1.1. Classical Mechanics

A classical mechanical system is described by the generalized coordinates

$$q(t) = \{ q_1(t), \ \dots q_N(t) \} \ = \{ q_i(t) \}$$

The action of the system is defined as

$$A[q] = \int_{t_a}^{t_b} dt \, L(q, \dot{q}, t) \tag{1.2}$$

where the Lagrangian L is at most quadratic in the generalized velocities

$$\dot{q} = \{ \dot{q}_1(t), \ \dots \dot{q}_N(t) \} = \left\{ \frac{d q_i}{d t} \right\}$$

According to the principle of least action, the classical path $q^{\rm cl}(t)$ extremizes A with respect to all neighboring paths

$$q(t) = q^{\rm cl}(t) + \delta q(t) \tag{1.3}$$

having the same end points $q(t_b)$ and $q(t_a)$.

The variation of A is defined as

$$\delta A[q] \equiv \left\{ A[q + \delta q] - A[q] \right\}_{\rm lin} \tag{1.4}$$

$$= \text{terms linear in } \delta q \text{ in the Taylor expansion of } A.$$

The classical path is given by

$$\delta A[q] \Big|_{q(t) = q^c(t)} = 0 \tag{1.5}$$

for all $\delta q(t) = q(t) - q^c(t)$ that satisfy

$$\delta q(t_a) = \delta q(t_b) = 0 \tag{1.6}$$

Putting (1.2) into (1.4) gives

$$\delta A[q] = \int_{t_a}^{t_b} dt \left\{ L(q + \delta q, \dot{q} + \delta \dot{q}, t) - L(q, \dot{q}, t) \right\}_{\rm lin}$$

$$= \int_{t_a}^{t_b} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

where sum over repeated indices is understood (Einstein's summation convention).

Using

$$\int_{t_a}^{t_b} dt \, \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \int_{t_a}^{t_b} dt \, \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i$$

$$= \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} dt \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \qquad \text{[Integration by part.]}$$

we have

$$\delta A[q] = \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \tag{1.7}$$

$$= \int_{t_a}^{t_b} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \qquad \text{[(1.6) used.]} \tag{1.7a}$$

(1.5) implies q^c is the solution to the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{1.8}$$

The Legendre transform of the Lagrangian is called the Hamiltonian

$$H \equiv p_i \dot{q}_i - L \tag{1.9}$$

where the generalized momentum p_i conjugate to q_i is defined as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{1.10}$$

The generalized velocities can be eliminated from (1.9) only if (1.10) can be inverted to give solutions

$$\dot{q}_i = v_i(q, p, t) \qquad \forall\ i = 1, \ \ldots, N \tag{1.11}$$

This in turn requires the Hessian matrix

$$H_{ij}(q, \dot{q}, t) \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \tag{1.12}$$

to be invertible (non-singular). In which case, we can write $H = H(q, p, t)$ so that (1.2) takes the canonical form

$$A[q, p] = \int_{t_a}^{t_b} d\,t \left[p_i\, v_i - H(q, p, t) \right] \tag{1.14}$$

The state of the system is now specified by the path $\{ q^{\text{cl}}(t), p^{\text{cl}}(t) \}$ in phase space that extremizes A. Analogeous to (1.3), we set

$$q(t) = q^{\text{cl}}(t) + \delta q(t) \qquad\qquad p(t) = p^{\text{cl}}(t) + \delta p(t) \tag{1.15}$$

with

$$\delta q(t_a) = \delta q(t_b) = 0$$

Variation on (1.14) gives

$$\delta A[q, p] = \int_{t_a}^{t_b} d\,t \left(\dot{q}_i\, \delta p_i + p_i\, \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right)$$

Integrating by part the $\delta \dot{q}_i$ terms gives

$$\delta A[q, p] = p_i\, \delta q_i \Big|_{t_a}^{t_b} + \int_{t_a}^{t_b} d\,t \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] \tag{1.16}$$

The classical path is therefore the solution of the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{1.17}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

As the system moves along a classical path, the action changes as a function of the end points according to (1.16) by

$$\delta A[q, p] = p_i\, \delta q_i \Big|_{t_a}^{t_b} \tag{1.18}$$

Anarbitrary function $O(q, p, t)$ changes along an arbitrary path at the rate

$$\frac{d}{d\,t} O(q, p, t) = \frac{\partial O}{\partial q_i} \dot{q}_i + \frac{\partial O}{\partial p_i} \dot{p}_i + \frac{\partial O}{\partial t} \tag{1.19}$$

Along the classical path given by (1.17),

$$\frac{d}{d\,t} O(q, p, t) = \frac{\partial O}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial O}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial O}{\partial t}$$

$$= \{ O, H \} + \frac{\partial O}{\partial t} \tag{1.20}$$

where the Poisson bracket is defined as

$$\{A, B\} \equiv \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i} \qquad (1.21)$$

Caution: Kleinert's used a Poisson bracket that is the negative of ours.

The crucial properties of the Poisson bracket are

Antisymmetry: $\{A, B\} = -\{B, A\}$ (1.22)

Jacobi identity: $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$ (1.23)

A and B are said to commute if $\{A, B\} = 0$.

A function $O(q, p)$ that has no explicit time dependency evolves according to

$$\frac{dO}{dt} = \{O, H\}$$

If it also commutes with H, then $\frac{dO}{dt} = 0$ so that it is a constant of motion. Since every quantity commutes with itself, a time-independent Hamiltonian

$$H = H(q, p) \qquad (1.26)$$

is a constant of motion.

The Hamilton equations (1.17) are special cases of (1.20) since

$$\dot{p}_i = \{p_i, H\} = \frac{\partial p_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j}\frac{\partial p_i}{\partial p_j} = -\frac{\partial H}{\partial q_j}\delta_{ij} = -\frac{\partial H}{\partial q_i} \qquad (1.24)$$

$$\dot{q}_i = \{q_i, H\} = \frac{\partial q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j}\frac{\partial q_i}{\partial p_j} = \delta_{ij}\frac{\partial H}{\partial p_j} = \frac{\partial H}{\partial p_i}$$

Brackets between the phase space variables are

$$\{q_i, p_j\} = \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_j}{\partial q_k}\frac{\partial q_i}{\partial p_k} = \delta_{ik}\,\delta_{jk} = \delta_{ij} \qquad (1.25)$$

$$\{q_i, q_j\} = \frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_j}{\partial q_k}\frac{\partial q_i}{\partial p_k} = 0$$

$$\{p_i, p_j\} = \frac{\partial p_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial p_j}{\partial q_k}\frac{\partial p_i}{\partial p_k} = 0$$

Consider a local, or point, transformation

$$q_i = f_i(Q, t) \qquad \forall\ i = 1, \ldots, N \qquad (1.27)$$

Here, local means all quantities involved are at the same time.

Since q & Q describe the same system, they are equivalent so that (1.27) must be invertible

$$Q_j = f_j^{-1}(q, t) \qquad \forall\ j = 1, \ldots, N \qquad (1.28)$$

which requires

$$\det\left(\frac{\partial f_i}{\partial Q_j}\right) \neq 0 \qquad (1.29)$$

Let

$$L(q, \dot{q}, t) = L\left[f(Q, t),\ \dot{f}(Q, t),\ t\right] \equiv L'\left(Q, \dot{Q}, t\right) \qquad (1.30)$$

then (1.2) becomes

$$A[Q] = \int_{t_a}^{t_b} dt\, L\left[f(Q, t),\ \dot{f}(Q, t),\ t\right] \qquad (1.31)$$

$$= \int_{t_a}^{t_b} dt\, L'\left(Q, \dot{Q}, t\right) \qquad (1.31a)$$

As (1.2) leads to (1.8), the variation on (1.31a) with

$$\delta Q(t_a) = \delta Q(t_b) = 0$$

gives

$$\frac{\partial L'}{\partial Q_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{Q}_j} = 0 \qquad (1.32)$$

Similarly, the variation on (1.31) gives the counterpart of (1.7)

$$\delta A[q] = \frac{\partial L}{\partial \dot{q}_i}\delta f_i \Bigg|_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta f_i \qquad (1.33)$$

Since allowing δq_i to be arbitrary implies δf_i are also arbitrary, while

$$\delta q_i(t_a) = \delta q_i(t_b) = 0 \qquad \text{implies} \qquad \delta f_i(t_a) = \delta f_i(t_b) = 0$$

setting (1.33) to zero reproduces the Euler-Lagrange equations (1.8). This means(1.32) also extremizes A and hence equivalent to (1.8). In other words, the Lagrangian formulism is independent of the coordinates.

Note that the local transformation (1.27) also specifies a transformation on the velocities

$$\dot{q}_i = \dot{f}_i(Q, t) = \frac{\partial f_i}{\partial Q_j}\dot{Q}_j + \frac{\partial f_i}{\partial t} \qquad (1.34)$$

A local transformation in the phase space takes the form

$$q_i = q_i(Q, P, t) \qquad \forall\ i = 1, \ldots, N \qquad (1.35)$$
$$p_i = p_i(Q, P, t)$$

together with the inverse

$$Q_j = Q_j(q, p, t) \qquad \forall\ j = 1, \ldots, N \qquad (1.36)$$
$$P_j = P_j(q, p, t)$$

Note that (1.36) inverts both equations in (1.35) so that Q_j cannot be written as q_j^{-1}, nor P_j as p_j^{-1}.

Under the transformation (1.35-6), we get

$$L(q, \dot{q}, t) = L'(Q, \dot{Q}, t) \qquad H(q, p, t) = \overline{H}(Q, P, t) \qquad (1.36a)$$

On the other hand, the Legendre transform of L' gives another Hamiltonian

$$H'(Q, P', t) = P_j'\dot{Q}_j - L' \qquad \text{where} \qquad P_j' = \frac{\partial L'}{\partial \dot{Q}_j} \qquad (1.36b)$$

As in (1.9) and (1.17), the Hamiltonian eqs

$$\dot{Q}_j = \frac{\partial H'}{\partial P_j'} \qquad \dot{P}_j' = -\frac{\partial H'}{\partial Q_j} \qquad (1.36c)$$

are equivalent to the Euler-Lagrange eqs obtained from L or L'.

Using $\overline{H}$, we can get another set of Hamiltonian eqs

$$\dot{Q}_j = \frac{\partial \overline{H}}{\partial P_j} \qquad \dot{P}_j = -\frac{\partial \overline{H}}{\partial Q_j} \qquad (1.38a)$$

If (1.36c) and (1.38a) are compatible, the transformation (1.35-6) is called canonical. Obviously, this is the case if $P = P'$.

A more general criterion is obtained using the fact that two Lagrangians, L' and $\tilde{L}$, are equivalent if they differ only by a total time derivative, i.e.,

$$\tilde{L} = L' + \frac{dF}{dt} \qquad (1.36d)$$

where F is an arbitrary function. This is so since

$$\tilde{A} = \int_{t_a}^{t_b} d\,t\,\tilde{L} = \int_{t_a}^{t_b} d\,t\,L' + F\Big|_{t_a}^{t_b} = A' + F\Big|_{t_a}^{t_b}$$

$$= A + F\Big|_{t_a}^{t_b} = \int_{t_a}^{t_b} d\,t\,L + F\Big|_{t_a}^{t_b} \tag{1.36e}$$

which means $\delta\tilde{A} = \delta A' = \delta A$ for variations with fixed end points.

Hence, if one can find an F such that $P_j = \dfrac{\partial \tilde{L}}{\partial \dot{Q}_j}$, one can write (1.36a) as

$$\tilde{H} = P_j\,\dot{Q}_j - \tilde{L} = P_j\,\dot{Q}_j - L' - \frac{d\,F}{d\,t} \tag{1.36f}$$

and the transformation is canonical with Hamilton eqs given by

$$\dot{Q}_j = \frac{\partial \tilde{H}}{\partial P_j} \qquad \dot{P}_j = -\frac{\partial \tilde{H}}{\partial Q_j} \tag{1.38}$$

Writing (1.36e) in terms of the Hamiltonians, we have

$$\int_{t_a}^{t_b} d\,t\left(P_j\,\dot{Q}_j - \tilde{H}\right) = \int_{t_a}^{t_b} d\,t\,(\,p_i\,\dot{q}_i - H\,) + F\Big|_{t_a}^{t_b} \tag{1.40a}$$

$$= \int_{t_a}^{t_b} d\,t\left[p_i\left(\frac{\partial q_i}{\partial P_j}\dot{P}_j + \frac{\partial q_i}{\partial Q_j}\dot{Q}_j + \frac{\partial q_i}{\partial t}\right) - H\right] + F\Big|_{t_a}^{t_b}$$

$$= \int_{t_a}^{t_b}\left[p_i\left(\frac{\partial q_i}{\partial P_j}d\,P_j + \frac{\partial q_i}{\partial Q_j}d\,Q_j + \frac{\partial q_i}{\partial t}d\,t\right) - H\,d\,t\right] + F\Big|_{t_a}^{t_b}$$

$$\rightarrow \qquad \int_{t_a}^{t_b}\left[\left(P_j - p_i\frac{\partial q_i}{\partial Q_j}\right)d\,Q_j - p_i\frac{\partial q_i}{\partial P_j}d\,P_j + \left(H - p_i\frac{\partial q_i}{\partial t} - \tilde{H}\right)d\,t\right] = F\Big|_{t_a}^{t_b} \tag{1.40}$$

which means

$$d\,F = \left(P_j - p_i\frac{\partial q_i}{\partial Q_j}\right)d\,Q_j - p_i\frac{\partial q_i}{\partial P_j}d\,P_j + \left(H - p_i\frac{\partial q_i}{\partial t} - \tilde{H}\right)d\,t$$

so that

$$\frac{\partial F}{\partial Q_j} = P_j - p_i\frac{\partial q_i}{\partial Q_j} \qquad \frac{\partial F}{\partial P_j} = -\,p_i\frac{\partial q_i}{\partial P_j} \qquad \frac{\partial F}{\partial t} = H - p_i\frac{\partial q_i}{\partial t} - \tilde{H}$$

From (1.36f) and (1.38), we expect the 2nd partial derivatives of F to be at least continuous and hence independent of the order of differentiation. Hence,

$$\frac{\partial^2 F}{\partial P_k\,\partial Q_j} = \delta_{j\,k} - \frac{\partial p_i}{\partial P_k}\frac{\partial q_i}{\partial Q_j} - p_i\frac{\partial^2 q_i}{\partial P_k\,\partial Q_j}$$

$$= \frac{\partial^2 F}{\partial Q_j\,\partial P_k} = -\frac{\partial p_i}{\partial Q_j}\frac{\partial q_i}{\partial P_k} - p_i\frac{\partial^2 q_i}{\partial Q_j\,\partial P_k}$$

$$\rightarrow \qquad \frac{\partial q_i}{\partial Q_j}\frac{\partial p_i}{\partial P_k} - \frac{\partial q_i}{\partial P_k}\frac{\partial p_i}{\partial Q_j} = \delta_{j\,k} \tag{1.41}$$

where we've assume the transformation (1.35) to have continuous 2nd partial derivatives.

$$\frac{\partial^2 F}{\partial Q_k\,\partial Q_j} = -\frac{\partial p_i}{\partial Q_k}\frac{\partial q_i}{\partial Q_j} - p_i\frac{\partial^2 q_i}{\partial Q_k\,\partial Q_j}$$

$$= \frac{\partial^2 F}{\partial Q_j\,\partial Q_k} = -\frac{\partial p_i}{\partial Q_j}\frac{\partial q_i}{\partial Q_k} - p_i\frac{\partial^2 q_i}{\partial Q_j\,\partial Q_k}$$

$$\rightarrow \qquad \frac{\partial q_i}{\partial Q_j}\frac{\partial p_i}{\partial Q_k} - \frac{\partial q_i}{\partial Q_k}\frac{\partial p_i}{\partial Q_j} = 0 \tag{1.41a}$$

$$\frac{\partial^2 F}{\partial P_k \partial P_j} = -\frac{\partial p_i}{\partial P_k}\frac{\partial q_i}{\partial P_j} - p_i \frac{\partial^2 q_i}{\partial P_k \partial P_j}$$
$$= \frac{\partial^2 F}{\partial P_j \partial P_k} = -\frac{\partial p_i}{\partial P_j}\frac{\partial q_i}{\partial P_k} - p_i \frac{\partial^2 q_i}{\partial P_j \partial P_k}$$
$$\to \quad \frac{\partial q_i}{\partial P_j}\frac{\partial p_i}{\partial P_k} - \frac{\partial q_i}{\partial P_k}\frac{\partial p_i}{\partial P_j} = 0 \qquad (1.41b)$$

$$\frac{\partial^2 F}{\partial t \partial Q_j} = \frac{\partial P_j}{\partial t} - \frac{\partial p_i}{\partial t}\frac{\partial q_i}{\partial Q_j} - p_i \frac{\partial^2 q_i}{\partial t \partial Q_j}$$
$$= \frac{\partial^2 F}{\partial Q_j \partial t} = \frac{\partial (H - \tilde{H})}{\partial Q_j} - \frac{\partial p_i}{\partial Q_j}\frac{\partial q_i}{\partial t} - p_i \frac{\partial^2 q_i}{\partial Q_j \partial t}$$
$$\to \quad \frac{\partial q_i}{\partial t}\frac{\partial p_i}{\partial Q_j} - \frac{\partial q_i}{\partial Q_j}\frac{\partial p_i}{\partial t} + \frac{\partial P_j}{\partial t} = \frac{\partial (H - \tilde{H})}{\partial Q_j} \qquad (1.42)$$

$$\frac{\partial^2 F}{\partial t \partial P_j} = -\frac{\partial p_i}{\partial t}\frac{\partial q_i}{\partial P_j} - p_i \frac{\partial^2 q_i}{\partial t \partial P_j}$$
$$= \frac{\partial^2 F}{\partial P_j \partial t} = \frac{\partial (H - \tilde{H})}{\partial P_j} - \frac{\partial p_i}{\partial P_j}\frac{\partial q_i}{\partial t} - p_i \frac{\partial^2 q_i}{\partial P_j \partial t}$$
$$\to \quad \frac{\partial q_i}{\partial t}\frac{\partial p_i}{\partial P_j} - \frac{\partial q_i}{\partial P_j}\frac{\partial p_i}{\partial t} = \frac{\partial (H - \tilde{H})}{\partial P_j} \qquad (1.42a)$$

Defining the Lagrange bracket by

$$(f, g) = \frac{\partial q_i}{\partial f}\frac{\partial p_i}{\partial g} - \frac{\partial q_i}{\partial g}\frac{\partial p_i}{\partial f} \qquad (1.43a)$$

we can write (1.141-b) as

$$(Q_j, P_k) = \delta_{jk}$$
$$(Q_j, Q_k) = 0$$
$$(P_j, P_k) = 0 \qquad (1.43)$$

Cautions: Our brackets (both Poisson & Lagrange) are the negative of those used by Kleinert.

Time-dependent coordinate transformations satisfying these equations are called symplectic.

The Jacobian matrix of the transformation can be written in block-matrix form as

$$\boldsymbol{J} = \begin{pmatrix} \mathbf{Qq} & \mathbf{Qp} \\ \mathbf{Pq} & \mathbf{Pp} \end{pmatrix} \qquad (1.44)$$

where

$$Qq_{ij} = \frac{\partial Q_i}{\partial q_j} \qquad Qp_{ij} = \frac{\partial Q_i}{\partial p_j}$$
$$Pq_{ij} = \frac{\partial P_i}{\partial q_j} \qquad Pp_{ij} = \frac{\partial P_i}{\partial p_j} \qquad i, j = 1, \ldots, N \qquad (1.44a)$$

Cautions: Our $\boldsymbol{J}$ describes the transformation $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{Q}, \boldsymbol{P})$ while Kleinert's $\boldsymbol{J}$ is for $(\boldsymbol{p}, \boldsymbol{q}) \to (\boldsymbol{P}, \boldsymbol{Q})$. They differ by a permutation of rows & columns.

It can be proved by direct calculation that the inverse of $\boldsymbol{J}$ is

$$\boldsymbol{J}^{-1} = \begin{pmatrix} \mathbf{qQ} & \mathbf{qP} \\ \mathbf{pQ} & \mathbf{pP} \end{pmatrix} \qquad (1.45)$$

where

$$qQ_{ij} = \frac{\partial q_i}{\partial Q_j} \qquad qP_{ij} = \frac{\partial q_i}{\partial P_j}$$

$$pQ_{ij} = \frac{\partial p_i}{\partial Q_j} \qquad pP_{ij} = \frac{\partial p_i}{\partial P_j} \qquad i,\ j = 1,\ \ldots,\ N \tag{1.45a}$$

For example

$$\boldsymbol{J}\,\boldsymbol{J}^{-1} = \begin{pmatrix} \mathbf{Qq} & \mathbf{Qp} \\ \mathbf{Pq} & \mathbf{Pp} \end{pmatrix} \begin{pmatrix} \mathbf{qQ} & \mathbf{qP} \\ \mathbf{pQ} & \mathbf{pP} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{Qq}\cdot\mathbf{qQ} + \mathbf{Qp}\cdot\mathbf{pQ} & \mathbf{Qq}\cdot\mathbf{qP} + \mathbf{Qp}\cdot\mathbf{pP} \\ \mathbf{Pq}\cdot\mathbf{qQ} + \mathbf{Pp}\cdot\mathbf{pQ} & \mathbf{Pq}\cdot\mathbf{qP} + \mathbf{Pp}\cdot\mathbf{pP} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial Q_i}{\partial q_k}\frac{\partial q_k}{\partial Q_j} + \frac{\partial Q_i}{\partial p_k}\frac{\partial p_k}{\partial Q_j} & \frac{\partial Q_i}{\partial q_k}\frac{\partial q_k}{\partial P_j} + \frac{\partial Q_i}{\partial p_k}\frac{\partial p_k}{\partial P_j} \\ \frac{\partial P_i}{\partial q_k}\frac{\partial q_k}{\partial Q_j} + \frac{\partial P_i}{\partial p_k}\frac{\partial p_k}{\partial Q_j} & \frac{\partial P_i}{\partial q_k}\frac{\partial q_k}{\partial P_j} + \frac{\partial P_i}{\partial p_k}\frac{\partial p_k}{\partial P_j} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial Q_i}{\partial Q_j} & \frac{\partial Q_i}{\partial P_j} \\ \frac{\partial P_i}{\partial Q_j} & \frac{\partial P_i}{\partial P_j} \end{pmatrix} = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta_{ij} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} = \boldsymbol{I}$$

where $\boldsymbol{I}$ with elements $I_{ij} = \delta_{ij}$ is the unit matrix of the appropriate dimensions.

The symplectic unit matrix is defined as

$$\boldsymbol{E} = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix} \tag{1.46}$$

with inverse

$$\boldsymbol{E}^{-1} = \boldsymbol{E}^{T} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{I} \\ \boldsymbol{I} & \mathbf{0} \end{pmatrix} \tag{1.46a}$$

Thus,

$$\boldsymbol{E}^{-1}\,\boldsymbol{J}^{-1}\,\boldsymbol{E} = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{qQ} & \mathbf{qP} \\ \mathbf{pQ} & \mathbf{pP} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} -\mathbf{qP} & \mathbf{qQ} \\ -\mathbf{pP} & \mathbf{pQ} \end{pmatrix} = \begin{pmatrix} -\mathbf{pP} & \mathbf{pQ} \\ \mathbf{qP} & -\mathbf{qQ} \end{pmatrix}$$

$$\left(\boldsymbol{E}^{-1}\,\boldsymbol{J}^{-1}\,\boldsymbol{E}\right)^{T}\boldsymbol{J}^{-1} = \begin{pmatrix} -\mathbf{pP}^{T} & \mathbf{qP}^{T} \\ \mathbf{pQ}^{T} & -\mathbf{qQ}^{T} \end{pmatrix} \begin{pmatrix} \mathbf{qQ} & \mathbf{qP} \\ \mathbf{pQ} & \mathbf{pP} \end{pmatrix}$$

$$= \begin{pmatrix} -\mathbf{pP}^{T}\cdot\mathbf{qQ} + \mathbf{qP}^{T}\cdot\mathbf{pQ} & -\mathbf{pP}^{T}\cdot\mathbf{qP} + \mathbf{qP}^{T}\cdot\mathbf{pP} \\ \mathbf{pQ}^{T}\cdot\mathbf{qQ} - \mathbf{qQ}^{T}\cdot\mathbf{pQ} & \mathbf{pQ}^{T}\cdot\mathbf{qP} - \mathbf{qQ}^{T}\cdot\mathbf{pP} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\partial p_k}{\partial P_i}\frac{\partial q_k}{\partial Q_j} + \frac{\partial q_k}{\partial P_i}\frac{\partial p_k}{\partial Q_j} & -\frac{\partial p_k}{\partial P_i}\frac{\partial q_k}{\partial P_j} + \frac{\partial q_k}{\partial P_i}\frac{\partial p_k}{\partial P_j} \\ \frac{\partial p_k}{\partial Q_i}\frac{\partial q_k}{\partial Q_j} - \frac{\partial q_k}{\partial Q_i}\frac{\partial p_k}{\partial Q_j} & \frac{\partial p_k}{\partial Q_i}\frac{\partial q_k}{\partial P_j} - \frac{\partial q_k}{\partial Q_i}\frac{\partial p_k}{\partial P_j} \end{pmatrix}$$

$$= \begin{pmatrix} (P_i,\ Q_j) & (P_i,\ P_j) \\ -(Q_i,\ Q_j) & -(Q_i,\ P_j) \end{pmatrix} \equiv \mathcal{L} \tag{1.48}$$

Similarly,

$$\boldsymbol{E}^{-1}\,\boldsymbol{J}\,\boldsymbol{E} = \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Qq} & \mathbf{Qp} \\ \mathbf{Pq} & \mathbf{Pp} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} -\mathbf{Qp} & \mathbf{Qq} \\ -\mathbf{Pp} & \mathbf{Pq} \end{pmatrix} = \begin{pmatrix} -\mathbf{Pp} & \mathbf{Pq} \\ \mathbf{Qp} & -\mathbf{Qq} \end{pmatrix}$$

$$\boldsymbol{J}\left(\boldsymbol{E}^{-1}\,\boldsymbol{J}\,\boldsymbol{E}\right)^{T} = \begin{pmatrix} \mathbf{Qq} & \mathbf{Qp} \\ \mathbf{Pq} & \mathbf{Pp} \end{pmatrix} \begin{pmatrix} -\mathbf{Pp}^{T} & \mathbf{Qp}^{T} \\ \mathbf{Pq}^{T} & -\mathbf{Qq}^{T} \end{pmatrix}$$

$$= \begin{pmatrix} -\mathbf{Qq}\cdot\mathbf{Pp}^T + \mathbf{Qp}\cdot\mathbf{Pq}^T & \mathbf{Qq}\cdot\mathbf{Qp}^T - \mathbf{Qp}\cdot\mathbf{Qq}^T \\ -\mathbf{Pq}\cdot\mathbf{Pp}^T + \mathbf{Pp}\cdot\mathbf{Pq}^T & \mathbf{Pq}\cdot\mathbf{Qp}^T - \mathbf{Pp}\cdot\mathbf{Qq}^T \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\partial Q_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} + \frac{\partial Q_i}{\partial p_k}\frac{\partial P_j}{\partial q_k} & \frac{\partial Q_i}{\partial q_k}\frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}\frac{\partial Q_j}{\partial q_k} \\ -\frac{\partial P_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} + \frac{\partial P_i}{\partial p_k}\frac{\partial P_j}{\partial q_k} & \frac{\partial P_i}{\partial q_k}\frac{\partial Q_j}{\partial p_k} - \frac{\partial P_i}{\partial p_k}\frac{\partial Q_j}{\partial q_k} \end{pmatrix}$$

$$= \begin{pmatrix} -\{Q_i, P_j\} & \{Q_i, Q_j\} \\ -\{P_i, P_j\} & \{P_i, Q_j\} \end{pmatrix} \equiv \mathcal{P} \tag{1.49}$$

$$\rightarrow \quad \mathcal{L}\,\mathcal{P} = \left(\boldsymbol{E}^{-1}\boldsymbol{J}^{-1}\boldsymbol{E}\right)^T \boldsymbol{J}^{-1}\boldsymbol{J}\left(\boldsymbol{E}^{-1}\boldsymbol{J}\boldsymbol{E}\right)^T$$
$$= \left(\boldsymbol{E}^{-1}\boldsymbol{J}^{-1}\boldsymbol{E}\right)^T \left(\boldsymbol{E}^{-1}\boldsymbol{J}\boldsymbol{E}\right)^T$$
$$= \boldsymbol{E}^{-1}\boldsymbol{J}^{-1\,T}\boldsymbol{E}\boldsymbol{E}^{-1}\boldsymbol{J}^T\boldsymbol{E}$$
$$= \boldsymbol{E}^{-1}\boldsymbol{J}^{-1\,T}\boldsymbol{J}^T\boldsymbol{E} = \boldsymbol{E}^{-1}\boldsymbol{E} = \boldsymbol{I}$$

$$\therefore \quad \mathcal{L}^{-1} = \mathcal{P} \tag{1.49a}$$

Given (1.43), (1.48) becomes

$$\mathcal{L} = \begin{pmatrix} -\boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix} = -\boldsymbol{I} \tag{1.49b}$$

(1.49) & (1.49a) then imply

$$\{Q_i, P_j\} = \delta_{ij}$$
$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \tag{1.47}$$

The Lagrange & Poisson brackets, (1.43) & (1.47), are therefore equivalent. Either can serve as the criteria for a canonical transformation.

Since [see (1.9) & (1.36a)]

$$L' = L = p_i\,\dot{q}_i - H \qquad\qquad H = \overline{H}$$

(1.36f) implies

$$P_j\,\dot{Q}_j - p_i\,\dot{q}_i = \tilde{H} - H + \frac{dF}{dt} \tag{1.50}$$

(1.48) & (1.49b) give

$$\det\mathcal{L} = \det\left[\left(\boldsymbol{E}^{-1}\boldsymbol{J}^{-1}\boldsymbol{E}\right)^T \boldsymbol{J}^{-1}\right]$$
$$= \left(\det\boldsymbol{J}^{-1}\right)^2 = (\det\boldsymbol{J})^{-2}$$
$$= (-)^{2N} = 1$$
$$\rightarrow \quad \det\boldsymbol{J}^{-1} = \det\boldsymbol{J} = \pm 1$$

Hence,

$$\int d\,q_i\,d\,p_i = \int d\,Q_j\,d\,P_j \left|\det\boldsymbol{J}^{-1}\right|$$
$$= \int d\,Q_j\,d\,P_j \tag{1.51}$$

Note that the use of $\left|\det\boldsymbol{J}^{-1}\right|$ implies the volume elements in both sets of coordinates have been chosen to have the same orientation.

In terms of the new coordinates Q & P, we define a new Poisson bracket

$$\{A, B\}' \equiv \frac{\partial A}{\partial Q_j}\frac{\partial B}{\partial P_j} - \frac{\partial B}{\partial Q_j}\frac{\partial A}{\partial P_j} \tag{1.52}$$

so that the equation of motion for any dynamical variable $O(Q, P, t)$ is [c.f. (1.38)]

$$\frac{dQ}{dt} = \{O, H\}' + \frac{\partial Q}{\partial t} \tag{1.53}$$

Note that (1.52) implies

$$\{Q_i, P_j\}' = \delta_{ij}$$
$$\{Q_i, Q_j\}' = \{P_i, P_j\}' = 0 \tag{1.54}$$

The canonical transformation (1.40a) can be written as

$$\int_{t_a}^{t_b} dt\,(p_i\,\dot{q}_i - H) = \int_{t_a}^{t_b} dt\left(P_j\,\dot{Q}_j - \tilde{H} - \frac{dF}{dt}\right) \tag{1.56}$$

Setting

$$F = F(q, Q, t) \tag{1.55}$$

gives

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_j}\dot{Q}_j + \frac{\partial F}{\partial t}$$

(1.56) then gives

$$p_i = -\frac{\partial F}{\partial q_i} \qquad P_j = \frac{\partial F}{\partial Q_j} \qquad \tilde{H} - H = -\frac{\partial F}{\partial t} \tag{1.56a}$$

(1.56a) thus specifies a canonical transformation of the type

$$p_i = p_i(q, Q, t) \qquad P_j = P_j(q, Q, t)$$

and is generated by F of the form (1.55).

Using

$$\frac{d}{dt}(P_j\,Q_j) = P_j\,\dot{Q}_j + \dot{P}_j\,Q_j \tag{1.56b}$$

we can write (1.56) as

$$\int_{t_a}^{t_b} dt\left(p_i\,\dot{q}_i + \dot{P}_j\,Q_j - H + \tilde{H}\right) = \int_{t_a}^{t_b} dt\,\frac{d}{dt}(P_j\,Q_j + F)$$
$$= \int_{t_a}^{t_b} dt\,\frac{dF'}{dt} \tag{1.56c}$$

where

$$F' = P_j\,Q_j + F \tag{1.56d}$$

Assuming

$$F' = F'(q, P, t) \tag{1.55a}$$

we have

$$\frac{dF'}{dt} = \frac{\partial F'}{\partial q_i}\dot{q}_i + \frac{\partial F'}{\partial P_j}\dot{P}_j + \frac{\partial F'}{\partial t}$$

(1.56c) then gives

$$p_i = \frac{\partial F'}{\partial q_i} \qquad Q_j = \frac{\partial F'}{\partial P_j} \tag{1.58}$$

$$\tilde{H} - H = \frac{\partial F'}{\partial t} \tag{1.59}$$

(1.58-9) thus specify a canonical transformation of the type

$$p_i = p_i(q, P, t) \qquad Q_j = Q_j(q, P, t)$$

and is generated by F' of the form (1.55a). Furthermore, (1.58) and (1.56d) show that F' is simply the Legendre transform of F.

Similarly, other Legendre transforms of F will generate other kinds of canonical transformations. In this sense, they are referred to as the generating functions of the transforms.

Consider now a special case $F' = F'(q, P, t)$ such that all $P_j = \alpha_j$ are constants. Thus,

$$\dot{P}_j = -\frac{\partial \tilde{H}}{\partial Q_j} = 0$$

and Q_j is said to be cyclic. This means, $\tilde{H} = \tilde{H}(\alpha)$ and we can set $\tilde{H} = 0$ without affecting the dynamics of the system. (1.59) thus becomes

$$-H(q, p, t) = \frac{\partial F'(q, \alpha, t)}{\partial t} \tag{1.60}$$

Using (1.58), we get a partial differential equation on F',

$$-H\left(q, \frac{\partial F'}{\partial q}, t\right) = \frac{\partial F'(q, \alpha, t)}{\partial t} \tag{1.61}$$

called the Hamilton-Jacobi equation.

A solution to (1.61) and (1.58-9) is the function

$$A(q, t) = \int_{t_0}^{t} d t' \left[p_i \dot{q}_i - H(q, p, t') \right] \tag{1.61a}$$

To check this claim,

$$\frac{\partial A}{\partial q_i} = -\int_{t_0}^{t} d t' \frac{\partial H}{\partial q_i} = \int_{t_0}^{t} d t' \dot{p}_i = p_i \tag{1.62}$$

$$\frac{\partial A}{\partial P_i} = 0 = Q_i$$

as in (1.58). Furthermore,

$$\frac{d A}{d t} = p_i \dot{q}_i - H \tag{1.63}$$

$$= \frac{\partial A}{\partial q_i} \dot{q}_i - H$$

$$\rightarrow \quad \frac{\partial A}{\partial t} = -H(q, p, t) \tag{1.64}$$

$$= -H\left(q, \frac{\partial A}{\partial p}, t\right)$$

as in (1.61).