2. Applications Of Vector Algebra To Analytic Geometry

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2.2. Lines In n-Space

$\mathbb{R}^n$ is an analytic model of $n$-D Euclidean space $E^n$. Thus, the geometric concepts and properties of $E^n$ are expressed in terms of $n$-tuples of real numbers in $\mathbb{R}^n$.

Definition (Point)

A point is a vector ($n$-tuple) in $\mathbb{R}^n$.

Definition (Line)

Let $P$ be a point and $A$ a non-zero vector. A line through $P$ and parallel to $A$ is the set of points

$$L(P; A) = \{P + tA : t \in \mathbb{R}\} = \{P + tA\}$$

The vector $A$ is called the direction vector of the line.

A point $Q$ is on the line $L(P; A)$ if $Q = P + tA$ for some $t$.

Properties

The line $L(O; A)$ is the linear span $L(A)$ of $A$.

Thus, we can write $L(P; A) = \{P + X : X \in L(A)\}$. [See Fig.2.1]
2.3. Some Simple Properties Of Straight Lines In $\mathbb{R}^n$

Definition (Parallelism of Vectors)

Two vectors $A$ and $B$ are parallel iff $A = \alpha B$ for some $\alpha \in \mathbb{R}$.

Theorem 2.1

Two lines $L(P; A)$ and $L(P; B)$ are equal iff $A$ and $B$ are parallel.

Proof of $\Rightarrow$

Let $L(P; A) = L(P; B)$. A point $X$ on the line can therefore be written as

$$X = P + \alpha A = P + \beta B$$

for some $\alpha, \beta \in \mathbb{R}$

Hence, $A = \gamma B$ where $\gamma = \frac{\beta}{\alpha} \in \mathbb{R}$. That is, $A$ and $B$ are parallel.

Proof of $\Leftarrow$

Let $A$ and $B$ be parallel so that $A = \alpha B$ for some $\alpha \in \mathbb{R}$.

If $Q$ is on $L(P; A)$, then $Q = P + tA$ for some $t \in \mathbb{R}$. Thus,

$$Q = P + tA = P + t\alpha B = P + \beta B$$

where $\beta = t\alpha \in \mathbb{R}$

Hence, $Q$ is also on $L(P; B)$. Since this is true for arbitrary $Q$, we have

$L(P; A) \subseteq L(P; B)$. Reversing the role of $A$ and $B$ gives $L(P; B) \subseteq L(P; A)$.

Hence, $L(P; A) = L(P; B)$.

Definition (Parallelism of Lines)

Two lines $L(P; A)$ and $L(P; B)$ are parallel if $A$ and $B$ are parallel.

Theorem 2.3 (Euclid's Parallel Postulate)

Given a line $L$ and a point $Q$ not on $L$, there is one and only one line $L'$ through $Q$ and parallel to $L$. 
Proof

Let $A$ be the direction vector of $L$. We can write $L' = L(Q, A)$. According to theorem 2.1, $L'$ is unique. Q.E.D.

Theorem 2.4 (2 distinct points determines a lines)

If $P \neq Q$, there is 1 and only 1 line containing both $P$ and $Q$.

The line can be written as $\{P + t(Q - P)\}$.

Proof

The line $L(P; Q - P) = \{P + t(Q - P)\}$ goes through both $P$ and $Q$ since they correspond to $t = 0$ and $t = 1$, respectively.

Let $L'$ be another line that goes through both $P$ and $Q$. Since $L'$ goes through $P$, we have $L' = L(P; A)$ for some nonzero $A$. Now, $L'$ also goes through $Q$ so that $P + \alpha A = Q$ for some $\alpha$. Hence, $\alpha A = Q - P$ so that $A$ and $Q - P$ are parallel. By theorem 2.2, $L' = L(P; Q - P)$. Q.E.D.

Example

From theorem 2.4, we see that $Q$ is on $L(P, A)$ iff $A$ and $Q - P$ are parallel, i.e., $Q - P = \alpha A$.

Consider the 3-space case with $P = (1, 2, 3)$ and $A = (2, -1, 5)$.

To test if $Q = (1, 1, 4)$ is on $L(P, A)$, we examine $Q - P = (0, -1, 1)$ and find that it is not a multiple of $A$. Hence, $Q$ is not on the line.

For $Q = (5, 0, 13)$, we have $Q - P = (4, -2, 10) = 2A$ so that $Q$ is on $L$.

Theorem 2.5 (Linear Dependence)

Two vectors $A, B \in \mathbb{R}^n$ are linearly dependent iff they lie on the same line through the origin.
Proof

If either vector is zero, the theorem holds trivially.
If both are nonzero, they are dependent iff \( B = \alpha A \) for some \( \alpha \in \mathbb{R} \).

However, \( B = \alpha A \) iff \( B \) is on the line \( L(0, A) \). Q.E.D.
2.4. Lines And Vector-Valued Functions In $n$-Space

The line $L(P; A)$ can be considered as the track of a moving particle with position $X$ at time $t$ given by

$$X(t) = P + tA$$  \hspace{1cm} (2.1)

Here $X(t)$ is an example of a vector-valued function of a real variable, i.e.,

$$X : R \to L(P, A) \subset R^n \text{ and } t \mapsto X(t)$$

The scalar $t$ is often called a parameter and (2.1) a vector parametric equation, or simply, a vector equation of the line $L(P; A)$.

Consider a line passing through points $P$ and $Q$. By theorem 2.4, it can be written as $L = L(P; Q - P)$ so that (2.1) becomes

$$X(t) = P + t(Q - P) = (1 - t)P + tQ$$

Note that $X(a) = X(b)$ if

$$\Leftrightarrow (1 - a)P + aQ = (1 - b)P + bQ$$

i.e.,

$$(a - b)(Q - P) = 0$$

which means $a = b$ since $Q - P = A \neq 0$. Hence, 2 points on the line are distinct iff their parameters are distinct.

Consider 3 points $X(a)$, $X(b)$ and $X(c)$. We say $X(c)$ is between $X(a)$ and $X(b)$ if $a < c < b$ or $b < c < a$.

A pair of points $P$ and $Q$ are congruent to another pair $P'$ and $Q'$ if

$$\|P - Q\| = \|P' - Q\|.$$  The norm $\|P - Q\|$ is called the distance between $P$ and $Q$. 
2.5. Lines In 3-Space And In 2-Space

Consider eq(2.1) for the case of 3-space.

Writing \( P = (p, q, r) \), \( A = (a, b, c) \) and \( X(t) = (x, y, z) \), we have

\[
x = p + ta \\
y = q + tb \\
z = r + tc
\]

which are known as **scalar parametric equations**, or simply, **parametric equations**, for the line.

For 2-space, only the 1st two equations in (2.2) are required. The parameter \( t \) can then be eliminated to give

\[
x - p = \frac{y - q}{b} a \Rightarrow b(x - p) - a(y - q) = 0
\]

which is called a **Cartesian equation** for the line. Provided \( a \neq 0 \), we can put it in the **point-slope form**,

\[
y - q = \frac{b}{a} (x - p)
\]

where the point \( (p, q) \) and slope \( \frac{b}{a} \) are easily discerned.

Eq(2.3) can also be written in terms of dot products, namely,

\[
N \cdot (X - P) = 0
\]

where \( N = (b, -a) \) is called a **normal vector** to the line since it is perpendicular to the direction vector \( A \), i.e., \( N \cdot A = ba - ab = 0 \). [cf. Fig.2.2]

**Theorem 2.6**

Let \( L \) be a line in \( \mathbb{R}^2 \) given by \( N \cdot (X - P) = 0 \) and let \( d = \frac{|P \cdot N|}{\|N\|} \).

Then \( \|X\| \leq d \quad \forall X \in L \).

Furthermore, \( \|X\| = d \) iff \( X \) is the projection of \( P \) along \( N \), i.e.,

\[
X = \frac{P \cdot N}{N \cdot N} N
\]

**Proof**

Since \( N \cdot X = N \cdot P \quad \forall X \in L \), we have, by the Cauchy-Schwarz inequality,
\[ |N \cdot P| = |N \cdot X| \leq \sqrt{|X \cdot X||N \cdot N|} = \|X\||N\| \]

\[ \Rightarrow \quad \|X\| \geq \frac{|P \cdot N|}{\|N\|} = d \]

Now, the equality holds iff \( X = \alpha N \) for some \( \alpha \in \mathbb{R} \). This means
\[ \alpha N \cdot N = P \cdot N \quad \Rightarrow \quad \alpha = \frac{P \cdot N}{N \cdot N} \quad \text{Q.E.D.} \]

**Corollary**

Let \( Q \) be a point not on \( L \), then
\[ \|X - Q\| \geq d = \frac{|(P - Q) \cdot N|}{\|N\|} \quad \forall X \in L \]

where the equality applies iff \( X - Q \) is the projection of \( P - Q \) and \( d \) is called the **distance** of \( Q \) to \( L \).
2.7. Planes In Euclidean $n$-Space

Since 2 lines define a plane, the linear span $L(A,B)$ is a plane defined by the lines $L(0,A)$ and $L(0,B)$. To make the plane pass through a point $P$, one needs only to add the vector $P$ to every vector in $L(A,B)$.

Definition: Plane in $\mathbb{R}^n$

A set $M$ of points is called a plane if there exists a point $P$ and 2 L.I. vectors $A$ and $B$ such that

$$M = \{ P + sA + tB \mid s, t \in \mathbb{R} \}$$

More specifically, $M$ is called a plane through $P$ spanned by $A$ and $B$.

Theorem 2.7

Two planes $M = \{ P + sA + tB \}$ and $M' = \{ P + sC + tD \}$ through the same point $P$ are equal iff $L(A,B) = L(C,D)$.

Proof

But the definition of a plane, it is obvious that $L(A,B) = L(C,D) \Rightarrow M = M'$. Conversely, if $M = M'$, then for any given $s$ and $t$, there exist $s'$ and $t'$ such that

$$P + sA + tB = P + s'C + t'D \Rightarrow sA + tB = s'C + t'D$$

i.e., $L(A,B) = L(C,D)$. Q.E.D.

Theorem 2.8

Two planes $M = \{ P + sA + tB \}$ and $M' = \{ Q + sA + tB \}$ spanned by the same vectors $A$ and $B$ are equal iff $Q$ is on $M$.

Proof
Definition: Parallelism

Two planes \( M = \{ P + sA + tB \} \) and \( M' = \{ P + sC + tD \} \) are said to be parallel if
\[
L(A, B) = L(C, D)
\]
Also, a vector \( X \) is parallel to the plane \( M \) if \( X \in L(A, B) \).

Theorem 2.9

Given a point \( Q \) not on plane \( M \), there is one and only one plane \( M' \) that contains \( Q \) and parallel to \( M \).

Proof

Theorem 2.10

Let \( P, Q, R \) be 3 points not on the same line. There is one and only one plane \( M \) containing these 3 points. Furthermore,
\[
M = \{ P + s( Q - P ) + t( R - P ) \} \quad (2.4)
\]

Proof

Theorem 2.11

3 vectors \( A, B, C \) are L.I. iff they lie on the same plane through \( O \).

Proof
2.8. Planes And Vector-Valued Functions

A plane $M = \{P + sA + tB\}$ may be considered as a mapping

$$X : \mathbb{R}^2 \rightarrow M$$

$$(s, t) \mapsto X(s, t) = P + sA + tB \quad (2.6)$$

where $X$ is called a **vector-valued function** of 2 real variables. The scalars $s$ and $t$ are called **parameters** and eq(2.6) is called a **parametric**, or **vector**, **equation** of the plane. For vectors in $\mathbb{R}^n$, eq(2.6) represents $n$ scalar equations. Taking as example the case $\mathbb{R}^3$, eq(2.6) represents $3$ scalar parametric equations. We have

$$P = (p_1, p_2, p_3) \quad A = (a_1, a_2, a_3) \quad B = (b_1, b_2, b_3) \quad X = (x, y, z)$$

$$x = p_1 + sa_1 + tb_1 \quad y = p_2 + sa_2 + tb_2 \quad z = p_3 + sa_3 + tb_3$$

Eliminating $s$ and $t$ gives a linear equation of the form

$$ax + by + cz = d$$

known as the **Cartesian equation of the plane**.

**Example**

Let $M = \{P + sA + tB\}$ with $P = (1, 2, 3)$, $A = (1, 2, 1)$ and $B = (1, -4, -1)$.

The vector equation is

$$X(s, t) = (1, 2, 3) + s(1, 2, 1) + t(1, -4, -1)$$

which is equivalent to 3 scalar parametric equations

$$x = 1 + s + t \quad y = 2 + 2s - 4t \quad z = 3 + s - t$$

To obtain the Cartesian equation, we solve $s, t$ from the 1st and 3rd eqs

$$s + t = x - 1 \quad s - t = z - 3$$

to get

$$s = \frac{1}{2}(x + z - 4) \quad t = \frac{1}{2}(x - z + 2)$$

Substituting them into the 2nd eq gives

$$y = 2 + (x + z - 4) - 2(x - z + 2)$$

or

$$y = -6 - x + 3z$$
2.10. The Cross Product Of Two Vectors In $\mathbb{R}^3$

Definition: Cross Product

The cross product of 2 vectors $A$ and $B$ in $\mathbb{R}^3$ is defined as another vector $C = A \times B$ with components

$$(A \times B)_i = c_i = \varepsilon_{ijk} a_j b_k$$

where the Einstein notation of implied summation over repeated indices is used and $\varepsilon_{ijk}$ is the totally anti-symmetric Levi-Civita symbol defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{even permutation of } 123 \\ -1 & \text{odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases}$$

i.e., $\varepsilon_{123} = \varepsilon_{231} = \cdots = 1$, $\varepsilon_{213} = \varepsilon_{321} = \cdots = -1$ and $\varepsilon_{113} = \varepsilon_{221} = \cdots = 0$.

Carrying out the summations, we have

$$A \times B = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

Theorem 2.12: Properties

$\forall A, B, C \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$,

(a) $A \times B = -(B \times A)$

(b) $A \times (B + C) = A \times B + A \times C$

(c) $\alpha (A \times B) = (\alpha A) \times B = A \times (\alpha B)$

(d) $A \cdot (A \times B) = 0$

(e) $B \cdot (A \times B) = 0$

(f) $\|A \times B\|^2 = \|A\|^2 \|B\|^2 - (A \cdot B)^2$

(g) $A \times B = 0 \iff A$ and $B$ are L.I.

Proof

Properties (a-c) follow immediately from definition.

For (d), we have
\[ A \cdot (A \times B) = \varepsilon_{ijk} a_i a_j b_k = -\varepsilon_{ijk} a_i a_j b_k \quad \text{[ } \varepsilon_{ijk} = -\varepsilon_{jik} \text{ ]} \]

\[ = -\varepsilon_{jik} a_i a_j b_k \quad \text{[ } i \leftrightarrow j \text{ ]} \]

\[ = 0 \quad \text{[ } x = -x \rightarrow x = 0 \text{ ]} \]

For (f),
\[ \|A \times B\|^2 = (A \times B) \cdot (A \times B) = \varepsilon_{ijk} a_j b_k \varepsilon_{ilm} a_l b_m \]

Using the identity
\[ \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \]

we have
\[ \|A \times B\|^2 = (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) a_j b_k a_l b_m = a_j b_k a_l b_m - a_l b_k a_j b_m \]

\[ = (A \cdot A)(B \cdot B) - (A \cdot B)^2 = \|A\|^2 \|B\|^2 - (A \cdot B)^2 \]

Using \( A \cdot B = \|A\| \|B\| \cos \theta \) where \( \theta \) is the angle between \( A \) and \( B \), we have
\[ \|A \times B\| = \|A\| \|B\| \sin \theta = \text{Area of parallelogram with sides } A \text{ and } B. \]

[see Fig.2.5]

For (g): using (f), we have
\[ A \times B = O \quad \Leftrightarrow \quad \|A\|^2 \|B\|^2 = (A \cdot B)^2 \]

By the Cauchy-Schwarz inequality, this means \( A \) and \( B \) are linearly dependent.

**Examples**

From property (a), we have
\[ A \times A = -A \times A = 0 \]

Also, writing the \( j \)th component of the \( i \)th coordinate unit vector as
\[ (e_i)_j = \delta_{ij} \]

we have
\[ (e_i \times e_j)_k = \varepsilon_{kln} (e_i)_l (e_j)_m = \varepsilon_{kln} \delta_{jl} \delta_{jm} = \varepsilon_{kij} = \varepsilon_{ijk} \]

\[ \Rightarrow \quad e_i \times e_j = \varepsilon_{ijk} e_k \quad (1) \]

Note that geometrically, (1) can be implemented in 2 ways; thus giving rise to the
right and left handed coordinate systems (see Fig.2.4)

**Theorem 2.13**

Let \( A \) and \( B \) be L.I. in \( \mathbb{R}^3 \), then

(a) \( A, B, \) and \( A \times B \) are L.I.

(b) Every vector in \( \mathbb{R}^3 \) orthogonal to both \( A \) and \( B \) are scalar multiples of \( A \times B \).

**Proof**

(a)
Let \( C = A \times B \). Since \( A \) and \( B \) are L.I., \( C \neq 0 \). Consider the equation

\[
\alpha A + \beta B + \gamma C = 0
\]

Taking \( C \cdot \) on both sides reduces it to \( \gamma C \cdot C = 0 \) so that \( \gamma = 0 \). Since \( A \) and \( B \) are L.I., we must have \( \alpha = \beta = 0 \). QED

(b)
Let \( N \) be orthogonal to both \( A \) and \( B \). By part (a), \( A, B, \) and \( C = A \times B \) are L.I. so that they form a basis in \( \mathbb{R}^3 \). Hence, we can write

\[
N = \alpha A + \beta B + \gamma C \quad (2)
\]

Taking \( N \cdot \) on (2) gives \( N \cdot N = \gamma N \cdot C \).

Taking \( C \cdot \) on (2) gives \( C \cdot N = \gamma C \cdot C \).

Hence, \( (N \cdot C)^2 = (N \cdot N)(C \cdot C) \) so that by the Cauchy-Schwarz inequality, the theorem is proved.
2.11. The Cross Product Expressed As A Determinant

An $n \times n$ determinant is defined as

$$\det[a_{ij}] = e_{i_1 \ldots i_n} a_{i_1} \cdots a_{i_n}$$

The cross product $C = A \times B$ can be written as

$$C = e_i (A \times B)_i = e_{ijk} a_j b_k = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
2.13. The Scalar Triple Product

The scalar triple product is given by

$$A \cdot (B \times C) = \varepsilon_{ijk} a_i b_j c_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

= volume of parallelepiped with sides $A$, $B$ and $C$.

Theorem 2.14

3 vectors $A$, $B$ and $C$ in $\mathbb{R}^3$ are linearly dependent $\iff A \cdot (B \times C) = 0$.

Proof

$\Rightarrow$

If $B$ and $C$ are linearly dependent, then $B \times C = 0$ and theorem is proved.

If $B$ and $C$ are L.I., then there exists $\alpha, \beta, \gamma$, not all zero, such that

$$\alpha A + \beta B + \gamma C = 0$$

Furthermore $\alpha \neq 0$ since otherwise $\alpha = \beta = \gamma = 0$. Taking $(B \times C)$· on both sides gives $\alpha A \cdot (B \times C) = 0$. QED.

$\Leftarrow$

Given $A \cdot (B \times C) = 0$, we have either $B \times C = 0$ or $A$ is orthogonal to $B \times C \neq 0$.

For the former case, $B$ and $C$ are dependent and the theorem is proved. For the latter case, theorem 2.13 says that $B$, $C$ and $B \times C$ form a basis so that

$$A = \alpha B + \beta C + \gamma B \times C$$

Taking $(B \times C)$· on both sides gives $0 = \gamma \|B \times C\|^2$ so that $\gamma = 0$, i.e.,

$$A = \alpha B + \beta C$$

QED.

A set of simultaneous equations
\[ a_1 x + b_1 y + c_1 z = d_1 \]
\[ a_2 x + b_2 y + c_2 z = d_2 \] \hspace{1cm} (2.11)
\[ a_3 x + b_3 y + c_3 z = d_3 \]
can be written as a single vector equation
\[ A x + B y + C z = D \] \hspace{1cm} (2.12)
in obvious notations.

Taking \((B \times C) \cdot\) on both sides gives
\[ A \cdot (B \times C) x = D \cdot (B \times C) \]

Provided \(A \cdot (B \times C) \neq 0\), we have
\[
x = \frac{D \cdot (B \times C)}{A \cdot (B \times C)} = \begin{vmatrix} d_1 & b_1 & c_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\]
and analogously for \(y\) and \(z\). This is known as the Cramer’s rule.

If \(A \cdot (B \times C) = 0\), then \(A, B, C\) lies in the same plane. There will be no solution unless \(D\) also lies in the same plane.
2.16. Normal Vectors To Planes In $\mathbb{R}^3$

Definition: Normal Vector

A non-zero vector $N \in \mathbb{R}^3$ is a **normal vector** to the plane $M = \{P + sA + tB\}$ if $N$ is perpendicular to both $A$ and $B$.

**Theorem 2.15**

Consider a plane $M = \{P + sA + tB\}$ in $\mathbb{R}^3$. Let $N = A \times B$. Then

(a) $N$ is a normal vector to $M$.

(b) $M = \{X \mid (X - P) \cdot N = 0\}$

**Proof**

(a) Follows from properties (d) and (e) of Theorem 2.12.

(b) Proof is analogous to that for the line discussed in section 2.5.

**Theorem 2.16**

Given a plane $M$ through $P$ with a normal vector $N$ and

$$d = \frac{|P \cdot N|}{\|N\|}$$

(2.17)

Then, $\|X\| \geq d \ \forall X \in M$, where the equal sign applies iff

$$X = tN \quad \text{with} \quad t = \frac{P \cdot N}{N \cdot N}$$

**Proof**

Proof is analogous to that for the line discussed in Theorem 2.6.

**Corollary**

The shortest distance from a point $Q$ to a plane $M$ is

$$d = \frac{|(Q - P) \cdot N|}{\|N\|}$$

Actually, $d$ is called the distance from $Q$ to $M$. 
2.17. Linear Cartesian Equations For Planes In $R^3$

Eq(2.16) can be expressed in terms of components. Setting

$$N = (a, b, c) \quad P = (x_1, y_1, z_1) \quad X = (x, y, z)$$

we have

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad (2.18)$$

which is known as the **Cartesian equation for the plane**. An alternative form is

$$ax + by + cz = d \quad (2.19)$$

where

$$d = ax_1 + by_1 + cz_1 = N \cdot P$$

Eq(2.19) is called a **linear equation** in $x$, $y$ and $z$. In general, a linear equation in $R^3$ represents a plane and vice versa.

Two planes with parallel normal vectors are said to be **parallel** to each other. Thus, the planes

$$ax + by + cz = d_1 \quad ax + by + cz = d_2$$

are parallel. Furthermore, the (shortest) distance between them is

$$\frac{|d_1 - d_2|}{\|N\|} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Two planes are **orthogonal** if their normal vectors are orthogonal. More generally, the angle between 2 planes is defined as the angle between their normal vectors.
2.19. The Conic Sections

A right circular cone can be generated by rotating a generator line $G$ about an axis $A$ with which it intersects at an angle $0 < \theta < \frac{\pi}{4}$ at the vertex $P$. Each of the upper and lower portions is called a nappe of the cone. The curves obtained by the intersection of a plane with the surface of the cone are called conic sections or simply conics. Depending on the angle of the plane, one can obtain 3 types of curves [see Fig.2.9]. With the help of the icecream-cone proof [see text], they can be identified with the parabola, ellipse and hyperbola defined in Fig.2.10.
2.20. Eccentricity Of Conic Sections

Description of the conics in vector algebra terms made use a quantity called $e$.

Definition: Conics

Given a directrix line $L$ and a focus point $F$ not on $L$, a conic section is a set of points $X$ satisfying

$$
\|X - F\| = ed(X, L) \quad (2.20)
$$

where $e$ is a positive real number called the eccentricity and $d(X, L)$ is the distance from $X$ to $L$. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$.

Alternative Form

Now, given a point $P$ on $L$, we can write

$$
d(X, L) = \frac{(X - P) \cdot N}{\|N\|} = (X - P) \cdot \hat{N}
$$

where $\hat{N}$ is the unit normal vector of $L$. Hence, (2.20) can be written as

$$
\|X - F\| = e|(X - P) \cdot \hat{N}| \quad (2.21)
$$

This can be simplified by setting $P = F + d\hat{N}$, where $d$ is the distance between $F$ and $L$ [see Fig.2.12]. Thus,

$$
\|X - F\| = e|(X - F) \cdot \hat{N} - d| \quad (2.22)
$$
2.21. Polar Equations For Conic Sections

Setting the focus at the origin, i.e., \( F = 0 \), eq(2.22) simplifies to

\[
\|X\| = e|X \cdot \hat{N} - d| \quad \text{(2.23)}
\]

which, in polar coordinates, becomes

\[
r = e|r \cos \theta - d| \quad \text{(2.24)}
\]

where \( r = \|X\| \) and \( X \cdot \hat{N} = r \cos \theta \) [see Fig.2.13].

If \( X \) is on the same side of the directrix as the focus, \( r \cos \theta - d < 0 \) so that (2.24) becomes

\[
r = e(d - r \cos \theta) \quad \Rightarrow \quad r = \frac{ed}{e \cos \theta + 1} \quad \text{(2.25)}
\]

If \( X \) is on the other side of the directrix from the focus, \( r \cos \theta - d > 0 \) and (2.24) becomes

\[
r = e(r \cos \theta - d) \quad \Rightarrow \quad r = \frac{ed}{e \cos \theta - 1} \quad \text{(2.26)}
\]

Since, here, \( \theta \) is restricted to \( 0 < \theta < \frac{\pi}{2} \) while \( r, e, d > 0 \), eq(2.26) cannot be satisfied for any \( \theta \) if \( e \leq 1 \). In other word, only hyperbola can have a branch on the other side of the directrix from the focus.

Consider now the branch indicated by (2.25).

If \( e < 1 \), there is no restriction on \( \theta \) so that \( 0 < \theta < 2\pi \) as befits an ellipse.

For \( e > 1 \), \( \theta \) is restricted to \( \theta \leq \cos^{-1}\left( -\frac{1}{e} \right) \).
2.23. Cartesian Equation For A General Conic

Eq(2.22) can be written as

$$\|X - F\| = e |X \cdot \widehat{N} - F \cdot \widehat{N} - d| = |eX \cdot \widehat{N} - a| \quad (2.29)$$

where $a = e (F \cdot N + d)$. Taking the square of (2.29) gives

$$\|X\|^2 - 2X \cdot F + \|F\|^2 = e^2 (X \cdot \widehat{N})^2 - 2eaX \cdot \widehat{N} + a^2 \quad (2.30)$$

Setting

$$X = (x, y) \quad F = (f_1, f_2) \quad \widehat{N} (n_1, n_2)$$

where $n_1^2 + n_2^2 = 1$, eq(2.30) becomes

$$x^2 + y^2 - 2(xf_1 + yf_2) + f_1^2 + f_2^2 = e^2 (xn_1 + yn_2)^2 - 2ea(xn_1 + yn_2) + a^2$$

$$\Rightarrow \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + G = 0 \quad (2.32)$$

with

$$A = 1 - e^2n_1^2 \quad B = -2e^n_1n_2 \quad C = 1 - e^2n_2^2 \quad (2.31)$$

$$D = 2(aen_1 - f_1) \quad E = 2(aen_2 - f_2) \quad G = f_1^2 + f_2^2 - a^2$$

Note that the discriminant

$$\Delta = B^2 - 4AC = \left(2e^n_1n_2\right)^2 - 4\left(1 - e^2n_1^2\right)\left(1 - e^2n_2^2\right)$$

$$= 4e^4n_1^2n_2^2 - 4\left(1 - e^2 + e^4n_1^2n_2^2\right) = 4\left(e^2 - 1\right) \quad (2.33')$$

so that eq(2.32) represents an ellipse, parabola, or hyperbola if $\Delta$ is negative, zero or positive, respectively. [Caution: our $\Delta$ is the negative of that given in the text].
2.24. Conic Sections Symmetric About The Origin

Consider the cartesian equation

\[ \|X\|^2 - 2X \cdot F + \|F\|^2 = e^2 \left( X \cdot \hat{N} \right)^2 - 2eaX \cdot \hat{N} + a^2 \]  

(2.30)

where

\[ a = e \left( d + F \cdot \hat{N} \right) \]  

(2.30a)

If (2.30) is to be symmetric about the origin, i.e., invariant under \( X \rightarrow -X \), all terms in odd powers of \( X \) must vanish. Hence

\[ F = ae\hat{N} \]  

(2.35)

Combining this with (2.30a) gives

\[ F \cdot \hat{N} = ae = \left( d + F \cdot \hat{N} \right) e^2 = \frac{e^2d}{1-e^2} \quad \text{if} \quad e \neq 1 \]

\[ \Rightarrow \quad a = \frac{ed}{1-e^2} \]  

(2.36)

Putting these back to (2.30) gives

\[ \|X\|^2 + a^2e^2 = e^2 \left( X \cdot \hat{N} \right)^2 + a^2 \]

\[ \Rightarrow \quad \|X\|^2 - e^2 \left( X \cdot \hat{N} \right)^2 = a^2 \left( 1 - e^2 \right) \quad \text{where} \quad e \neq 1 \]  

(2.37)

which is clearly symmetric about the origin. Since (2.37) is also invariant under \( \hat{N} \rightarrow -\hat{N} \), there are 2 pairs of symmetrically placed directrices and foci. According to (2.35), the foci are at points \( \pm ae\hat{N} \). From exercise 7 in sec 2.28, the directrices intersect the line joining the foci at points \( \pm \frac{a}{e} \hat{N} \).

To find the intersects of the curve (2.37) on the line joining the foci, we set \( X = \beta\hat{N} \) so that (2.37) becomes

\[ \beta^2 - e^2 \beta^2 = a^2 \left( 1 - e^2 \right) \]

\[ \Rightarrow \quad \beta = \pm a \]

These points at \( X = \pm a\hat{N} \) are called the vertices of the central conics given by (2.37). The segment that joins them are called the major axis if the conic is an ellipse, the transverse axis if the conic is a hyperbola.

To find the intersects of the curve (2.37) on the line that passes through the origin and perpendicular to the major/transverse axis, we set \( X = \gamma\hat{M} \) where \( \hat{M} \cdot \hat{N} = 0 \).
Eq(2.37) then becomes

$$\gamma^2 = a^2(1 - e^2) \quad \Rightarrow \quad \gamma = \pm a\sqrt{1 - e^2}$$

Since $X$ and hence $\gamma$ must be real, there is no solution for $e > 1$. For ellipses with $e < 1$, the segment that joins these points are called the **minor axis**.
2.25. Cartesian Equations For The Ellipse And The Hyperbola in Standard Position

Consider again the central conics

\[ \|X\|^2 - e^2 (X \cdot \tilde{N})^2 = a^2 (1 - e^2) \] (2.37)

Setting \( X = (x, y) \) and \( \tilde{N} = (1, 0) \), we have

\[ x^2 + y^2 - e^2 x^2 = a^2 (1 - e^2) \]

\[ \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1 \] (2.38)

which is called the standard form of the central conics. The foci are at \( \pm ae\tilde{N} = (\pm ae, 0) \). The directrices are vertical lines intersecting the \( x \)-axis at points \( \pm \frac{a}{e} \tilde{N} = \left( \pm \frac{a}{e}, 0 \right) \).

For an ellipse with \( e < 1 \), we set \( b = a\sqrt{1 - e^2} \) so that (2.38) becomes

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] (2.39)

and

\[ c = ae = a\sqrt{1 - \frac{b^2}{a^2}} = \sqrt{a^2 - b^2} \]

For a hyperbola with \( e > 1 \), we set \( b = |a|\sqrt{e^2 - 1} \) so that (2.38) becomes

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \] (2.40)

and

\[ c = |a|e = |a|\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{a^2 + b^2} \]

Now, as \( |x| \to \infty \), (2.40) becomes

\[ \frac{y^2}{b^2} \approx \frac{x^2}{a^2} \quad \text{or} \quad y \approx \pm \frac{b}{|a|} x \]
These give 2 lines with tangents $\pm \frac{b}{|a|}$ that pass through the origin. They are called \textbf{asymptotes} of the hyperbola. See Fig. 2.14.
2.26. Cartesian Equations For The Parabola

Consider the conics equation for a parabola

\[ \|X - F\| = \|(X - P) \cdot \hat{N}\| \]  \hspace{1cm} (2.21)

Taking \( X = (x, y), \ \hat{N} = (1, 0), \ F = (c, 0) \) and the directrix a vertical line at \( x = \alpha \), (2.21) becomes

\[ \sqrt{(x - c)^2 + y^2} = |x - \alpha| \]

\[ \Rightarrow \quad (x - c)^2 + y^2 = (x - \alpha)^2 \]

or

\[ y^2 = 2(c - \alpha) x + \alpha^2 - c^2 \]  \hspace{1cm} (2.43a)

The standard form of a parabola is obtained by setting \( \alpha = -c \) so that (2.43a) becomes

\[ y^2 = 4cx \]  \hspace{1cm} (2.43)

where the vertex is at the origin. A parabola with vertex at \( (x_0, y_0) \) is simply

\[ (y - y_0)^2 = 4c(x - x_0) \]

Note that a parabola does not have any asymptotes.
2.28. Miscellaneous Exercises On Conic Sections