I.8. Heisenberg Picture of Quantum Mechanics

In the Schrodinger picture, time evolution is carried by the states and the canonical operators $\hat{\boldsymbol{x}}$ & $\hat{\boldsymbol{p}}$ are time independent.

In the Heisenberg picture, the situation is reversed. The states are time independent while time evolution is carried by the time dependent canonical operators $x_H(t)$ & $p_H(t)$. Canonical commutation rules are imposed only at "equal times" :

$$[x_{Hi}(t), p_{Hj}(t)] = i\hbar\,\delta_{ij} \tag{1.268}$$
$$[x_{Hi}(t), x_{Hj}(t)] = [p_{Hi}(t), p_{Hj}(t)] = 0$$

At this stage, these Heisenberg operators are to be taken as matrices with possibly continuous indexing.

According to Heisenberg, quantum versions of classical equations can be obtained by replacing Poisson brackets with $(i\hbar)^{-1}$ times the corresonding commutators.
[Reminder: our Poisson brackets differ from Kleinert's by a minus sign.]

For example, the 1-D Hamiltonian eqs. become

$$\dot{x}_H(t) = \frac{1}{i\hbar}[x_H(t), H_H]$$
$$\dot{p}_H(t) = \frac{1}{i\hbar}[p_H(t), H_H] \tag{1.269}$$

where

$$H_H = H[x_H(t), p_H(t), t] \tag{1.270}$$

Similarly, any classical observable $O(x, p, t)$ can be quantized as the Heisenberg operator

$$O_H(t) \equiv O[x_H(t), p_H(t), t] \tag{1.271}$$

provided all operator ordering issues are resolved. Evolution of $O_H(t)$ obeys

$$\frac{d\,O_H}{d\,t} = \frac{1}{i\hbar}[O_H, H_H] + \frac{\partial\,O_H}{\partial\,t} \tag{1.272}$$

These rules are referred to as Heisenberg's oorrespondence principle.

The elements of the matrix O_H are related to the Schrodinger operator

$$\hat{O}(t) \equiv O(\hat{x}, \hat{p}, t) \tag{1.273}$$

by

$$O_H(t)_{ab} \equiv \left\langle a(t) \,\middle|\, \hat{O}(t) \,\middle|\, b(t) \right\rangle \tag{1.274}$$

where $|\,a\rangle$, $|\,b\rangle$, ... are members of a complete basis of the Hilbert space.

It is convenient to define the Heisenberg state as the Schrodinger state at $t = 0$:

$$|\,a_H\rangle = |\,a(0)\rangle$$

so that

$$|\,a(t)\rangle = \hat{U}(t, 0)\,|\,a_H\rangle \tag{1.275}$$

(1.274) thus becomes

$$O_H(t)_{ab} = \left\langle a_H \,\middle|\, \hat{U}^+(t, 0)\,\hat{O}(t)\,\hat{U}(t, 0) \,\middle|\, b_H \right\rangle$$
$$\equiv \left\langle a_H \,\middle|\, \hat{O}_H(t) \,\middle|\, b_H \right\rangle \tag{1.279}$$

where we've defined the Heisenberg picture operator as

$$\hat{O}_H(t) \equiv \hat{U}^+(t, 0)\,\hat{O}(t)\,\hat{U}(t, 0) \tag{1.278}$$

In particular,

$$\hat{x}_H(t) \equiv \hat{U}^+(t, 0)\, \hat{x}\, \hat{U}(t, 0) \qquad (1.277)$$

$$\hat{p}_H(t) \equiv \hat{U}^+(t, 0)\, \hat{p}\, \hat{U}(t, 0) \qquad (1.276)$$

(1.279) implies

$$\frac{d}{d\,t} O_H(t)_{a\,b} = \left\langle a_H \;\middle|\; \frac{d}{d\,t} \hat{O}_H(t) \;\middle|\; b_H \right\rangle \qquad (1.280)$$

(1.278) gives

$$\begin{aligned}
\frac{d}{d\,t} \hat{O}_H(t) &= \left(\frac{d}{d\,t} \hat{U}^+(t, 0)\right) \hat{O}(t)\, \hat{U}(t, 0) + \hat{U}^+(t, 0) \left(\frac{\partial}{\partial\, t} \hat{O}(t)\right) \hat{U}(t, 0) \\
&\quad + \hat{U}^+(t, 0)\, \hat{O}(t) \frac{d}{d\,t} \hat{U}(t, 0) \\
&= \left(\frac{d}{d\,t} \hat{U}^+(t, 0)\right) \hat{U}(t, 0)\, \hat{U}^+(t, 0)\, \hat{O}(t)\, \hat{U}(t, 0) \\
&\quad + \hat{U}^+(t, 0) \left(\frac{\partial}{\partial\, t} \hat{O}(t)\right) \hat{U}(t, 0) \\
&\quad + \hat{U}^+(t, 0)\, \hat{O}(t)\, \hat{U}(t, 0)\, \hat{U}^+(t, 0) \frac{d}{d\,t} \hat{U}(t, 0) \\
&= \left(\frac{d}{d\,t} \hat{U}^+(t, 0)\right) \hat{U}(t, 0)\, \hat{O}_H(t) + \left(\frac{\partial}{\partial\, t} \hat{O}(t)\right)_H \\
&\quad + \hat{O}_H(t)\, \hat{U}^+(t, 0) \frac{d}{d\,t} \hat{U}(t, 0) \\
&= -\frac{1}{i\,\hbar} \hat{U}^+(t, 0)\, \hat{H}^+(t)\, \hat{U}(t, 0)\, \hat{O}_H(t) + \left(\frac{\partial}{\partial\, t} \hat{O}(t)\right)_H \\
&\quad + \frac{1}{i\,\hbar} \hat{O}_H(t)\, \hat{U}^+(t, 0)\, \hat{H}(t)\, \hat{U}(t, 0)
\end{aligned}$$

Thus, if $\hat{H}$ is hermitian,

$$\frac{d}{d\,t} \hat{O}_H(t) = \frac{1}{i\,\hbar}\left[\hat{O}_H(t),\, \hat{U}^+(t, 0)\, \hat{H}(t)\, \hat{U}(t, 0)\right] + \left(\frac{\partial}{\partial\, t} \hat{O}(t)\right)_H \qquad (1.282)$$

$$= \frac{1}{i\,\hbar}\left[\hat{O}_H(t),\, \hat{H}_H(t)\right] + \left(\frac{\partial}{\partial\, t} \hat{O}(t)\right)_H \qquad (1.283)$$

which is just the operator form of the matrix eq. (1.272).