1.2. Relativistic Mechanics in Curved Spacetime

Ref: H.Goldstein, et al, "Classical Mechanics", 3rd ed., Addison Wesley (2000)

The classical action of a relativistic spinless point particle in a curved 4-D spacetime is (see Goldstein, §7.10)

$$\mathcal{A} = \int d\tau\, L$$
$$= -M c \int d\tau \sqrt{g_{\mu\nu}\, \dot{q}^{\mu}(\tau)\, \dot{q}^{\nu}(\tau)} \tag{1.66}$$

where

$$L = -M c \sqrt{g_{\mu\nu}\, \dot{q}^{\mu}(\tau)\, \dot{q}^{\nu}(\tau)} \qquad \dot{q} = \frac{d q}{d \tau} \tag{1.66a}$$

where τ is a scalar parameter along the trajectory. If τ is the proper time, then

$$g_{\mu\nu}\, \dot{q}^{\mu}\, \dot{q}^{\nu} = c^2 \tag{1.66b}$$

so that the components of $\dot{q}$ are not independent. However, (1.66b) is not a true dynamical constraint since it can be taken as the definition of the proper time. Obviously, correct results can be obtained by treating all $\dot{q}^{\mu}$ as independent as long as (1.66b) is applied only after all variations are done.

With

$$\frac{\partial L}{\partial q^{\mu}} = -\frac{M c}{2 \sqrt{g_{\alpha\beta}\, \dot{q}^{\alpha}\, \dot{q}^{\beta}}} \frac{\partial g_{\sigma\nu}}{\partial q^{\mu}} \dot{q}^{\sigma}\, \dot{q}^{\nu}$$
$$= \frac{(M c)^2}{2 L} \frac{\partial g_{\sigma\nu}}{\partial q^{\mu}} \dot{q}^{\sigma}\, \dot{q}^{\nu}$$

$$\frac{\partial L}{\partial \dot{q}^{\mu}} = -\frac{M c}{2 \sqrt{g_{\alpha\beta}\, \dot{q}^{\alpha}\, \dot{q}^{\beta}}} g_{\sigma\nu}\left(\delta^{\sigma}_{\mu}\, \dot{q}^{\nu} + \dot{q}^{\sigma}\, \delta^{\nu}_{\mu}\right)$$
$$= \frac{(M c)^2}{2 L} \left(g_{\mu\nu}\, \dot{q}^{\nu} + \dot{q}^{\sigma}\, g_{\sigma\mu}\right)$$
$$= \frac{(M c)^2}{L} g_{\mu\nu}\, \dot{q}^{\nu}$$

the Euler-Lagrange eq becomes

$$\frac{1}{2 L} \frac{\partial g_{\sigma\nu}}{\partial q^{\mu}} \dot{q}^{\sigma}\, \dot{q}^{\nu} - \frac{d}{d t}\left(\frac{g_{\mu\nu}\, \dot{q}^{\nu}}{L}\right) = 0 \tag{1.67}$$

(1.67) is the final result of the variations on the action. It is now legitimate to apply (1.66b) so that for τ being the proper time

$$\frac{1}{2} \frac{\partial g_{\sigma\nu}}{\partial q^{\mu}} \dot{q}^{\sigma}\, \dot{q}^{\nu} - \frac{d}{d t}\left(g_{\mu\nu}\, \dot{q}^{\nu}\right) = 0 \tag{1.68}$$

Assuming g is not explicitly time dependent,

$$\dot{g}_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial q^{\sigma}} \dot{q}^{\sigma}$$

so that

$$\frac{d}{d t}\left(g_{\mu\nu}\, \dot{q}^{\nu}\right) = \frac{\partial g_{\mu\nu}}{\partial q^{\sigma}} \dot{q}^{\sigma}\, \dot{q}^{\nu} + g_{\mu\nu}\, \ddot{q}^{\nu}$$

and (1.68) becomes

$$g_{\mu\nu}\,\ddot{q}^{\nu} = \left(\frac{1}{2}\frac{\partial g_{\sigma\nu}}{\partial q^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial q^{\sigma}}\right)\dot{q}^{\sigma}\,\dot{q}^{\nu} \tag{1.69}$$

The Christoffel symbol is defined as

$$\overline{\Gamma}_{\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial g_{\nu\sigma}}{\partial q^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial q^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial q^{\sigma}}\right) \tag{1.70}$$

and the Christoffel symbol of the 2nd kind as

$$\overline{\Gamma}_{\mu\nu}{}^{\sigma} = g^{\sigma\alpha}\,\overline{\Gamma}_{\mu\nu\alpha} \tag{1.71}$$

Using

$$\frac{\partial g_{\mu\nu}}{\partial q^{\sigma}}\,\dot{q}^{\sigma}\,\dot{q}^{\nu} = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial q^{\sigma}} + \frac{\partial g_{\mu\sigma}}{\partial q^{\nu}}\right)\dot{q}^{\sigma}\,\dot{q}^{\nu}$$

(1.69) becomes

$$g_{\mu\nu}\,\ddot{q}^{\nu} + \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial q^{\sigma}} + \frac{\partial g_{\mu\sigma}}{\partial q^{\nu}} - \frac{\partial g_{\sigma\nu}}{\partial q^{\mu}}\right)\dot{q}^{\sigma}\,\dot{q}^{\nu} = 0$$

$$g_{\mu\nu}\,\ddot{q}^{\nu} + \overline{\Gamma}_{\sigma\nu\mu}\,\dot{q}^{\sigma}\,\dot{q}^{\nu} = 0$$

$$g^{\alpha\mu}\,g_{\mu\nu}\,\ddot{q}^{\nu} + g^{\alpha\mu}\,\overline{\Gamma}_{\sigma\nu\mu}\,\dot{q}^{\sigma}\,\dot{q}^{\nu} = 0$$

$$\ddot{q}^{\alpha} + \overline{\Gamma}_{\sigma\nu}{}^{\alpha}\,\dot{q}^{\sigma}\,\dot{q}^{\nu} = 0 \tag{1.72}$$

Since the solutions of this equation minimize the length of a curve in spacetime, they are called geodesics.