I.7. Properties of the Time Evolution Operator

The group properties of $\hat{U}$ were declared in §1.6 on physical grounds. Here, mathematical proofs are supplied.

a) Fundamental Composition Law

$$\hat{U}(t,t') = \hat{U}(t,t_1)\,\hat{U}(t_1,t') \qquad \forall\ t > t_1 > t' \tag{1.254}$$

For $\hat{H}$ time-independent, proof is trivial using (1.232).

For $\hat{H}$ time-dependent, (1.252) gives

$$\begin{aligned}\hat{U}(t,t_1)\,\hat{U}(t_1,t') &= \hat{T}\left[\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t} d\,t_2\,\hat{H}(t_2)\right)\right]\hat{T}\left[\exp\left(-\frac{i}{\hbar}\int_{t'}^{t_1} d\,t_3\,\hat{H}(t_3)\right)\right] \\ &= \hat{T}\left[\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t} d\,t_2\,\hat{H}(t_2)\right)\exp\left(-\frac{i}{\hbar}\int_{t'}^{t_1} d\,t_3\,\hat{H}(t_3)\right)\right] \\ &= \hat{T}\left[\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t} d\,t_2\,\hat{H}(t_2)-\frac{i}{\hbar}\int_{t'}^{t_1} d\,t_3\,\hat{H}(t_3)\right)\right] \\ &= \hat{T}\left[\exp\left(-\frac{i}{\hbar}\int_{t'}^{t} d\,t_2\,\hat{H}(t_2)\right)\right] \\ &= \hat{U}(t,t')\end{aligned} \tag{1.255}$$

b) Unitarity

Evolution of the system is governed by $\hat{U}(t,t')$ with $t > t'$. This is therefore called the causal (or retarded) case. In contrast, $t < t'$ is called the anticausal (or advanced) case. The latter case may be unobservable physically, it is obviously valid mathematically.

Setting $t = t'$ in (1.254) gives

$$1 = \hat{U}(t,t_1)\,\hat{U}(t_1,t)$$

$$\rightarrow \qquad \hat{U}(t,t_1)^{-1} = \hat{U}(t_1,t) \tag{1.256}$$

Let

$$|\,\Psi(t)\,\rangle = \hat{U}(t,t')\ |\,\Psi(t')\,\rangle \tag{1.257}$$

then

$$\begin{aligned}\hat{U}(t,t')^{-1}\ |\,\Psi(t)\,\rangle &= \hat{U}(t,t')^{-1}\,\hat{U}(t,t')\ |\,\Psi(t')\,\rangle \\ &= |\,\Psi(t')\,\rangle\end{aligned} \tag{1.258}$$

On the other hand, if we allow anticausal evolution, then

$$|\,\Psi(t')\,\rangle = \hat{U}(t',t)\ |\,\Psi(t)\,\rangle \tag{1.258a}$$

Since (1.258) & (1.258a) are valid for all $\Psi(t)$, we recover (1.256).

For $\hat{H}$ time-independent, (1.232) gives

$$\hat{U}(\,t',t\,) = e^{-i(t'-t)\hat{H}/\hbar} = \hat{U}(t,t')^{-1}$$

and

$$\hat{U}^{+}(\,t,t'\,) = e^{i(t-t')\hat{H}^{+}/\hbar}$$

Hence, if $\hat{H}$ is hermitian, then

$$\hat{U}(t,t')^{-1} = \hat{U}^{+}(\,t,t'\,) \tag{1.260}$$

and $\hat{U}$ is unitary.

In order to prove (1.260) for a time-dependent hamiltonian, we define the time-antiordering operator as one that re-orders any product of operators so that the time arguments of the ones on the left are always earlier than those on the right:

$$\hat{\bar{T}}\left[\hat{O}_1(t_1)\ldots\hat{O}_n(t_n)\right]=\hat{O}_{i_1}(t_{i_1})\ldots\hat{O}_{i_n}(t_{i_n}) \qquad t_{i_1}\leq\ldots\leq t_{i_n} \qquad (1.262a)$$

Using (1.241), we have

$$\begin{aligned}\left\{\hat{T}\left[\hat{O}_1(t_1)\ldots\hat{O}_n(t_n)\right]\right\}^+ &= \left\{\hat{O}_{i_n}(t_{i_n})\ldots\hat{O}_{i_1}(t_{i_1})\right\}^+\\ &=\hat{O}^+_{i_1}(t_{i_n})\ldots\hat{O}^+_{i_n}(t_{i_n})\\ &=\hat{\bar{T}}\left[\hat{O}^+_1(t_1)\ldots\hat{O}^+_n(t_n)\right]\\ &=\hat{\bar{T}}\left[\hat{O}^+_n(t_n)\ldots\hat{O}^+_1(t_1)\right] \qquad (1.262)\\ &=\hat{\bar{T}}\left\{\left[\hat{O}_1(t_1)\ldots\hat{O}_n(t_n)\right]^+\right\}\end{aligned}$$

The iteration result (1.239) is valid irregardless of which one of t & t' is larger. Hence, for $\hat{U}(t', t)$ with $t>t'$, we need only replace $\hat{T}$ with $\hat{\bar{T}}$ when applying (1.252):

$$\begin{aligned}\hat{U}(t',t) &= \hat{\bar{T}}\exp\left[-\frac{i}{\hbar}\int_t^{t'}dt_1\,\hat{H}(t_1)\right] \qquad t>t'\\ &=\hat{\bar{T}}\exp\left[\frac{i}{\hbar}\int_{t'}^{t}dt_1\,\hat{H}(t_1)\right] \qquad (1.261)\\ &=\hat{U}^{-1}(t,t')\end{aligned}$$

where the last equality comes from (1.256).

On the other hand, using (1.262), we have

$$\begin{aligned}\hat{U}^+(t,t') &= \left\{\hat{T}\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}dt_1\,\hat{H}(t_1)\right]\right\}^+ \qquad t>t'\\ &=\hat{\bar{T}}\exp\left[\frac{i}{\hbar}\int_{t'}^{t}dt_1\,\hat{H}^+(t_1)\right]\end{aligned}$$

If $\hat{H}(t)$ is hermitian, then

$$\hat{U}^{-1}(t,t')=\hat{U}^+(t,t') \qquad (1.263)$$

i.e., $\hat{U}$ is unitary, as promised.

c) Schrodinger Equation for $\hat{U}(t, t')$

As already stated in § 1.6,

$$|\,\Psi(t)\rangle=\hat{U}(t,t')\,|\,\Psi(t')\rangle \qquad (1.264)$$

implies $\hat{U}$ satisfies an equivalent of the Schrodinger eq.

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t')=\hat{H}(t)\,\hat{U}(t,t') \qquad (1.265)$$

Taking the adjoint, we get

$$-i\hbar\frac{\partial}{\partial t}\hat{U}^+(t,t')=\hat{U}^+(t,t')\,\hat{H}^+(t)$$

If $\hat{H}(t)$ is hermitian, then

$$-i\hbar\frac{\partial}{\partial t}\hat{U}^+(t,t')=\hat{U}^{-1}(t,t')\,\hat{H}(t) \qquad (1.266)$$

Setting $t = t'$ in (1.264) gives

$$\hat{U}(t,\ t) = 1 \qquad (1.267)$$

which serves as the initial condition for (1.265) and (1.266).

Eqs (1.254), (1.260) & (1.267) make the set of all $\hat{U}(t,\ t')$ an Abelian group.