangle \, G^J{}_I \end{aligned}$$

where

$$G^J{}_I = G^{j_1 \cdots j_n}{}_{i_1 \ldots i_n} = g^{j_1}{}_{i1} \cdots \; g^{j_n}{}_{i_n}.$$

For $g \in \mathrm{GL}(m, \mathbb{C})$, the matrix $\left(g^j{}_i\right)$ forms a $m - D$ representation.
The matrix $\left(G^J{}_I\right)$ then forms a $n\,m - D$ representation of $\mathrm{GL}(m, \mathbb{C})$.

$$G \mid x \rangle = \mid x_G \rangle = x^I \, G \left| I \right\rangle = x^I \left| J \right\rangle G^J{}_I = x_G{}^J \left| J \right\rangle$$

$\longrightarrow \quad x_G{}^J = G^J{}_I \; x^I$

Obviously, this representation is in general reducible.

Representation of S_n on V_m^n

$\forall \; p \in S_n$, define operator P on V_m^n as

$$P \mid x \rangle = \mid x_P \rangle$$

$\ni \quad x_P{}^I = x^{p\,I} \qquad\qquad p\,I = p\{\, i_1 \ldots i_n \} = \left\{ i_{p_1} \ldots i_{p_n} \right\}$

ie. $\quad x_P{}^{i_1 \cdots i_n} = x^{\, i_{p_1} \cdots i_{p_n}}$

Since

$$\mid x \rangle = x^I \left| I \right\rangle \qquad\qquad \mid x_P \rangle = x_P{}^I \left| I \right\rangle$$

we have

$$\begin{aligned} \mid x_P \rangle &= P \mid x \rangle = x^I \; P \left| I \right\rangle \\ &= x_P{}^J \left| J \right\rangle = x^{p\,J} \left| J \right\rangle = x^I \left| p^{-1} \, I \right\rangle \end{aligned}$$

$\longrightarrow \quad P \mid I \rangle = \left| p^{-1} \, I \right\rangle$

ie. $\quad P \mid i_1 \ldots i_n \rangle = \left| p^{-1} \{ i_1 \ldots i_n \} \right\rangle = \left| i_{p_1^{-1}} \ldots i_{p_n^{-1}} \right\rangle$

The matrix representation $\left(P^I{}_J\right)$ of p on V_m^n is defined by

$$P \mid I \rangle = \mid J \rangle \, P^J{}_I = \left| p^{-1} \, I \right\rangle$$

ie. $\quad \mid j_1 \ldots j_n \rangle \, P^{j_1 \cdots j_n}{}_{i_1 \ldots i_n} = \left| i_{p_1^{-1}} \ldots i_{p_n^{-1}} \right\rangle$

$\longrightarrow \quad P^J{}_I = P^{j_1 \cdots j_n}{}_{i_1 \ldots i_n} = \delta^j{}_{i_{p_1^{-1}} 1} \ldots \delta^j{}_{i_{p_n^{-1}} n} = \delta^j{}_{i_1 p_1} \cdots \; \delta^j{}_{i_n p_n}$

Obviously, this representation is in general reducible.

Definition: Symmetry Preserving Transformations

Let $\left(D^I{}_J\right)$ be the matrix representation of a linear transformation D on V_m^n.
D is **symmetry preserving** $\iff \quad D^{p\,I}{}_{p\,J} = D^I{}_J \;\; \forall \; p \in S_n$

eg. Elements of both $\mathrm{GL}(m, \mathbb{C})$ & S_n are symmetry preserving.

Theorem: $\boldsymbol{G\,P = P\,G}$

Let

$g \in \mathrm{GL}(m.\mathbb{C}),\ \ p \in S_n$

$\Longrightarrow \quad G\,P = P\,G \qquad \text{on } \ V_m^n$

- **Definition:** **Tensors of Symmetry** Θ_λ^p

Tensors of symmetry Θ_λ^p are elements of the set $\left\{ e_\lambda^p \left| \alpha \right\rangle,\ \ |\ \alpha\ \right\rangle \in V_m^n \left. \right\}$
where e_λ^p is the Young symmetrizer of the Young tableau Θ_λ^p.

- **Definition:** **Tensors of Symmetry Class** λ

Tensors of symmetry class λ are elements of the set $\{ r\, e_\lambda \,|\, \alpha \rangle ;\ \ r \in \mathcal{S}_n\,,\ \ |\ \alpha\ \rangle \in V_m^n \}$
where $\mathcal{S}_n$ is the group algebra of S_n
e_λ is the Young symmetrizer of the normal Young tableau Θ_λ.

The presence of r in the definition means that the symmetry class is characterized by the Young diagrams instead of individual tableaux.

- **Definition:** $\boldsymbol{T_\lambda(\alpha)}$

For a given $|\ \alpha\ \rangle \in V_m^n$, we define

$T_\lambda(\alpha) \equiv \{ r\, e_\lambda \,|\, \alpha \rangle ;\ \ r \in \mathcal{S}_n \}$

where $\mathcal{S}_n$ is the group algebra of S_n
e_λ is the Young symmetrizer of the normal Young tableau Θ_λ.

- **Theorem:** $\boldsymbol{T_\lambda(\alpha)}$ **is invariant under** $\boldsymbol{S_n}$
- **Theorem:** **Rep of** $\boldsymbol{S_n}$ **on** $\boldsymbol{T_\lambda(\alpha)}$ **= IR generated by** $\boldsymbol{e_\lambda}$ **on** $\boldsymbol{\mathcal{S}_n}$
- **Theorem:** $\boldsymbol{T_\lambda(\alpha) = T_\lambda(\beta)}$ **or** $\boldsymbol{T_\lambda(\alpha) \bigcap T_\lambda(\beta) = \Phi}$
- **Theorem:** $\boldsymbol{T_\lambda(\alpha) \bigcap T_\mu(\beta) = \Phi}$ **if** $\boldsymbol{\lambda \neq \mu}$