1.14. Particle on the Surface of a Sphere

For a particle moving on the surface of a sphere of radius r, the natural coordinates are spherical with

$$x^1 = x = r\sin\theta\cos\varphi \quad x^2 = y = r\sin\theta\,\sin\varphi \qquad x^3 = z = r\cos\theta \quad (1.402)$$

and

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} \qquad \varphi = \tan^{-1}\frac{y}{x} \qquad (1.402a)$$

With $\dot r = 0$, we have

$$\dot x = r\left(\cos\theta\cos\varphi\,\dot\theta - \sin\theta\sin\varphi\,\dot\varphi\right)$$
$$\dot y = r\left(\cos\theta\sin\varphi\,\dot\theta + \sin\theta\cos\varphi\,\dot\varphi\right)$$
$$\dot z = -r\sin\theta\,\dot\theta$$

so that

$$\dot x^2 + \dot y^2 + \dot z^2 = r^2\left(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2\right)$$

and

$$d\,s^2 = r^2\left(d\,\theta^2 + \sin^2\theta\;d\,\varphi^2\right) \qquad (1.402b)$$

$$\to \qquad \mathrm{g} = (g_{\mu\nu}) = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix} \qquad (1.402c)$$

$$g = \det \mathrm{g} = r^4\sin^2\theta \qquad (1.402d)$$

For a “free” particle, the Lagrangian is therefore

$$L = \frac{1}{2}M\,r^2\left(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2\right) \qquad (1.403)$$

The canonical momenta are

$$p_\theta = \frac{\partial L}{\partial\,\dot\theta} = M\,r^2\,\dot\theta \qquad p_\varphi = \frac{\partial L}{\partial\,\dot\varphi} = M\,r^2\sin^2\theta\,\dot\varphi \qquad (1.404)$$

so that the classical Hamiltonian becomes

$$H = p_\theta\,\dot\theta + p_\varphi\,\dot\varphi - L$$
$$= p_\theta\frac{p_\theta}{M\,r^2} + p_\varphi\frac{p_\varphi}{M\,r^2\sin^2\theta} - \frac{1}{2}M\,r^2\left[\left(\frac{p_\theta}{M\,r^2}\right)^2 + \sin^2\theta\left(\frac{p_\varphi}{M\,r^2\sin^2\theta}\right)^2\right]$$
$$= \frac{1}{2\,M\,r^2}\left(p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2\right) \qquad (1.405)$$

According to the canonical quantization rule (1.387), we have

$$\hat p_\theta = \frac{\hbar}{i}g^{-1/4}\,\partial_\theta g^{1/4} = \frac{\hbar}{i}\frac{1}{\sin^{1/2}\theta}\partial_\theta\sin^{1/2}\theta \qquad (1.406)$$

$$\hat p_\varphi = \frac{\hbar}{i}g^{-1/4}\,\partial_\varphi g^{1/4} = \frac{\hbar}{i}\partial_\varphi \qquad (1.406a)$$

However, as shown in §1.13, these momenta do not lead to the correct Hamiltonian unless arbitrary factors of $g^{1/4}$ an $g^{-1/4}$ are inserted into $\hat p_\nu\,g^{\nu\mu}\,\hat p_\mu = -\hbar^2\,\Delta_{\text{can}}$.

Furthermore, the transformation (1.402-a) is not invertible at certain points such as those with $\theta = 0$. Hence, we cannot safely transform the Cartesian quantization rules

$$\left[\hat x^i,\,\hat p_j\right] = i\,\hbar\,\delta^i_j \qquad (1.407)$$

$$\left[\hat x^i,\,\hat x^j\right] = \left[\hat p_i,\,\hat p_j\right] = 0$$

into their curvilinear counterparts.

Consider now the classical angular momentum

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p} \qquad (1.408)$$

with Cartesian components

$$L_i = \epsilon_{ijk}\, x_j\, p_k \qquad (1.408a)$$

where

$$\epsilon_{ijk} = \begin{cases} (-)^P & P = \text{number of permutations of } (1,\ 2,\ 3) \text{ that leads to } (i,\ j,\ k) \\ 0 & (i,\ j,\ k) \text{ is not } a \text{ permutation of } (1,\ 2,\ 3) \end{cases}$$

The quantum version is simply

$$\hat{\boldsymbol{L}} = \hat{\boldsymbol{x}} \times \hat{\boldsymbol{p}} \qquad (1.409)$$

with Cartesian components

$$\hat{L}_i = \epsilon_{ijk}\, \hat{x}_j\, \hat{p}_k \qquad (1.408a)$$

There is no problem of operator ordering since there are no terms with $j = k$ in (1.408a)

Using (1.408a), we have

$$\begin{aligned}\left[\hat{L}_i,\ \hat{L}_j\right] &= \epsilon_{ikl}\,\epsilon_{jmn}\left[\hat{x}_k\,\hat{p}_l\,,\ \hat{x}_m\,\hat{p}_n\right] \\ &= \epsilon_{ikl}\,\epsilon_{jmn}\left(\left[\hat{x}_k\,,\ \hat{x}_m\,\hat{p}_n\right]\hat{p}_l + \hat{x}_k\left[\hat{p}_l\,,\ \hat{x}_m\,\hat{p}_n\right]\right) \\ &= \epsilon_{ikl}\,\epsilon_{jmn}\left(\hat{x}_m\left[\hat{x}_k\,,\ \hat{p}_n\right]\hat{p}_l + \hat{x}_k\left[\hat{p}_l\,,\ \hat{x}_m\right]\hat{p}_n\right) \\ &= i\,\hbar\,\epsilon_{ikl}\,\epsilon_{jmn}\left(\hat{x}_m\,\delta_{kn}\,\hat{p}_l - \hat{x}_k\,\delta_{lm}\,\hat{p}_n\right) \\ &= i\,\hbar\left(\epsilon_{ikl}\,\epsilon_{jmk}\,\hat{x}_m\,\hat{p}_l - \epsilon_{ikm}\,\epsilon_{jmn}\,\hat{x}_k\,\hat{p}_n\right)\end{aligned}$$

Using

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}$$

we have

$$\epsilon_{ijk}\,\epsilon_{ijn} = \delta_{jj}\,\delta_{kn} - \delta_{jn}\,\delta_{kj} = 2\,\delta_{kn}$$

so that

$$\begin{aligned}\epsilon_{ijk}\left[\hat{L}_i,\ \hat{L}_j\right] &= i\,\hbar\left(\epsilon_{ijk}\,\epsilon_{ikl}\,\epsilon_{jmk}\,\hat{x}_m\,\hat{p}_l - \epsilon_{ijk}\,\epsilon_{ikm}\,\epsilon_{jmn}\,\hat{x}_k\,\hat{p}_n\right) \\ &= i\,\hbar\left(-2\,\delta_{jl}\,\epsilon_{jmk}\,\hat{x}_m\,\hat{p}_l + 2\,\delta_{jm}\,\epsilon_{jmn}\,\hat{x}_k\,\hat{p}_n\right) \\ &= 2\,i\,\hbar\left(-\epsilon_{jmk}\,\hat{x}_m\,\hat{p}_j + \epsilon_{jjn}\,\hat{x}_k\,\hat{p}_n\right) \\ &= 2\,i\,\hbar\,\hat{J}_k\end{aligned}$$

$$\begin{aligned}\rightarrow \qquad \epsilon_{mnk}\,\epsilon_{ijk}\left[\hat{L}_i,\ \hat{L}_j\right] &= (\delta_{mi}\,\delta_{nj} - \delta_{mj}\,\delta_{ni})\left[\hat{L}_i,\ \hat{L}_j\right] \\ &= \left[\hat{L}_m,\ \hat{L}_n\right] - \left[\hat{L}_n,\ \hat{L}_m\right] = 2\left[\hat{L}_m,\ \hat{L}_n\right] \\ &= 2\,i\,\hbar\epsilon_{mnk}\,\hat{J}_k\end{aligned}$$

i.e.,

$$\left[\hat{L}_i,\ \hat{L}_j\right] = i\,\hbar\,\epsilon_{ijk}\,\hat{L}_k \qquad (1.410)$$

$\hat{\boldsymbol{L}}$ is called the generators of the rotation group since $e^{i\,\theta\,\boldsymbol{n}\cdot\hat{\boldsymbol{L}}/\hbar}$ is the operator for a rotation about the axis $\boldsymbol{n}$ by an angle θ. Noting that classically,

$$\{L_i,\ L_j\} = \epsilon_{ijk}\,L_k$$

(1.410) is therefore a quantization rule on the Poisson brackets of the generators of the rotation group. It is also the prototype of the group quantization mentioned in §1.13.

Using (1.402), we have, for $r = \text{constant}$,

$$\hat{p}_1 = \frac{\hbar}{i}\,\frac{\partial}{\partial x} = \frac{\hbar}{i}\left(\frac{\partial\,\theta}{\partial\,x}\,\frac{\partial}{\partial\,\theta} + \frac{\partial\,\varphi}{\partial\,x}\,\frac{\partial}{\partial\,\varphi}\right) = \frac{\hbar}{i\,r}\left(\cos\theta\cos\varphi\,\frac{\partial}{\partial\,\theta} - \frac{\sin\varphi}{\sin\theta}\,\frac{\partial}{\partial\,\varphi}\right)$$

$$\hat{p}_2 = \frac{\hbar}{i}\frac{\partial}{\partial y} = \frac{\hbar}{i}\left(\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}\right) = \frac{\hbar}{i\,r}\left(\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\hat{p}_3 = \frac{\hbar}{i}\frac{\partial}{\partial z} = \frac{\hbar}{i}\left(\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi}\right) = -\frac{\hbar}{i\,r}\sin\theta\frac{\partial}{\partial\theta}$$

so that

$$\hat{L}_1 = \hat{x}_2\,\hat{p}_3 - \hat{x}_3\,\hat{p}_2$$
$$= \frac{\hbar}{i}\left[-\sin^2\theta\sin\varphi\frac{\partial}{\partial\theta} - \cos\theta\left(\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]$$
$$= -\frac{\hbar}{i}\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \qquad (1.412)$$

$$\hat{L}_2 = \hat{x}_3\,\hat{p}_1 - \hat{x}_1\,\hat{p}_3$$
$$= \frac{\hbar}{i}\left[\cos\theta\left(\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) + \sin^2\theta\cos\varphi\frac{\partial}{\partial\theta}\right]$$
$$= \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \qquad (1.412a)$$

$$\hat{L}_3 = \hat{x}_1\,\hat{p}_2 - \hat{x}_2\,\hat{p}_1$$
$$= \frac{\hbar}{i}\left[\sin\theta\cos\varphi\left(\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right.$$
$$\left. - \sin\theta\sin\varphi\left(\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]$$
$$= \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \qquad (1.412b)$$

$$\rightarrow \qquad \hat{L}_1^2 = -\hbar^2\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$
$$= -\hbar^2\left(\sin^2\varphi\frac{\partial^2}{\partial\theta^2} - \csc^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \cot\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\right.$$
$$+ \cot\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}$$
$$\left. - \cot^2\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \cot^2\theta\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\right)$$

$$= -\hbar^2\left\{\sin^2\varphi\frac{\partial^2}{\partial\theta^2} - \sin\varphi\cos\varphi\left[\left(\csc^2\theta + \cot^2\theta\right)\frac{\partial}{\partial\varphi} - 2\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}\right]\right.$$
$$\left. + \cot\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \cot^2\theta\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\right\}$$

Hint: Let the operator act on a function ψ if the above derivation seems unclear.
Using a symbolic manipulation package such as *Mathematica* is also an option.

$$\hat{L}_2^2 = -\hbar^2\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$= -\hbar^2 \left(\cos^2\varphi \frac{\partial^2}{\partial \theta^2} + \csc^2\theta \sin\varphi \cos\varphi \frac{\partial}{\partial \varphi} - \cot\theta \sin\varphi \cos\varphi \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \right.$$
$$+ \cot\theta \sin^2\varphi \frac{\partial}{\partial \theta} - \cot\theta \sin\varphi \cos\varphi \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi}$$
$$\left. + \cot^2\theta \sin\varphi \cos\varphi \frac{\partial}{\partial \varphi} + \cot^2\theta \sin^2\varphi \frac{\partial^2}{\partial \varphi^2} \right)$$
$$= -\hbar^2 \left\{ \cos^2\varphi \frac{\partial^2}{\partial \theta^2} + \sin\varphi \cos\varphi \left[\left(\csc^2\theta + \cot^2\theta \right) \frac{\partial}{\partial \varphi} - 2\cot\theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \right] \right.$$
$$\left. + \cot\theta \sin^2\varphi \frac{\partial}{\partial \theta} + \cot^2\theta \sin^2\varphi \frac{\partial^2}{\partial \varphi^2} \right\}$$

$$\hat{L}_3^2 = -\hbar^2 \frac{\partial^2}{\partial \varphi^2}$$

$$\therefore \quad \hat{\boldsymbol{L}}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$$
$$= -\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \left(1 + \cot^2\theta \right) \frac{\partial^2}{\partial \varphi^2} \right]$$
$$= -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2} \right) \qquad (1.415a)$$

which is proportional to the angular part of the Laplacian in spherical coordinates.

Classically,

$$L_1 = M(y\dot{z} - z\dot{y})$$
$$= M r^2 \left[-\sin^2\theta \sin\varphi \dot{\theta} - \cos\theta \left(\cos\theta \sin\varphi \dot{\theta} + \sin\theta \cos\varphi \dot{\varphi} \right) \right]$$
$$= M r^2 \left(-\sin\varphi \dot{\theta} - \sin\theta \cos\theta \cos\varphi \dot{\varphi} \right) \qquad (1.413)$$

$$L_2 = M(z\dot{x} - x\dot{z})$$
$$= M r^2 \left[\cos\theta \left(\cos\theta \cos\varphi \dot{\theta} - \sin\theta \sin\varphi \dot{\varphi} \right) + \sin^2\theta \cos\varphi \dot{\theta} \right]$$
$$= M r^2 \left(\cos\varphi \dot{\theta} - \sin\theta \cos\theta \sin\varphi \dot{\varphi} \right) \qquad (1.413a)$$

$$L_3 = M(x\dot{y} - y\dot{x})$$
$$= M r^2 \left[\sin\theta \cos\varphi \left(\cos\theta \sin\varphi \dot{\theta} + \sin\theta \cos\varphi \dot{\varphi} \right) \right.$$
$$\left. - \sin\theta \sin\varphi \left(\cos\theta \cos\varphi \dot{\theta} - \sin\theta \sin\varphi \dot{\varphi} \right) \right]$$
$$= M r^2 \sin^2\theta \dot{\varphi} \qquad (1.413b)$$

Using the polar variables, we get

$$\boldsymbol{L}^2 = L_1^2 + L_2^2 + L_3^2$$
$$= M^2 r^4 \left(\dot{\theta}^2 + \sin^2\theta \cos^2\theta \dot{\varphi}^2 + \sin^4\theta \dot{\varphi}^2 \right)$$
$$= M^2 r^4 \left(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2 \right)$$

Using the Cartesian variables, we get

$$\boldsymbol{L}^2 = M^2 \left[\left(x^2 + y^2 \right) \dot{z}^2 + \left(x^2 + z^2 \right) \dot{y}^2 + \left(y^2 + z^2 \right) \dot{x}^2 - 2 y \dot{y} z \dot{z} - 2 x \dot{x} z \dot{z} - 2 x \dot{x} y \dot{y} \right]$$
$$= M^2 \left[r^2 \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - (x\dot{x} + y\dot{y} + z\dot{z})^2 \right]$$
$$= M^2 r^2 \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

since for rotations,

$$x\dot{x} + y\dot{y} + z\dot{z} = \boldsymbol{x} \cdot \dot{\boldsymbol{x}} = 0$$

Hence,

$$H = \frac{1}{2} M\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2 M r^2} \boldsymbol{L}^2 \tag{1.414}$$

Using the quantized form (1.415a), we have

$$\begin{aligned} \hat{H} &= \frac{1}{2 M r^2} \hat{\boldsymbol{L}}^2 \\ &= -\frac{\hbar^2}{2 M r^2}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2}\right) \end{aligned} \tag{1.415}$$

The eigenfunctions of $\hat{\boldsymbol{L}}^2$ are the spherical harmonics [see any text book on mathematical physics or quantum mechanics],

$$Y_{l\,m}(\theta,\, \varphi) = (-)^m \sqrt{\frac{2\,l+1}{4\,\pi} \frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\, e^{i\,m\,\varphi} \tag{1.416}$$

where $P_l^m(z)$ are the associated Legendre polynomials

$$P_l^m(z) = \frac{1}{2^l\, l!} \left(1 - z^2\right)^{m/2} \frac{d^{\,l+m}}{d\, z^{l+m}} \left(z^2 - 1\right)^l \qquad (-l \le m \le l) \tag{1.417}$$

The $Y_{l\,m}$'s are orthonormal:

$$\begin{aligned} \langle\, l\, m \mid l'\, m' \,\rangle &\equiv \int_0^{\pi} d\,\theta \int_0^{2\,\pi} d\,\varphi \sin\theta\, Y_{l\,m}^{*}(\theta,\, \varphi)\, Y_{l'\,m'}(\theta,\, \varphi) \\ &= \delta_{l\,l'}\, \delta_{m\,m'} \end{aligned} \tag{1.418}$$

To recapitulate, the canonical quantization leads to a Hamiltonian

$$\hat{H}_{\text{can}} = \frac{1}{2\,M} \hat{p}_\nu\, g^{\nu\,\mu}\, \hat{p}_\mu = -\frac{\hbar^2}{2\,M} \Delta_{\text{can}}$$

that differs from the correct Hamiltonian (1.415). However, the correct result can be obtained by the proper distribution of $g^{-1/4}$ & $g^{1/4}$ factors. Using (1.402d), we have

$$g^{1/4} = r \sin^{1/2}\theta \qquad\qquad g^{-1/4} = \frac{1}{r \sin^{1/2}\theta} \tag{1.419}$$

Since the sphere is embedded in a Euclidean space, we see that (1.415) is simply

$$\hat{H} = -\frac{\hbar^2}{2\,M} \Delta \tag{1.420}$$

where the Laplace-Beltrami operator Δ associated with the metric

$$\mathbb{g} = (g_{\mu\nu}) = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{1.421}$$

is equal to the angular part of the Laplacian ∇^2 in spherical coordinates:

$$\Delta = \frac{1}{r^2}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2}\right) \tag{1.422}$$