1.5. Quantum Mechanics of General Lagrangian Systems

Let x^i & q^μ be the Cartesian & general coordinates, respectively.

Given

$$x^i = x^i(q) \qquad\qquad i, \mu = 1, \ldots, D$$

we have

$$\partial_\mu \equiv \frac{\partial}{\partial q^\mu} = \frac{\partial x^i}{\partial q^\mu}\frac{\partial}{\partial x^i} = e^i{}_\mu \partial_i \tag{1.356}$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$ and the transformation matrix $\mathbb{e}$ with i^{th} row and μ^{th} column element

$$e^i{}_\mu = \frac{\partial x^i}{\partial q^\mu} = \partial_\mu x^i \tag{1.357}$$

is called the basis D-ad (e.g., triad for $D = 3$ and tetrad for $D = 4$). It transforms the Cartesian (or old) basis to the general (or new) one. The matrix $\tilde{\mathbb{e}}$ with i^{th} row and μ^{th} column element

$$e_i{}^\mu = \frac{\partial q^\mu}{\partial x^i} = \partial_i q^\mu \tag{1.357a}$$

is called the reciprocal D-ad & satisfies

$$e_i{}^\mu e^i{}_\nu = \partial_i q^\mu \partial_\nu x^i = \frac{\partial q^\mu}{\partial x^i}\frac{\partial x^i}{\partial q^\nu} = \frac{\partial q^\mu}{\partial q^\nu} = \delta^\mu_\nu \tag{1.358}$$

$$e^i{}_\mu e_j{}^\mu = \partial_\mu x^i \partial_j q^\mu = \frac{\partial x^i}{\partial q^\mu}\frac{\partial q^\mu}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta^i_j \tag{1.358a}$$

which means

$$\tilde{\mathbb{e}}^T \mathbb{e} = \mathbb{e}^T \tilde{\mathbb{e}} = I \;\rightarrow\qquad \mathbb{e}^{-1} = \tilde{\mathbb{e}}^T$$

where T denotes the transpose operation.

Note that the convention is that the 1st (or row) index in $\mathbb{e}$ and $\tilde{\mathbb{e}}$ denotes the Cartesian (or old) coordinates, while the 2nd (or colum) index denotes the general (or new) coordinates. Summation always involves a pair of upper and lower indices.

Since only contravariant components of the coordinates are used, lower (upper) index corresponds to the denominator (nominator) position. Eqs(1.356-7a) can therefore be written down by inspection. Note that the placement of the indices in (1.356) is also in accordance with the usual convention for a transformation between bases.

Hence,

$$e_i{}^\mu \partial_\mu = e_i{}^\mu e^j{}_\mu \partial_j = \delta^j_i \partial_j = \partial_i \tag{1.359}$$

Cartesian component of the momentum operator is given by

$$\hat{p}_i = \frac{\hbar}{i}\partial_i = \frac{\hbar}{i} e_i{}^\mu \partial_\mu \tag{1.360}$$

The kinetic energy operator, commonly quoted as

$$\hat{T} = \frac{1}{2M}\hat{\mathbf{p}}^2 = -\frac{\hbar^2}{2M}\nabla^2 \tag{1.361}$$

is strictly speaking, valid only in Cartesian coordinates

$$\hat{T} = \frac{1}{2M} g^{ij}\hat{p}_i\hat{p}_j = -\frac{\hbar^2}{2M}\Delta \tag{1.362}$$

where $g^{ij} = \delta^{ij}$ is the inverse of the Euclidean metric tensor $g_{ij} = \delta_{ij}$ in Cartesian coordinates and the

Laplacian is defined as

$$\Delta = g^{ij}\,\partial_i \partial_j \tag{1.363a}$$

In terms of the general coordinates,

$$\begin{aligned}\Delta &= g^{ij}\, e_i{}^{\mu}\, \partial_\mu e_j{}^{\nu}\, \partial_\nu \\ &= g^{ij}\, e_i{}^{\mu} (\partial_\mu e_j{}^{\nu})\, \partial_\nu + g^{ij}\, e_i{}^{\mu}\, e_j{}^{\nu}\, \partial_\mu \partial_\nu \\ &= e^{j\mu} (\partial_\mu e_j{}^{\nu})\, \partial_\nu + e^{j\mu}\, e_j{}^{\nu}\, \partial_\mu \partial_\nu \end{aligned} \tag{1.363}$$

where

$$g^{ij}\, e_i{}^{\mu} = e^{j\mu}$$

The Euclidean metric tensor in curvilinear coordinates is

$$g_{\mu\nu} = e^i{}_\mu\, e^j{}_\nu\, g_{ij} = \frac{\partial x^i}{\partial q^\mu} \frac{\partial x^j}{\partial q^\nu}\, \delta_{ij} = e^i{}_\mu\, e_{i\nu} = e_{j\mu}\, e^j{}_\nu \tag{1.364}$$

with inverse

$$g^{\mu\nu} = e_i{}^{\mu}\, e_j{}^{\nu}\, g^{ij} = \frac{\partial q^\mu}{\partial x^i} \frac{\partial q^\nu}{\partial x^j}\, \delta^{ij} = e^{j\mu}\, e_j{}^{\nu} = e_i{}^{\mu}\, e^{i\nu} \tag{1.365}$$

With the help of (1.358a) and (1.358), we have

$$\begin{aligned} g^{\mu\nu}\, g_{\nu\lambda} &= e_i{}^{\mu}\, e_j{}^{\nu}\, g^{ij}\, e^k{}_\nu\, e^n{}_\lambda\, g_{kn} = e_i{}^{\mu}\, \delta_j^k\, g^{ij}\, e^n{}_\lambda\, g_{kn} \\ &= e_i{}^{\mu}\, g^{ij}\, e^n{}_\lambda\, g_{jn} = e_i{}^{\mu}\, \delta^i_n\, e^n{}_\lambda = e_i{}^{\mu}\, e^i{}_\lambda \\ &= \delta^\mu_\lambda \end{aligned}$$

as expected.

The affine connection is defined as

$$\begin{aligned} \Gamma_{\mu\nu}{}^{\lambda} &= (\partial_\mu e^i{}_\nu)\, e_i{}^{\lambda} = e_i{}^{\lambda}\, \partial_\mu e^i{}_\nu \\ &= \partial_\mu (e_i{}^{\lambda}\, e^i{}_\nu) - e^i{}_\nu\, \partial_\mu e_i{}^{\lambda} \\ &= -e^i{}_\nu\, \partial_\mu e_i{}^{\lambda} \end{aligned} \tag{1.366}$$

(1.363) becomes

$$\begin{aligned} \Delta &= e^{j\mu} (\partial_\mu e_j{}^{\nu})\, \partial_\nu + g^{ij}\, e_i{}^{\mu}\, e_j{}^{\nu}\, \partial_\mu \partial_\nu \\ &= -\Gamma_\mu{}^{\mu\nu}\, \partial_\nu + g^{\mu\nu}\, \partial_\mu \partial_\nu \end{aligned} \tag{1.367}$$

The metric tensor can also be defined using the line element as follows

$$d s^2 = g_{ij}\, d x^i\, d x^j \tag{1.369}$$

$$\begin{aligned} &= g_{ij} \frac{\partial x^i}{\partial q^\mu} \frac{\partial x^j}{\partial q^\nu}\, d q^\mu\, d q^\nu \\ &= g_{ij}\, e^i{}_\mu\, e^j{}_\nu\, d q^\mu\, d q^\nu \\ &= g_{\mu\nu}\, d q^\mu\, d q^\nu \end{aligned} \tag{1.370}$$

The volume element is given by

$$\begin{aligned} d^D x &\equiv \bigwedge_{i=1}^{D} d x^i = \bigwedge_{i=1}^{D} e^i{}_\mu\, d q^\mu \\ &= (\det \mathbb{e}) \bigwedge_{\mu=1}^{D} d q^\mu \\ &\equiv (\det \mathbb{e})\, d^D q \end{aligned} \tag{1.371a}$$

where $\bigwedge$ is the (anti-symmetric) exterior product [see any text on differential geometry]. For convenience, we have also assumed that the two sets of exterior products have the same orientation.

Let $\mathbb{g}$ be the matrix with elements $g_{\mu\nu}$. The relation

$$g_{\mu\nu} = g_{ij}\, e^i{}_\mu\, e^j{}_\nu$$

has the matrix form

$$\mathbb{g} = \mathbb{e}^T \mathbb{I} \mathbb{e} = \mathbb{e}^T \mathbb{e}$$

where $\mathbb{I}$ is the unit matrix that represents g_{ij}. Taking the determinant, we have

$$g = \det \mathbb{e}^T \cdot \det \mathbb{e} = (\det \mathbb{e})^2$$

where

$$g = \det \mathbb{g} = \det (g_{\mu\nu}) \tag{1.372}$$

(1.371a) thus becomes

$$d^D x = \sqrt{g}\, d^D q \tag{1.371}$$

The derivative of a determinant is found in the following manner. Consider the matrix $\mathbb{a} = (a_{ij})$. By the Laplace expansion of the i^{th} row (implicit summation suspended)

$$a = \det \mathbb{a} = \sum_j (-)^{i+j} a_{ij} M_{ij} \tag{1.372a}$$

where M_{ij} is the (i, j) minor defined as the determinant of the matrix obtained by striking out the i^{th} row and j^{th} column of $\mathbb{a}$. On the other hand, Cramer's rule gives

$$\mathbb{a}^{-1}_{ij} = \frac{1}{a} (-)^{i+j} M_{ji} \tag{1.372b}$$

Since M_{ik} for all k are independent of a_{ij}, we have

$$\frac{\partial a}{\partial a_{ij}} = (-)^{i+j} M_{ij} = a\, \mathbb{a}^{-1}_{ji} \tag{1.372c}$$

Hence, with implicit summation rule restored,

$$\begin{aligned}
g^{-1/2} \partial_\mu g^{1/2} &= \frac{1}{2g} \partial_\mu g = \frac{1}{2g} \frac{\partial g}{\partial g_{\lambda\kappa}} \partial_\mu g_{\lambda\kappa} \\
&= \frac{1}{2} g^{\lambda\kappa} \partial_\mu g_{\lambda\kappa} \qquad (1.373) \\
&= \frac{1}{2} e_i^{\ \lambda} e_j^{\ \kappa} g^{ij} \partial_\mu (g_{mn} e^m_{\ \lambda} e^n_{\ \kappa}) \quad [\text{ (1.364-5) used }] \\
&= \frac{1}{2} e_i^{\ \lambda} e_j^{\ \kappa} g^{ij} g_{mn} [(\partial_\mu e^m_{\ \lambda}) e^n_{\ \kappa} + e^m_{\ \lambda} \partial_\mu e^n_{\ \kappa}] \\
&= \frac{1}{2} (e_i^{\ \lambda} \delta^n_j g^{ij} g_{mn} \partial_\mu e^m_{\ \lambda} + \delta^m_i e_j^{\ \kappa} g^{ij} g_{mn} \partial_\mu e^n_{\ \kappa}) \\
&= \frac{1}{2} (e_i^{\ \lambda} g^{ij} g_{mj} \partial_\mu e^m_{\ \lambda} + e_j^{\ \kappa} g^{ij} g_{in} \partial_\mu e^n_{\ \kappa}) \\
&= \frac{1}{2} (e_i^{\ \lambda} \delta^i_m \partial_\mu e^m_{\ \lambda} + e_j^{\ \kappa} \delta^j_n \partial_\mu e^n_{\ \kappa}) \\
&= \frac{1}{2} (e_i^{\ \lambda} \partial_\mu e^i_{\ \lambda} + e_j^{\ \kappa} \partial_\mu e^j_{\ \kappa}) \\
&= e_i^{\ \lambda} \partial_\mu e^i_{\ \lambda} \\
&= \Gamma_{\mu\lambda}^{\ \ \lambda} \equiv \Gamma_\mu \qquad [\text{ (1.366) used. }] \qquad (1.374)
\end{aligned}$$

Again, from (1.366), we have

$$\Gamma_\mu^{\ \mu\nu} = -g^{\mu\lambda} e^i_{\ \lambda} \partial_\mu e_i^{\ \nu} = -e^{i\mu} \partial_\mu e_i^{\ \nu}$$

$$\Gamma_\mu^{\ \nu\mu} = -g^{\nu\lambda} e^i_{\ \lambda} \partial_\mu e_i^{\ \mu} = -e^{i\nu} \partial_\mu e_i^{\ \mu}$$

$$\begin{aligned}
\partial_\mu g^{\mu\nu} &= \partial_\mu (g^{ij} e_i^{\ \mu} e_j^{\ \nu}) = g^{ij} [(\partial_\mu e_i^{\ \mu}) e_j^{\ \nu} + e_i^{\ \mu} \partial_\mu e_j^{\ \nu}] \\
&= (\partial_\mu e_i^{\ \mu}) e^{i\nu} + e^{j\mu} \partial_\mu e_j^{\ \nu}
\end{aligned}$$

$$\rightarrow \quad \Gamma_\mu{}^{\mu\nu} + \Gamma_\mu{}^{\nu\mu} = -\partial_\mu g^{\mu\nu} \tag{1.375}$$

Using (1.357-7a), we can write (1.366) as

$$\begin{aligned}\Gamma_{\mu\nu}{}^\lambda &= e_i{}^\lambda \partial_\mu e^i{}_\nu = \frac{\partial q^\lambda}{\partial x^i} \frac{\partial}{\partial q^\mu} \frac{\partial x^i}{\partial q^\nu} \\ &= \frac{\partial q^\lambda}{\partial x^i} \frac{\partial}{\partial q^\nu} \frac{\partial x^i}{\partial q^\mu} = e_i{}^\lambda \partial_\nu e^i{}_\mu \\ &= \Gamma_{\nu\mu}{}^\lambda \end{aligned} \tag{1.375a}$$

(1.375) can thus be written as

$$\begin{aligned}\Gamma_\mu{}^{\mu\nu} &= -\partial_\mu g^{\mu\nu} - \Gamma_\mu{}^{\nu\mu} \\ &= -\partial_\mu g^{\mu\nu} - g^{\nu\lambda} \Gamma_{\mu\lambda}{}^\mu \\ &= -\partial_\mu g^{\mu\nu} - g^{\nu\lambda} \Gamma_{\lambda\mu}{}^\mu && \text{[(1.375a) used.]} \\ &= -\partial_\mu g^{\mu\nu} - g^{\nu\lambda} g^{-1/2} \partial_\lambda g^{1/2} && \text{[(1.373-4) used.]} \\ &= -\partial_\mu g^{\mu\nu} - g^{\mu\nu} g^{-1/2} \partial_\mu g^{1/2} && \text{[} g^{\mu\nu} = g^{\nu\mu} \text{ used.]} \\ &= -\frac{1}{\sqrt{g}} \partial_\mu \left(g^{\mu\nu} \sqrt{g} \right) \end{aligned} \tag{1.376}$$

(1.367) becomes

$$\begin{aligned}\Delta &= \frac{1}{\sqrt{g}} \left[\partial_\mu \left(g^{\mu\nu} \sqrt{g} \right) \right] \partial_\nu + g^{\mu\nu} \partial_\mu \partial_\nu \\ &= \frac{1}{\sqrt{g}} \partial_\mu \left(g^{\mu\nu} \sqrt{g}\, \partial_\nu \right) \end{aligned} \tag{1.377}$$

which is called the Laplace-Beltrami operator.

Accordingly, the Schrodinger eq. in curvilinear coordinates becomes

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(q, t) &= \hat{H} \psi(q, t) \\ &= \left[-\frac{\hbar^2}{2M} \Delta + V(q) \right] \psi(q, t) \end{aligned} \tag{1.379}$$

with Δ given by (1.377).

Using (1.371), the scalar product becomes

$$\langle \psi(t) \mid \phi(t) \rangle = \int d^D q \sqrt{g}\; \psi^*(q, t)\, \phi(q, t) \tag{1.380}$$

As will be demonstrated in the following, (1.379) cannot be obtained from a classical Lagrangian by the quantization rules used in the Cartesian coordinate case.

Consider the classical Cartesian Lagrangian

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2} M \dot{\boldsymbol{x}}^2 - V(\boldsymbol{x}) \tag{1.381}$$

With the velocities transforming as

$$\dot{x}^i = e^i{}_\mu \dot{q}^\mu \tag{1.382}$$

we have

$$\begin{aligned}\dot{\boldsymbol{x}}^2 &= g_{ij} \dot{x}^i \dot{x}^j = g_{ij} e^i{}_\mu \dot{q}^\mu e^j{}_\nu \dot{q}^\nu \\ &= g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu && \text{[(1.364) used.]} \end{aligned}$$

and (1.381) becomes

$$L(q, \dot{q}) = \frac{1}{2} M g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu - V(q) \qquad (1.383)$$

Comparing with (1.12), we see that the Hessian metric is given by

$$H_{\mu\nu} = M g_{\mu\nu} \qquad (1.384)$$

The canonical momenta are

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{q}^\mu} \\ &= \frac{1}{2} M g_{\lambda\nu} \left(\delta_\mu^\lambda \dot{q}^\nu + \dot{q}^\lambda \delta_\mu^\nu \right) = \frac{1}{2} M \left(g_{\mu\nu} \dot{q}^\nu + g_{\lambda\mu} \dot{q}^\lambda \right) \\ &= M g_{\mu\nu} \dot{q}^\nu \qquad [\, g^{\mu\nu} = g^{\nu\mu} \text{ used. }] \qquad (1.385) \end{aligned}$$

The canonical commutation rules are

$$\left[\hat{q}^\mu, \hat{p}_\nu \right] = i\hbar\, \delta_\nu^\mu \qquad (1.386)$$
$$\left[\hat{q}^\mu, \hat{q}^\nu \right] = \left[\hat{p}_\mu, \hat{p}_\nu \right] = 0$$

where $\hat{q}^\mu$, $\hat{p}_\nu$ should be hermitian operators.

Many solutions to (1.386) were proposed over the years but none yet was found to be completely satisfactory.

To begin, a family of solutions to (1.386) is given by

$$\hat{p}_\mu = \frac{\hbar}{i} g^{-z} \partial_\mu g^z \qquad \hat{q}^\mu = q^\mu \qquad (1.387a)$$

since

$$\begin{aligned} \left[\hat{q}^\mu, \hat{p}_\nu \right] &= \frac{\hbar}{i} \left(q^\mu g^{-z} \partial_\nu g^z - g^{-z} \partial_\nu (g^z q^\mu) \right) \\ &= i\hbar\, \partial_\nu q^\mu \\ &= i\hbar\, \delta_\nu^\mu \qquad (1.387b) \end{aligned}$$

$$\left[\hat{p}_\mu, \hat{p}_\nu \right] = -\hbar^2 \left(g^{-z} \partial_\mu \partial_\nu g^z - g^{-z} \partial_\nu \partial_\mu g^z \right) = 0$$

By definition, the adjoint of $\hat{p}_\mu$ is given by

$$\left\langle \psi \mid \hat{p}_\mu^+ \mid \phi \right\rangle = \left\langle \hat{p}_\mu \psi \mid \phi \right\rangle$$

i.e.,

$$\begin{aligned} \int d^D q \sqrt{g}\; \psi^* \hat{p}_\mu^+ \phi &= \int d^D q \sqrt{g} \left(\frac{\hbar}{i} g^{-z} \partial_\mu (g^z \psi) \right)^* \phi \\ &= \int d^D q \sqrt{g} \left(-\frac{\hbar}{i} g^{-z} \partial_\mu (g^z \psi^*) \right) \phi \\ &= \int d^D q\; g^z \psi^* \frac{\hbar}{i} \partial_\mu \left(\sqrt{g}\, g^{-z} \phi \right) \\ &= \int d^D q\; \sqrt{g}\; \psi^* \frac{\hbar}{i} g^{z-1/2} \partial_\mu \left(g^{-z+1/2} \phi \right) \qquad (1.388) \end{aligned}$$

$$\rightarrow \qquad \hat{p}_\mu^+ = \frac{\hbar}{i} g^{z-1/2} \partial_\mu g^{-z+1/2} \qquad (1.387c)$$

Hence, if $\hat{p}_\mu$ is to be hermitian, then

$$z = -z + \frac{1}{2} \qquad \rightarrow \qquad z = \frac{1}{4}$$

which was the solution chosen by Kleinert :

$$\hat{p}_\mu = \frac{\hbar}{i} g^{-1/4} \partial_\mu g^{1/4} \qquad\qquad \hat{q}^\mu = q^\mu \qquad (1.387)$$

Analogous to the derivation of (1.374), we have

$$\frac{i}{\hbar} \hat{p}_\mu \psi = g^{-1/4} \partial_\mu \left(g^{1/4} \psi\right) = \partial_\mu \psi + \frac{1}{4g} (\partial_\mu g) \psi$$

$$= \partial_\mu \psi + \frac{1}{2} \Gamma_\mu \psi$$

$$\rightarrow \qquad \hat{p}_\mu = \frac{\hbar}{i} \left(\partial_\mu + \frac{1}{2} \Gamma_\mu\right) \qquad (1.389)$$

From (1.385), we have

$$\frac{1}{M} g^{\lambda\mu} p_\mu = g^{\lambda\mu} g_{\mu\nu} \dot{q}^\nu = \delta^\lambda_\nu \dot{q}^\nu = \dot{q}^\lambda \qquad (1.389a)$$

Hence,

$$p_\mu \dot{q}^\mu = \frac{1}{M} p_\mu g^{\mu\nu} p_\nu$$

$$g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = \frac{1}{M^2} g_{\mu\nu} g^{\mu\lambda} p_\lambda g^{\nu\kappa} p_\kappa = \frac{1}{M^2} \delta^\lambda_\nu p_\lambda g^{\nu\kappa} p_\kappa$$

$$= \frac{1}{M^2} p_\nu g^{\nu\kappa} p_\kappa$$

and the classical Hamiltonian associated with the Lagrangian in (1.383) becomes

$$H = p_\mu \dot{q}^\mu - L$$

$$= \frac{1}{2M} g^{\mu\nu} p_\mu p_\nu + V \qquad (1.390)$$

Since $g^{\mu\nu}$ is in general a function of q, quantization of (1.390) runs into the problem of operator ordering.

The choice

$$\hat{H}_{\text{can}} = \frac{1}{2M} \hat{p}_\mu g^{\mu\nu} \hat{p}_\nu + V \qquad (1.391)$$

known as the canonical Hamiltonian, has the advantage of being hermitian. This follows trivially from the hermicity of $\hat{p}_\mu$:

$$\hat{T}^+_{\text{can}} = \frac{1}{2M} \hat{p}_\nu g^{\nu\mu} \hat{p}_\mu = \hat{T}_{\text{can}} \qquad (1.391a)$$

$$= -\frac{\hbar^2}{2M} \Delta_{\text{can}}$$

where the canonical Laplacian [see (1.389)]

$$\Delta_{\text{can}} \psi = \left(\partial_\mu + \frac{1}{2} \Gamma_\mu\right) g^{\mu\nu} \left(\partial_\nu + \frac{1}{2} \Gamma_\nu\right) \psi \qquad (1.392)$$

$$= \partial_\mu (g^{\mu\nu} \partial_\nu \psi) + \frac{1}{2} \partial_\mu (g^{\mu\nu} \Gamma_\nu \psi) + \frac{1}{2} \Gamma_\mu g^{\mu\nu} \partial_\nu \psi + \frac{1}{4} g^{\mu\nu} \Gamma_\mu \Gamma_\nu \psi$$

differs from the Laplace-Beltrami operator

$$\Delta \psi = \frac{1}{\sqrt{g}} \partial_\mu \left(g^{\mu\nu} \sqrt{g}\, \partial_\nu \psi\right)$$

$$= \left(\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \right) g^{\mu\nu} \partial_\nu \psi + \partial_\mu (g^{\mu\nu} \partial_\nu \psi)$$
$$= \Gamma_\mu g^{\mu\nu} \partial_\nu \psi + \partial_\mu (g^{\mu\nu} \partial_\nu \psi)$$

by

$$(\Delta - \Delta_{\text{can}}) \psi = \frac{1}{2} \Gamma_\mu g^{\mu\nu} \partial_\nu \psi - \frac{1}{2} \partial_\mu (g^{\mu\nu} \Gamma_\nu \psi) - \frac{1}{4} g^{\mu\nu} \Gamma_\mu \Gamma_\nu \psi$$
$$= -\frac{1}{2} [\partial_\mu (g^{\mu\nu} \Gamma_\nu)] \psi - \frac{1}{4} g^{\mu\nu} \Gamma_\mu \Gamma_\nu \psi \qquad (1.393)$$

Note that we’ve inserted an arbitray function ψ into the equations above to avoid any ambiguities as to the range of ∂_μ.

By judicial insertion of pairs of $g^{-1/4}$ & $g^{1/4}$ into (1.391a), one can recover the correct Hamiltonian. Thus, using (1.387), we have

$$\frac{1}{2M} g^{-1/4} \hat{p}_\mu g^{1/4} g^{\mu\nu} g^{1/4} \hat{p}_\nu g^{-1/4} = -\frac{\hbar^2}{2M} g^{-1/2} \partial_\mu \left(g^{1/2} g^{\mu\nu} \partial_\nu \right)$$
$$= -\frac{\hbar^2}{2M} \Delta = \hat{T} \qquad (1.394a)$$

(1.394a) also has the correct classical limit since the $g^{-1/4}$ & $g^{1/4}$ pairs cancel out if $\hat{p}_\mu$ becomes a mere function. However, the relation between $\hat{T}$ and $\hat{p}_\mu$ becomes uncomprehensible.

As an example, consider a free particle in a plane described by radial coordinates $q^1 = r$ and $q^2 = \varphi$ so that

$$x^1 = x = r \cos\varphi \qquad x^2 = y = r \sin\varphi \qquad (1.395)$$

and

$$r = \sqrt{x^2 + y^2} \qquad \varphi = \tan^{-1} \frac{y}{x} \qquad (1.395a)$$

From the line element

$$d s^2 = d x^2 + d y^2 = d r^2 + r^2 d \varphi^2 \qquad (1.395b)$$

we obtain the metric tensor

$$\mathbb{g} = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad (1.396)$$

determinant

$$g = \det \mathbb{g} = r^2 \qquad (1.397)$$

and inverse

$$\mathbb{g}^{-1} = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \qquad (1.398)$$

The Laplace-Beltrami operator (1.377) becomes

$$\Delta = \frac{1}{r} \left[\partial_r (r \partial_r) + \partial_\varphi \left(\frac{1}{r^2} r \partial_\varphi \right) \right]$$
$$= \frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial_\varphi^2 \qquad (1.399)$$
$$= \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\varphi^2$$

in agreement with the familiar Laplacian in polar coordinates.

Using (1.373), we have

$$\Gamma_r = \frac{1}{r}\partial_r r = \frac{1}{r} \qquad \Gamma_\varphi = \frac{1}{r}\partial_\varphi r = 0$$

Since $\mathfrak{g}^{-1} = (g^{\mu\nu})$ is diagonal, the canonical Laplacian (1.392) reduces to

$$\begin{aligned}\Delta_{\text{can}}\,\psi &= \left(\partial_r + \frac{1}{2}\Gamma_r\right) g^{rr}\left(\partial_r + \frac{1}{2}\Gamma_r\right)\psi + \left(\partial_\varphi + \frac{1}{2}\Gamma_\varphi\right) g^{\varphi\varphi}\left(\partial_\varphi + \frac{1}{2}\Gamma_\varphi\right)\psi \\ &= \left(\partial_r + \frac{1}{2r}\right)\left(\partial_r + \frac{1}{2r}\right)\psi + \partial_\varphi\left(\frac{1}{r^2}\partial_\varphi\psi\right) \qquad (1.400a) \\ &= \partial_r^2\psi + \frac{1}{2r}\partial_r\psi + \partial_r\left(\frac{1}{2r}\psi\right) + \frac{1}{4r^2}\psi + \frac{1}{r^2}\partial_\varphi^2\psi \\ &= \partial_r^2\psi + \frac{1}{r}\partial_r\psi - \frac{1}{4r^2}\psi + \frac{1}{r^2}\partial_\varphi^2\psi\end{aligned}$$

or

$$\Delta_{\text{can}} = \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{4r^2} + \frac{1}{r^2}\partial_\varphi^2 \qquad (1.400)$$

Comparing with (1.399), we have

$$\Delta - \Delta_{\text{can}} = \frac{1}{4r^2} \qquad (1.401)$$

Note that this discrepancy between Δ and Δ_{can} arises even though there is no apparent ordering problem in the canonical expression (1.400a), since $g^{\mu\nu}$ can be placed anywhere without affecting the result.

See Kleinert's text (final paragraphs in the section) for a discussion on applications to general coordinates in non-Euclidean spaces. Of particular importance is the concept of group quantization for which quantization rules are applied to the Poisson brackets among q^μ and the generators of the symmetry group of the system. One notable example is the quantization of angular momentum in Euclidean space.