1.9. Interaction Picture and Perturbation Expansion

Consider

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{1.284}$$

where $\hat{H}_0$ is the free (or unperturbed) Hamiltonian for which

$$i\hbar \frac{\partial}{\partial t} |\, \psi(t) \rangle = \hat{H}_0 \,|\, \psi(t) \rangle$$

can be solved exactly, and $\hat{V}$ is the interaction potential (or perturbation).

We shall assume $\hat{H}_0$ to be time-independent and hermitian.

In Dirac's interaction picture, time evolution due to $\hat{H}_0$ is removed from the states

$$|\, \psi_I(t) \rangle \equiv e^{i\hat{H}_0 t/\hbar} \,|\, \psi(t) \rangle \tag{1.285}$$

$$\equiv \hat{U}_I(t, t') \,|\, \psi_I(t') \rangle \tag{1.287}$$

Using

$$|\, \psi(t) \rangle = \hat{U}(t, t') \,|\, \psi(t') \rangle$$

$$= \hat{U}(t, t')\, e^{-i\hat{H}_0 t'/\hbar} \,|\, \psi_I(t') \rangle$$

we get

$$\hat{U}_I(t, t') = e^{i\hat{H}_0 t/\hbar}\, \hat{U}(t, t')\, e^{-i\hat{H}_0 t'/\hbar} \tag{1.286a}$$

In case $\hat{H}$ is time-independent, we have

$$\hat{U}_I(t, t') = e^{i\hat{H}_0 t/\hbar}\, e^{-i\hat{H}(t-t')/\hbar}\, e^{-i\hat{H}_0 t'/\hbar}$$

$$= e^{i\hat{H}_0 t/\hbar}\, e^{-i\hat{H} t/\hbar}\, e^{i\hat{H} t'/\hbar}\, e^{-i\hat{H}_0 t'/\hbar} \tag{1.286}$$

Analogous to (1.286a), we set

$$\hat{O}_I(t) = e^{i\hat{H}_0 t/\hbar}\, \hat{O}(t)\, e^{-i\hat{H}_0 t/\hbar} \tag{1.286b}$$

and in particular,

$$\hat{V}_I(t) = e^{i\hat{H}_0 t/\hbar}\, \hat{V}\, e^{-i\hat{H}_0 t/\hbar} \tag{1.289}$$

Obviously, setting $\hat{V} = 0$ turns the interaction picture into the Heisenberg one.

(1.286a) gives

$$i\hbar \frac{\partial}{\partial t} \hat{U}_I(t, t') = -H_0\, e^{i\hat{H}_0 t/\hbar}\, \hat{U}(t, t')\, e^{-i\hat{H}_0 t'/\hbar} + e^{i\hat{H}_0 t/\hbar}\, \hat{H}\, \hat{U}(t, t')\, e^{-i\hat{H}_0 t'/\hbar}$$

$$= e^{i\hat{H}_0 t/\hbar}\, \hat{V}\, \hat{U}(t, t')\, e^{-i\hat{H}_0 t'/\hbar}$$

$$= e^{i\hat{H}_0 t/\hbar}\, \hat{V}\, e^{-i\hat{H}_0 t/\hbar}\, e^{i\hat{H}_0 t/\hbar}\, \hat{U}(t, t')\, e^{-i\hat{H}_0 t'/\hbar}$$

$$= \hat{V}_I(t)\, \hat{U}_I(t, t') \tag{1.288}$$

which is the equation of motion for $\hat{U}_I$.

As for $\hat{U}$, (1.288) can be solved by iteration starting with

$$\hat{U}_I(t, t') = 1 + \frac{1}{i\hbar} \int_{t'}^{t} d t_1\, \hat{V}_I(t_1)\, \hat{U}_I(t_1, t') \tag{1.290}$$

where we've used

$$\hat{U}_I(t', t') = 1$$

Continuing with the iteration, we have

$$\hat{U}_I(t, t') = 1 + \frac{1}{i\hbar} \int_{t'}^{t} d t_1\, \hat{V}_I(t_1) + \frac{1}{(i\hbar)^2} \int_{t'}^{t} d t_1 \int_{t'}^{t_1} d t_2\, \hat{V}_I(t_1)\, \hat{V}_I(t_2)\, \hat{U}_I(t_2, t')$$

$$= 1 + \frac{1}{i\hbar}\int_{t'}^{t} d\,t_1\,\hat{V}_I(t_1) + \frac{1}{(i\hbar)^2}\int_{t'}^{t} d\,t_1 \int_{t'}^{t_1} d\,t_2\,\hat{V}_I(t_1)\,\hat{V}_I(t_2)\,\hat{U}_I(t_2, t')$$

$$+ \frac{1}{(i\hbar)^3}\int_{t'}^{t} d\,t_1 \int_{t'}^{t_1} d\,t_2 \int_{t'}^{t_2} d\,t_3\,\hat{V}_I(t_1)\,\hat{V}_I(t_2)\,\hat{V}_I(t_3)\,\hat{U}_I(t_3, t')$$

and so on.

Using (1.289), we get

$$\hat{U}_I(t, t') = 1 + \frac{1}{i\hbar}\int_{t'}^{t} d\,t_1\, e^{i\hat{H}_0 t_1/\hbar}\,\hat{V}\,e^{-i\hat{H}_0 t_1/\hbar}\,\hat{U}_I(t_1, t') \qquad (1.291)$$

and

$$\hat{U}_I(t, t') = 1 + \frac{1}{i\hbar}\int_{t'}^{t} d\,t_1\, e^{i\hat{H}_0 t_1/\hbar}\,\hat{V}\,e^{-i\hat{H}_0 t_1/\hbar} \qquad (1.292)$$

$$+ \frac{1}{(i\hbar)^2}\int_{t'}^{t} d\,t_1 \int_{t'}^{t_1} d\,t_2\, e^{i\hat{H}_0 t/_1\hbar}\,\hat{V}\,e^{-i\hat{H}_0(t_1-t_2)/\hbar}\,\hat{V}\,e^{-i\hat{H}_0 t_2/\hbar} + \ldots$$

Rewriting (1.286a) as

$$\hat{U}(t, t') = e^{-i\hat{H}_0 t/\hbar}\,\hat{U}_I(t, t')\,e^{i\hat{H}_0 t'/\hbar}$$

(1.291) becomes

$$\hat{U}(t, t') = e^{-i\hat{H}_0 t/\hbar}\,e^{i\hat{H}_0 t'/\hbar} + \frac{1}{i\hbar}\int_{t'}^{t} d\,t_1\, e^{-i\hat{H}_0(t-t_1)/\hbar}\,\hat{V}\,\hat{U}(t_1, t')$$

$$= \hat{U}_0(t, t') + \frac{1}{i\hbar}\int_{t'}^{t} d\,t_1\,\hat{U}_0(t, t_1)\,\hat{V}\,\hat{U}(t_1, t') \qquad (1.293a)$$

where

$$\hat{U}_0(t, t') = e^{-i\hat{H}_0(t-t')/\hbar}$$

For $\hat{H}$ time-independent, (1.293a) becomes

$$e^{-i\hat{H}(t-t')/\hbar} = e^{-i\hat{H}_0(t-t')/\hbar} + \frac{1}{i\hbar}\int_{t'}^{t} d\,t_1\, e^{-i\hat{H}_0(t-t_1)/\hbar}\,\hat{V}\,e^{-i\hat{H}(t_1-t')/\hbar} \qquad (1.294)$$

which can be solved recurcively to give

$$e^{-i\hat{H}(t-t')/\hbar} = e^{-i\hat{H}_0(t-t')/\hbar} + \frac{1}{i\hbar}\int_{t'}^{t} d\,t_1\, e^{-i\hat{H}_0(t-t_1)/\hbar}\,\hat{V}\,e^{-i\hat{H}_0(t_1-t')/\hbar}$$

$$+ \frac{1}{(i\hbar)^2}\int_{t'}^{t} d\,t_1 \int_{t'}^{t_1} d\,t_2\, e^{-i\hat{H}_0(t-t_1)/\hbar}\,\hat{V}\,e^{-i\hat{H}_0(t_1-t_2)/\hbar}\,\hat{V}\,e^{-i\hat{H}_0(t_2-t')/\hbar} + \ldots \quad (1.293)$$

Note that the lowest-order correction agrees with the previous formula (1.253).