1.15. Spinning Top

Some derivations in this section are straightforward but tedious. *Mathematica* codes for them can be found in the file "1.15._Code.nb".

For a (symmetric) spinning top, we start with the classical Hamiltonian

$$H = \frac{1}{2\mathcal{I}_\xi}\left(L_\xi^2 + L_\eta^2\right) + \frac{1}{2\mathcal{I}_\zeta} L_\zeta^2 \qquad (1.423)$$

where $\mathcal{I}_\xi = \mathcal{I}_\eta$ and $\mathcal{I}_\zeta$ are the moments of inertia along the principal axes.

Classically, the total angular momentum of a collection of mass points is given by

$$\boldsymbol{L} = \sum_\nu \boldsymbol{x}_\nu \times \boldsymbol{p}_\nu \qquad (1.424)$$

which is easily quantized as

$$\hat{\boldsymbol{L}} = \sum_\nu \hat{\boldsymbol{x}}_\nu \times \hat{\boldsymbol{p}}_\nu \qquad (1.425)$$

Since operators associated with different particles commute, it is easy to show that the components of $\hat{\boldsymbol{L}}$ satisfy the commutation rules (1.410).

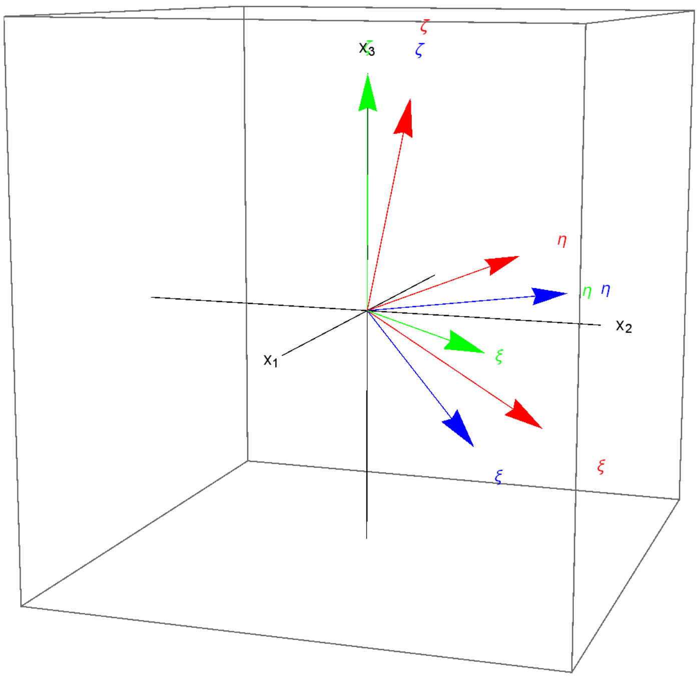
Placing the center of mass at the coordinate origin makes spatial orientions the only degrees of freedom. All motions are therefore rotations that can be specified by 3×3 orthogonal matrices of the form

$$\mathbb{R}(\alpha, \beta, \gamma) = \mathbb{R}_3(\alpha)\,\mathbb{R}_2(\beta)\,\mathbb{R}_3(\gamma) \qquad (1.426)$$

where α, β, γ are the Euler angles with

$$0 \le \alpha,\ \gamma < 2\pi \qquad \text{and} \qquad 0 \le \beta < \pi \qquad (1.426a)$$

Convention: matrices are denoted by double-stroked letters.



With reference to the graph above, α, β, γ are defined as follows:

Let $R_k(\theta)$ be the rotation about the k-axis by an angle θ.

1. Start with the fixed Cartesian axes (x_1, x_2, x_3).

2. Rotate by $R_3(\alpha)$ to get the (ξ, η, ζ) axes.
   ζ and x_3 therefore coincide. α is the angle between x_1 and ξ.
3. Rotate by $R_\eta(\beta)$ to get the (ξ, η, ζ) axes.
   η and η therefore coincide. β is the angle between $x_3 = \zeta$ and ζ.
4. Rotate by $R_\zeta(\gamma)$ to get the (ξ, η, ζ) axes.
   ζ and ζ therefore coincide. γ is the angle between $\eta = \eta$ and η.

The matrix

$$\mathbb{R}_i(\theta) = e^{-\theta\,\mathbb{G}_i} \qquad (1.427)$$

denotes a rotation about the (fixed) x_i-axis by an angle θ and the matrix

$$\mathbb{G}_i = (G_{jk}) = (\epsilon_{ijk}) \qquad (1.428)$$

is its generator. Note we did not use Kleinert's version of (1.427) because it seemed too constrived for $\hbar$ and i to appear in an orthogonal matrix for a classical theory.

Caution: Euler angles are defined through rotations about body (not-fixed) axes [see Goldstein].

Proof of (1.427-8) is straightforward. We'll do it for the case $i = 3$, we have

$$\mathbb{R}_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For $\theta \to 0$,

$$\mathbb{R}_3(\theta) \approx \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \mathbb{I} - \theta\,\mathbb{G}_3$$

$$\to \qquad \mathbb{G}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\epsilon_{3jk})$$

Let

$$\boldsymbol{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

be a position vector. The rotation of this vector about the x_i-axis by an angle θ results in a vector given by

$$\boldsymbol{r}' = \mathbb{R}_i(\theta)\,\boldsymbol{r} \qquad (1.429a)$$

The rotation of a wave function $\psi(\boldsymbol{r})$ about the x_i-axis by an angle θ results in a wave function

$$\psi'(\boldsymbol{r}) \equiv \psi\left[\mathbb{R}_i^{-1}(\theta)\,\boldsymbol{r}\right] \qquad (1.429b)$$

which is easily verified graphically by comparing the contours of ψ and ψ' for $\boldsymbol{r}$ on a plane perpendicular to the rotation axis.

The associated rotation operator in Hilbert space is defined by

$$\psi'(\boldsymbol{r}) = \hat{U}_i(\theta)\,\psi(\boldsymbol{r}) \qquad (1.429c)$$
$$\equiv e^{-\theta\,\hat{G}_i}\,\psi(\boldsymbol{r})$$

For $\theta \to 0$, we have

$$\psi(\boldsymbol{r} - \theta\,\mathbb{G}_i\,\boldsymbol{r}) \approx \psi(\boldsymbol{r}) - \theta\,\mathbb{G}_i\,\boldsymbol{r}\cdot\nabla\,\psi(\boldsymbol{r}) \qquad \text{[Taylor expansion.]}$$
$$\approx \left(1 - \theta\,\hat{G}_i\right)\psi(\boldsymbol{r}) \qquad \text{[(1.429c) used.]}$$

$$\to \qquad \hat{G}_i = \mathbb{G}_i\,\boldsymbol{r}\cdot\nabla = (\mathbb{G}_i)_{jk}\,x_k\,\partial_j = \epsilon_{ijk}\,x_k\,\partial_j$$
$$= -\frac{i}{\hbar}\,\hat{L}_i \qquad (1.429d)$$

$\therefore \qquad \hat{U}_i(\theta) = e^{-i\theta \hat{L}_i/\hbar}$ (1.429e)

Thus, the quantized version of (1.426) is

$$\hat{U}(\alpha, \beta, \gamma) = e^{-i\alpha\hat{L}_3/\hbar}\, e^{-i\beta\hat{L}_2/\hbar}\, e^{-i\gamma\hat{L}_3/\hbar} \qquad (1.429)$$

$$\rightarrow \qquad \hat{U}^{+}(\alpha, \beta, \gamma) = e^{i\gamma\hat{L}_3/\hbar}\, e^{i\beta\hat{L}_2/\hbar}\, e^{i\alpha\hat{L}_3/\hbar}$$

Therefore, $\hat{U}(\alpha, \beta, \gamma)$ is unitary.

From (1.426) and (1.427), we get

$$\mathbb{R}(\alpha, \beta, \gamma) = e^{-\alpha\mathbb{G}_3}\, e^{-\beta\mathbb{G}_2}\, e^{-\gamma\mathbb{G}_3}$$

$$\rightarrow \qquad \partial_\alpha \mathbb{R} = -\mathbb{G}_3 \mathbb{R} \qquad (1.430a)$$

$$\partial_\beta \mathbb{R} = -e^{-\alpha\mathbb{G}_3}\, \mathbb{G}_2\, e^{-\beta\mathbb{G}_2}\, e^{-\gamma\mathbb{G}_3}$$
$$= -e^{-\alpha\mathbb{G}_3}\, \mathbb{G}_2\, e^{\alpha\mathbb{G}_3}\, \mathbb{R} \qquad (1.430b)$$

Using the Lie's expansion formula for matrices,

$$e^{-\mathbb{A}}\, \mathbb{B}\, e^{\mathbb{A}} = \mathbb{I} - [\mathbb{A}, \mathbb{B}] + \frac{1}{2!}[\mathbb{A}, [\mathbb{A}, \mathbb{B}]] + \ldots \qquad (1.432)$$

we have

$$e^{-\alpha\mathbb{G}_3}\, \mathbb{G}_2\, e^{\alpha\mathbb{G}_3} = \mathbb{I} - \alpha[\mathbb{G}_3, \mathbb{G}_2] + \frac{\alpha^2}{2}[\mathbb{G}_3, [\mathbb{G}_3, \mathbb{G}_2]] - \ldots \qquad (1.432a)$$

Using (1.428) to write out the explicit form of $\mathbb{G}_i$, we have

$$\mathbb{G}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \mathbb{G}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \mathbb{G}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (1.432b)$$

$$\rightarrow \qquad [\mathbb{G}_1, \mathbb{G}_2] = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathbb{G}_3$$

$$[\mathbb{G}_2, \mathbb{G}_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = -\mathbb{G}_1$$

$$[\mathbb{G}_3, \mathbb{G}_1] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -\mathbb{G}_2$$

In summary,

$$[\mathbb{G}_i, \mathbb{G}_j] = -\epsilon_{ijk}\, \mathbb{G}_k \qquad (1.432c)$$

The same result can also be obtained using (1.428) directly:

$$[\mathbb{G}_i, \mathbb{G}_j]_{mn} = \epsilon_{imk}\,\epsilon_{jkn} - \epsilon_{jmk}\,\epsilon_{ikn}$$
$$= \delta_{in}\,\delta_{mj} - \delta_{ij}\,\delta_{mn} + \delta_{ji}\,\delta_{mn} - \delta_{jn}\,\delta_{mi}$$
$$= \delta_{in}\,\delta_{mj} - \delta_{jn}\,\delta_{mi}$$
$$= -\epsilon_{ijk}\,\epsilon_{kmn}$$
$$= -(\epsilon_{ijk}\,\mathbb{G}_k)_{mn}$$

which is simply (1.432c) in component form.

(1.432a) thus becomes

$$e^{-\alpha\mathbb{G}_3}\, \mathbb{G}_2\, e^{\alpha\mathbb{G}_3} = \mathbb{I} - \alpha\,\mathbb{G}_1 - \frac{\alpha^2}{2!}\mathbb{G}_2 + \frac{\alpha^3}{3!}\mathbb{G}_1 + \frac{\alpha^4}{4!}\mathbb{G}_2 - \ldots$$
$$= \cos\alpha\, \mathbb{G}_2 - \sin\alpha\, \mathbb{G}_1 \qquad (1.431)$$

which turns (1.430b) into

$$\partial_\beta \mathbb{R} = (-\cos\alpha\, \mathbb{G}_2 + \sin\alpha\, \mathbb{G}_1)\, \mathbb{R} \qquad (1.430c)$$

Finally,

$$\partial_\gamma \mathbb{R} = -\, e^{-\alpha \mathbb{G}_3}\, e^{-\beta \mathbb{G}_2}\, \mathbb{G}_3\, e^{-\gamma \mathbb{G}_3}$$
$$= -\, e^{-\alpha \mathbb{G}_3}\, e^{-\beta \mathbb{G}_2}\, \mathbb{G}_3\, e^{\beta \mathbb{G}_2}\, e^{\alpha \mathbb{G}_3}\, \mathbb{R} \qquad (1.430d)$$

Using

$$e^{-\beta \mathbb{G}_2}\, \mathbb{G}_3\, e^{\beta \mathbb{G}_2} = \cos\beta\, \mathbb{G}_3 + \sin\beta\, \mathbb{G}_1 \qquad (1.433)$$
$$e^{-\alpha \mathbb{G}_3}\, \mathbb{G}_1\, e^{\alpha \mathbb{G}_3} = \cos\alpha\, \mathbb{G}_1 + \sin\alpha\, \mathbb{G}_2$$

(1.430d) becomes

$$\partial_\gamma \mathbb{R} = -\, e^{-\alpha \mathbb{G}_3} \left(\cos\beta\, \mathbb{G}_3 + \sin\beta\, \mathbb{G}_1\right) e^{\alpha \mathbb{G}_3}\, \mathbb{R}$$
$$= -\left[\cos\beta\, \mathbb{G}_3 + \sin\beta\, (\cos\alpha\, \mathbb{G}_1 + \sin\alpha\, \mathbb{G}_2)\right] \mathbb{R} \qquad (1.430e)$$

Since (1.430a) is valid for all $\mathbb{R}(\alpha, \beta, \gamma)$, the operator form of $\mathbb{G}_3$ is

$$\hat{G}_3 = -\partial_\alpha \qquad (1.434a)$$

Eliminating $\mathbb{G}_2$ from (1.430c & e) gives

$$\left(\sin\alpha\, \partial_\beta - \frac{\cos\alpha}{\sin\beta}\, \partial_\gamma\right) \mathbb{R} = (\cos\alpha \cot\beta\, \mathbb{G}_3 + \mathbb{G}_1)\, \mathbb{R}$$

which gives

$$\hat{G}_1 = \cos\alpha \cot\beta\, \partial_\alpha + \sin\alpha\, \partial_\beta - \frac{\cos\alpha}{\sin\beta}\, \partial_\gamma \qquad (1.434b)$$

Similarly, eliminating $\mathbb{G}_1$ from (1.430c & e) gives

$$\left(\cos\alpha\, \partial_\beta + \frac{\sin\alpha}{\sin\beta}\, \partial_\gamma\right) \mathbb{R} = (-\sin\alpha \cot\beta\, \mathbb{G}_3 - \mathbb{G}_2)\, \mathbb{R}$$

which gives

$$\hat{G}_2 = \sin\alpha \cot\beta\, \partial_\alpha - \cos\alpha\, \partial_\beta - \frac{\sin\alpha}{\sin\beta}\, \partial_\gamma \qquad (1.434c)$$

By (1.429d), we have

$$\hat{L}_1 = i\hbar \left(\cos\alpha \cot\beta\, \partial_\alpha + \sin\alpha\, \partial_\beta - \frac{\cos\alpha}{\sin\beta}\, \partial_\gamma\right)$$
$$\hat{L}_2 = i\hbar \left(\sin\alpha \cot\beta\, \partial_\alpha - \cos\alpha\, \partial_\beta - \frac{\sin\alpha}{\sin\beta}\, \partial_\gamma\right) \qquad (1.434)$$
$$\hat{L}_3 = -i\hbar\, \partial_\alpha$$

Putting (1.434) into (1.429) gives $\hat{U}(\alpha, \beta, \gamma)$ entirely in terms of the Euler angles.

Reminder: under a rotation $\mathbb{R}$, a vector $\boldsymbol{r}$ becomes another vector

$$\boldsymbol{r}' = \mathbb{R}\, \boldsymbol{r}$$

and a matrix $\mathbb{A}$ becomes another matrix

$$\mathbb{A}' = \mathbb{R}\, \mathbb{A}\, \mathbb{R}^{-1}$$

Let the body axes (ξ, η, ζ) coincide initially with the fixed Cartesian axes (x_1, x_2, x_3), respectively. After a rotation $\mathbb{R}(\alpha, \beta, \gamma)$, we have

$$\boldsymbol{\xi} = \mathbb{R}\, \boldsymbol{x}_1 \qquad \boldsymbol{\eta} = \mathbb{R}\, \boldsymbol{x}_2 \qquad \boldsymbol{\zeta} = \mathbb{R}\, \boldsymbol{x}_3$$

where $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ are unit vectors along the body axes and

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\rightarrow \qquad \xi_j = \mathbb{R}_{j1} \qquad \eta_j = \mathbb{R}_{j2} \qquad \zeta_j = \mathbb{R}_{j3} \qquad (1.436a)$$

Likewise, the generators along the Cartesian axes are transformed, like vectors, into ones along the body axes:

$$\begin{aligned}\mathbb{G}_\xi &= \mathbb{R}\,\mathbb{G}_1\,\mathbb{R}^{-1} = \mathbb{G}_j\,\mathbb{R}_{j1}\\ \mathbb{G}_\eta &= \mathbb{R}\,\mathbb{G}_2\,\mathbb{R}^{-1} = \mathbb{G}_j\,\mathbb{R}_{j2}\\ \mathbb{G}_\zeta &= \mathbb{R}\,\mathbb{G}_3\,\mathbb{R}^{-1} = \mathbb{G}_j\,\mathbb{R}_{j3}\end{aligned} \qquad (1.436b)$$

Doing the matrix multiplications in (1.426) gives

$$\mathbb{R} = \begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\gamma & -\sin\gamma & 0\\ \sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{pmatrix} \quad (1.436c)$$

$$= \begin{pmatrix}\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\\ -\cos\gamma\sin\beta & \sin\beta\sin\gamma & \cos\beta\end{pmatrix}$$

Hence,

$$\begin{aligned}\mathbb{G}_\xi &= (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma)\,\mathbb{G}_1 + (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma)\,\mathbb{G}_2\\ &\quad -\cos\gamma\sin\beta\,\mathbb{G}_3\\ &= \cos\beta\cos\gamma\,(\cos\alpha\,\mathbb{G}_1 + \sin\alpha\,\mathbb{G}_2) + \sin\gamma\,(-\sin\alpha\,\mathbb{G}_1 + \cos\alpha\,\mathbb{G}_2) - \cos\gamma\sin\beta\,\mathbb{G}_3\\ \mathbb{G}_\eta &= (-\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma)\,\mathbb{G}_1 + (-\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma)\,\mathbb{G}_2\\ &\quad +\sin\beta\sin\gamma\,\mathbb{G}_3\\ &= -\cos\beta\sin\gamma\,(\cos\alpha\,\mathbb{G}_1 + \sin\alpha\,\mathbb{G}_2) + \cos\gamma\,(-\sin\alpha\,\mathbb{G}_1 + \cos\alpha\,\mathbb{G}_2) + \sin\beta\sin\gamma\,\mathbb{G}_3\\ \mathbb{G}_\zeta &= \cos\alpha\sin\beta\,\mathbb{G}_1 + \sin\alpha\sin\beta\,\mathbb{G}_2 + \cos\beta\,\mathbb{G}_3\\ &= \sin\beta\,(\cos\alpha\,\mathbb{G}_1 + \sin\alpha\,\mathbb{G}_2) + \cos\beta\,\mathbb{G}_3\end{aligned} \qquad (1.436)$$

Using (1.434b & c), we have

$$\begin{aligned}\cos\alpha\,\hat{G}_1 + \sin\alpha\,\hat{G}_2 &= \cos\alpha\left(\cos\alpha\cot\beta\,\partial_\alpha + \sin\alpha\,\partial_\beta - \frac{\cos\alpha}{\sin\beta}\,\partial_\gamma\right)\\ &\quad +\sin\alpha\left(\sin\alpha\cot\beta\,\partial_\alpha - \cos\alpha\,\partial_\beta - \frac{\sin\alpha}{\sin\beta}\,\partial_\gamma\right)\\ &= \cot\beta\,\partial_\alpha - \frac{1}{\sin\beta}\,\partial_\gamma\\ -\sin\alpha\,\hat{G}_1 + \cos\alpha\,\hat{G}_2 &= -\sin\alpha\left(\cos\alpha\cot\beta\,\partial_\alpha + \sin\alpha\,\partial_\beta - \frac{\cos\alpha}{\sin\beta}\,\partial_\gamma\right)\\ &\quad +\cos\alpha\left(\sin\alpha\cot\beta\,\partial_\alpha - \cos\alpha\,\partial_\beta - \frac{\sin\alpha}{\sin\beta}\,\partial_\gamma\right)\\ &= -\partial_\beta\end{aligned}$$

Hence, the operator form of (1.436) becomes

$$\begin{aligned}\hat{G}_\xi &= \cos\beta\cos\gamma\left(\cot\beta\,\partial_\alpha - \frac{1}{\sin\beta}\,\partial_\gamma\right) - \sin\gamma\,\partial_\beta + \cos\gamma\sin\beta\,\partial_\alpha\\ &= \frac{\cos\gamma}{\sin\beta}\,\partial_\alpha - \sin\gamma\,\partial_\beta - \cos\gamma\cot\beta\,\partial_\gamma\\ \hat{G}_\eta &= -\cos\beta\sin\gamma\left(\cot\beta\,\partial_\alpha - \frac{1}{\sin\beta}\,\partial_\gamma\right) - \cos\gamma\,\partial_\beta - \sin\beta\sin\gamma\,\partial_\alpha\\ &= -\frac{\sin\gamma}{\sin\beta}\,\partial_\alpha - \cos\gamma\,\partial_\beta + \sin\gamma\cot\beta\,\partial_\gamma\end{aligned}$$

$$\hat{G}_\zeta = \sin\beta\left(\cot\beta\,\partial_\alpha - \frac{1}{\sin\beta}\,\partial_\gamma\right) - \cos\beta\,\partial_\alpha$$
$$= -\partial_\gamma$$

Analogous to (1.429d), we have

$$\hat{L}_\xi = i\hbar\,\hat{G}_\xi = i\hbar\left(\frac{\cos\gamma}{\sin\beta}\,\partial_\alpha - \sin\gamma\,\partial_\beta - \cos\gamma\,\cot\beta\,\partial_\gamma\right)$$
$$\hat{L}_\eta = i\hbar\,\hat{G}_\eta = i\hbar\left(-\frac{\sin\gamma}{\sin\beta}\,\partial_\alpha - \cos\gamma\,\partial_\beta + \sin\gamma\,\cot\beta\,\partial_\gamma\right) \qquad (1.437)$$
$$\hat{L}_\zeta = i\hbar\,\hat{G}_\zeta = -i\hbar\,\partial_\gamma$$

It is sometimes convenient to use a different notation:

$$(\tilde{\mathbb{G}}_1,\ \tilde{\mathbb{G}}_2,\ \tilde{\mathbb{G}}_3) = (\mathbb{G}_\xi,\ \mathbb{G}_\eta,\ \mathbb{G}_\zeta) \qquad (\boldsymbol{y}_1,\ \boldsymbol{y}_2,\ \boldsymbol{y}_3) = (\boldsymbol{\xi},\ \boldsymbol{\eta},\ \boldsymbol{\zeta})$$

(1.436b) then simplifies to

$$\tilde{\mathbb{G}}_i = \mathbb{R}\,\mathbb{G}_i\,\mathbb{R}^{-1} = \mathbb{G}_j\,\mathbb{R}_{j\,i} \qquad (1.439)$$

Using (1.432c), we have

$$\left[\tilde{\mathbb{G}}_i,\ \tilde{\mathbb{G}}_j\right] = [\mathbb{G}_m,\ \mathbb{G}_n]\,\mathbb{R}_{m\,i}\,\mathbb{R}_{n\,j}$$
$$= -\epsilon_{m\,n\,l}\,\mathbb{G}_l\,\mathbb{R}_{m\,i}\,\mathbb{R}_{n\,j}$$

(1.436a) shows that $\mathbb{R}_{m\,i}$ is the m^{th} component of $\boldsymbol{y}_i$. Hence

$$\epsilon_{m\,n\,l}\,\mathbb{R}_{m\,i}\,\mathbb{R}_{n\,j} = \epsilon_{m\,n\,l}\,(\boldsymbol{y}_i)_m\,(\boldsymbol{y}_j)_n = (\boldsymbol{y}_i \times \boldsymbol{y}_j)_l = \epsilon_{i\,j\,k}\,(\boldsymbol{y}_k)_l = \epsilon_{i\,j\,k}\,\mathbb{R}_{l\,k}$$

$$\to \qquad \left[\tilde{\mathbb{G}}_i,\ \tilde{\mathbb{G}}_j\right] = -\epsilon_{i\,j\,k}\,\mathbb{G}_l\,\mathbb{R}_{l\,k} = -\epsilon_{i\,j\,k}\,\tilde{\mathbb{G}}_k \qquad (1.438a)$$

Setting

$$\left(\hat{\tilde{L}}_1,\ \hat{\tilde{L}}_2,\ \hat{\tilde{L}}_3\right) = \left(\hat{L}_\xi,\ \hat{L}_\eta,\ \hat{L}_\zeta\right) = i\hbar\left(\hat{G}_\xi,\ \hat{G}_\eta,\ \hat{G}_\zeta\right) = i\hbar\left(\hat{\tilde{G}}_1,\ \hat{\tilde{G}}_2,\ \hat{\tilde{G}}_3\right)$$

the operator form of (1.438a) gives

$$\left[\hat{\tilde{L}}_i,\ \hat{\tilde{L}}_j\right] = -\hbar^2\left[\hat{\tilde{G}}_i,\ \hat{\tilde{G}}_j\right] = \hbar^2\,\epsilon_{i\,j\,k}\,\hat{\tilde{G}}_k = -i\hbar\,\epsilon_{i\,j\,k}\,\hat{\tilde{L}}_k \qquad (1.438)$$

which differs by a minus sign from the usual commutators relation (1.410).

Using (1.437), we have

$$\hat{L}_\xi^2 + \hat{L}_\eta^2 = -\frac{\hbar^2}{2\sin^2\beta}\left(2\,\frac{\partial^2}{\partial\,\alpha^2} + \sin 2\beta\,\frac{\partial}{\partial\,\beta} + (1 - \cos 2\beta)\,\frac{\partial^2}{\partial\,\beta^2}\right.$$
$$\left. - 4\cos\beta\,\frac{\partial^2}{\partial\,\alpha\,\partial\,\gamma} + 2\cos^2\beta\,\frac{\partial^2}{\partial\,\gamma^2}\right)$$
$$\hat{L}_\zeta^2 = -\hbar^2\,\frac{\partial^2}{\partial\,\gamma^2}$$

Hence,

$$\hat{\boldsymbol{L}}^2 = \hat{L}_\xi^2 + \hat{L}_\eta^2 + \hat{L}_\zeta^2$$
$$= -\frac{\hbar^2}{2\sin^2\beta}\left(2\,\frac{\partial^2}{\partial\,\alpha^2} + (1 - \cos 2\beta)\,\frac{\partial^2}{\partial\,\beta^2} + \sin 2\beta\,\frac{\partial}{\partial\,\beta}\right. \qquad (1.440a)$$
$$\left. - 4\cos\beta\,\frac{\partial^2}{\partial\,\alpha\,\partial\,\gamma} + 2\,\frac{\partial^2}{\partial\,\gamma^2}\right)$$

The Hamiltonian (1.423) becomes

$$\hat{H} = -\frac{\hbar^2}{4\sin^2\beta}\frac{1}{\mathcal{I}_\xi}\left(2\frac{\partial^2}{\partial\alpha^2} + (1-\cos 2\beta)\frac{\partial^2}{\partial\beta^2} + \sin 2\beta\frac{\partial}{\partial\beta}\right.$$
$$\left. -4\cos\beta\frac{\partial^2}{\partial\alpha\,\partial\gamma} + 2\cos^2\beta\frac{\partial^2}{\partial\gamma^2}\right) - \frac{\hbar^2}{2\mathcal{I}_\zeta}\frac{\partial^2}{\partial\gamma^2}$$
$$= \frac{1}{2\mathcal{I}_\xi}\hat{\boldsymbol{L}}^2 + \frac{\hbar^2}{2}\left(\frac{1}{\mathcal{I}_\xi} - \frac{1}{\mathcal{I}_\zeta}\right)\frac{\partial^2}{\partial\gamma^2} \qquad (1.440b)$$

It is well known that $\hat{\boldsymbol{L}}^2$ and $\hat{L}_3$ share the eigenstates $|\,L, m\rangle$ with

$$\hat{\boldsymbol{L}}^2\,|\,L, m\rangle = L(L+1)\,\hbar^2\,\Big|\,L\,m\Big\rangle \qquad L = 0, 1, \ldots \qquad (1.440c)$$
$$\hat{L}_3\,|\,L, m\rangle = m\,\hbar^2\,\Big|\,L\,m\Big\rangle \qquad m = -j, \ldots, j$$

For the spinning top, let $\boldsymbol{r}_0$ be position vector of some reference point on the top and set

$$\hat{U}(\alpha, \beta, \gamma)\,|\,\boldsymbol{r}_0\rangle = |\,R(\alpha, \beta, \gamma)\,\boldsymbol{r}_0\rangle \equiv |\,\alpha, \beta, \gamma\,\rangle \qquad (1.440d)$$

Operating with $\langle\,\boldsymbol{r}_0\,|\,\hat{U}^{-1}$ on (1.440c) gives

$$\langle\,\boldsymbol{r}_0\,|\,\hat{U}^{-1}\,\hat{\boldsymbol{L}}^2\,|\,L\,m\rangle = L(L+1)\,\hbar^2\,\langle\,\boldsymbol{r}_0\,|\,\hat{U}^{-1}\,|\,L\,m\rangle$$
$$\rightarrow \qquad \langle\,\alpha, \beta, \gamma\,|\,\hat{\boldsymbol{L}}^2\,|\,L\,m\rangle = L(L+1)\,\hbar^2\,\langle\,\alpha, \beta, \gamma\,|\,L\,m\rangle$$

In terms of the wave function

$$\psi_{L\,m}(\alpha, \beta, \gamma) = \langle\,\alpha, \beta, \gamma\,|\,L\,m\rangle \qquad (1.440e)$$

this becomes

$$\hat{\boldsymbol{L}}^2(\alpha, \beta, \gamma)\,\psi_{L\,m}(\alpha, \beta, \gamma) = L(L+1)\,\hbar^2\,\psi_{L\,m}(\alpha, \beta, \gamma) \qquad (1.440f)$$

where $\hat{\boldsymbol{L}}^2(\alpha, \beta, \gamma)$ is a differential operator in coordinates α, β, γ. Obviously, similarly equations also hold for $\hat{L}_3$ and $\hat{H}$.

On the other hand, operating with $\langle\,L\,m'\,|\,\hat{U}$ on (1.440c) gives

$$\langle\,L\,m'\,|\,\hat{U}\,\hat{\boldsymbol{L}}^2\,\hat{U}^{-1}\,\hat{U}\,|\,L\,m\rangle = L(L+1)\,\hbar^2\,\langle\,j\,m'\,|\,\hat{U}\,|\,L\,m\rangle$$
$$\rightarrow \qquad \hat{\boldsymbol{L}}^2(\alpha, \beta, \gamma)\,d^L_{m\,m'}(\alpha, \beta, \gamma) = L(L+1)\,\hbar^2\,D^j_{m\,m'}(\alpha, \beta, \gamma) \qquad (1.440g)$$

where

$$D^L_{m\,m'}(\alpha, \beta, \gamma) \equiv \langle\,L\,m'\,|\,\hat{U}(\alpha, \beta, \gamma)\,|\,L\,m\rangle \qquad (1.440h)$$

and $\hat{\boldsymbol{L}}^2\,(\alpha, \beta, \gamma)$ is the differential operator form of $\hat{U}\,\hat{\boldsymbol{L}}^2\,\hat{U}^{-1}$.

Comparing with (1.440g) with (1.440f), we see that $\psi_{j\,m}(\alpha, \beta, \gamma)$ is proportional to $D^L_{m\,m'}(\alpha, \beta, \gamma)$ and that the complete set of quantum numbers should be L, m, m'.

Using (1.429), (1.440h) becomes

$$D^L_{m\,m'}(\alpha, \beta, \gamma) = \langle\,L\,m\,|\,e^{-i\,\alpha\,\hat{L}_3/\hbar}\,e^{-i\,\beta\,\hat{L}_2/\hbar}\,e^{-i\,\gamma\,\hat{L}_3/\hbar}\,|\,L\,m'\rangle$$
$$= d^L_{m\,m'}(\beta)\,e^{-i\,m\,\alpha - i\,m'\,\gamma} \qquad (1.442)$$

where

$$d^L_{m\,m'}(\beta) = \langle\,L\,m\,|\,e^{-i\,\beta\,\hat{L}_2/\hbar}\,|\,L\,m'\rangle$$

On the other hand, using (1.442) on (1.440a) gives

$$\rightarrow \qquad \hat{\boldsymbol{L}}^2\,d^j_{m\,m'}(\beta) = -\frac{\hbar^2}{2\sin^2\beta}\left(-2\,m^2 + (1-\cos 2\beta)\frac{\partial^2}{\partial\beta^2} + \sin 2\beta\frac{\partial}{\partial\beta}\right.$$
$$\left. + 4\,m\,m'\cos\beta - 2\,m'^2\right)d^L_{m\,m'}(\beta)$$

$$= -\hbar^2 \left(\frac{\partial^2}{\partial \beta^2} + \cot\beta \frac{\partial}{\partial \beta} - \frac{m^2 + m'^2 - 2 m m' \cos\beta}{\sin^2 \beta} \right) d^L_{m m'}(\beta)$$
$$= L(L+1)\, \hbar^2 \, d^L_{m m'}(\beta)$$

Therefore,

$$\left(\frac{d^2}{d \beta^2} + \cot\beta \frac{d}{d \beta} + L(L+1) - \frac{m^2 + m'^2 - 2 m m' \cos\beta}{\sin^2 \beta} \right) d^L_{m m'}(\beta) = 0 \quad (1.451)$$

Let

$$d^L_{m m'}(\beta) = \left(\cos \frac{\beta}{2}\right)^{m+m'} \left(\sin \frac{\beta}{2}\right)^{m-m'} g(\beta) \qquad (1.451a)$$

then (1.451) becomes

$$\frac{d^2 g}{d \beta^2} + \left[(2m+1)\cot\beta - \frac{2m'}{\sin\beta} \right] \frac{d g}{d \beta} + [\, L(L+1) - m(m+1) \,]\, g = 0 \qquad (1.451b)$$

Setting $x = \cos\beta$ turns it into

$$(1-x)^2 \frac{d^2 g}{d x^2} + 2\,[\, m' - (m+1)\, x \,] \frac{d g}{d x} + [\, L(L+1) - m(m+1) \,]\, g = 0 \qquad (1.451c)$$

Comparing with the solution $P_n^{(a,b)}$ of the Jacobi equation

$$\left(1 - x^2\right) y'' + [\, b - a - (a+b+2)\, x \,]\, y' + n\,(n+a+b+1)\, y = 0 \qquad (1.451d)$$

we see that

$$2m' = b - a \qquad\qquad 2\,(m+1) = a + b + 2$$
$$L(L+1) - m(m+1) = n\,(n+a+b+1)$$

The solution with $n \geq 0$ is

$$a = m - m' \qquad\qquad b = m + m' \qquad\qquad n = L - m$$
$$\rightarrow \quad g(x) = P_{L-m}^{(m-m',\, m+m')}(x) \qquad (1.451e)$$

$P_n^{(a,b)}$ are called the Jacobi polynomials and can be written as

$$P_n^{(a,b)}(x) = \frac{(-)^n}{n!} \frac{\Gamma(n+b+1)}{\Gamma(b+1)} F\left(-n,\, n+1+a+b\,;\; 1+b\,;\; \frac{1}{2}(1+x)\right) \quad (1.449)$$

where the hypergeometric functions are defined as

$$F(\, a,\, b;\; c\,;\, x\,) = 1 + \frac{a\, b}{c} x + \frac{a(a+1)\, b(b+1)}{c(c+1)} \frac{x^2}{2!} + \ldots \qquad (1.450)$$

To summarize, solutions to (1.451) are

$$d^L_{m m'}(\beta) = A^L_{m m'} \left(\cos \frac{\beta}{2}\right)^{m+m'} \left(\sin \frac{\beta}{2}\right)^{m-m'} P_{L-m}^{(m-m',\, m+m')}(\cos\beta) \quad (1.443)$$

where

$$A^L_{m m'} = \sqrt{\frac{(L+m')!\,(L-m')!}{(L+m)!\,(L-m)!}} \qquad (1.443a)$$

is a normalization constant chosen such that

$$\int d\tau\, D^{L_1 *}_{m_1 m_1'}\, D^{L_2}_{m_2 m_2'} = \frac{8\pi^2}{2L_1 + 1} \delta_{L_1 L_2}\, \delta_{m_1 m_1'}\, \delta_{m_2 m_2'} \qquad (1.453)$$

where

$$\int d\tau = \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma \qquad (1.453a)$$

is the measure for the inner product

$$\langle \psi \mid \phi \rangle = \int d\tau\, \psi^* \phi \tag{1.452}$$

Caution: Note the subtle differences between ours & Kleinert's (1.443).
Numerical justification of our choice can be found in "1.15.Code.nb".

(1.440b) indicates that $D^L_{m\,m'}$ of (1.442) are also the eigenstates of $\hat{H}$ with eigenvalues

$$E_{L\,m'} = \frac{\hbar^2}{2\,\mathcal{I}_\xi} L(L+1) + \frac{\hbar^2}{2}\left(\frac{1}{\mathcal{I}_\xi} - \frac{1}{\mathcal{I}_\zeta}\right) m'^2 \tag{1.440}$$

It is well known that the values of L can be generalized to half-integers. In which case, the simultaneous eigenstates of $\hat{\boldsymbol{L}}^2$ and $\hat{L}_3$ are usually denoted as $\mid j\,m\rangle$.

For $j = \frac{1}{2}$, (1.443) becomes

$$\mathbb{d}^{1/2} = \left(d^{1/2}_{m\,m'}\right) = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \tag{1.444}$$

where $m, m' = -\frac{1}{2}$ or $\frac{1}{2}$.

For generators that satisfy the commutation relations (1.432c), one can use

$$\mathbb{G}_j = i\,\frac{1}{2}\,\mathbb{s}_j \qquad j = 1, 2, 3$$

where the Pauli spin matrices are given by

$$\mathbb{s}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbb{s}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathbb{s}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.445}$$

so that (1.426) becomes

$$\mathbb{R} = \mathbb{D}^{1/2}(\alpha, \beta, \gamma) = e^{-i\,\alpha\,\mathbb{s}_3/2}\, e^{-i\,\beta\,\mathbb{s}_2/2}\, e^{-i\,\gamma\,\mathbb{s}_3/2} \tag{1.446}$$

where the elements of $\mathbb{D}^{1/2}$ are expected to be given by (1.442) with $m, m' = -\frac{1}{2}$ or $\frac{1}{2}$.

Using (1.445), we have

$$(\mathbb{s}_3)^2 = \mathbb{I} \qquad \rightarrow \qquad (\mathbb{s}_3)^{2n} = \mathbb{I} \qquad \& \qquad (\mathbb{s}_3)^{2n+1} = \mathbb{s}_3$$

Hence,

$$\begin{aligned} e^{-i\,\alpha\,\mathbb{s}_3/2} &= \sum_{k=0}^{\infty} \frac{(-i\,\alpha/2)^k}{k!} (\mathbb{s}_3)^k \\ &= \sum_{n=0}^{\infty} \frac{(-i\,\alpha/2)^{2n}}{(2n)!}\,\mathbb{I} + \sum_{n=0}^{\infty} \frac{(-i\,\alpha/2)^{2n+1}}{(2n+1)!}\,\mathbb{s}_3 \\ &= \sum_{n=0}^{\infty} \frac{(-)^n (\alpha/2)^{2n}}{(2n)!}\,\mathbb{I} - i \sum_{n=0}^{\infty} \frac{(-)^n (\alpha/2)^{2n+1}}{(2n+1)!}\,\mathbb{s}_3 \\ &= \cos\frac{\alpha}{2}\,\mathbb{I} - i \sin\frac{\alpha}{2}\,\mathbb{s}_3 \qquad (1.447) \\ &= \begin{pmatrix} e^{-i\,\alpha/2} & 0 \\ 0 & e^{i\,\alpha/2} \end{pmatrix} \end{aligned}$$

Similarly,

$$(\mathbb{s}_2)^2 = \mathbb{I} \qquad \rightarrow \qquad (\mathbb{s}_2)^{2n} = \mathbb{I} \qquad \& \qquad (\mathbb{s}_2)^{2n+1} = \mathbb{s}_2$$

Hence,

$$\begin{aligned} e^{-i\beta \mathbb{s}_2/2} &= \sum_{k=0}^{\infty} \frac{(-i\beta/2)^k}{k!} (\mathbb{s}_2)^k \\ &= \sum_{n=0}^{\infty} \frac{(-i\beta/2)^{2n}}{(2n)!} \mathbb{I} + \sum_{n=0}^{\infty} \frac{(-i\beta/2)^{2n+1}}{(2n+1)!} \mathbb{s}_2 \\ &= \sum_{n=0}^{\infty} \frac{(-)^n(\beta/2)^{2n}}{(2n)!} \mathbb{I} - i \sum_{n=0}^{\infty} \frac{(-)^n(\beta/2)^{2n+1}}{(2n+1)!} \mathbb{s}_2 \\ &= \cos\frac{\beta}{2} \mathbb{I} - i \sin\frac{\beta}{2} \mathbb{s}_2 \qquad (1.447) \\ &= \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} = \mathbb{d}^{1/2} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{D}^{1/2}(\alpha, \beta, \gamma) &= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\alpha/2} e^{-i\gamma/2} \cos\frac{\beta}{2} & e^{-i\alpha/2} e^{i\gamma/2}\left(-\sin\frac{\beta}{2}\right) \\ e^{i\alpha/2} e^{-i\gamma/2} \sin\frac{\beta}{2} & e^{i\alpha/2} e^{i\gamma/2} \cos\frac{\beta}{2} \end{pmatrix} \end{aligned}$$

which agrees with (1.442), as claimed earlier.

For $j = 1$, (1.443) becomes

$$\mathbb{d}^1 = \left(d^1_{m\,m'}\right) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix} \qquad (1.448)$$

where $m, m' = -1, 0, 1$.

Corresponding matrix with respect to the Cartesian basis $(\hat{\boldsymbol{x}}_i)$ is simply $\mathbb{R}_2(\beta)$. Expressing the basis $\{\,|\,1\,m\rangle\}$ as 3-D unit vectors $\{\hat{\boldsymbol{\epsilon}}_m\}$, we have

$$\hat{\boldsymbol{\epsilon}}_{\pm 1} = \mp\frac{1}{2}(\hat{\boldsymbol{x}} \pm i\,\hat{\boldsymbol{y}}) \qquad \hat{\boldsymbol{\epsilon}}_0 = \hat{\boldsymbol{z}}$$

so that

$$R(\beta)\,\hat{\boldsymbol{\epsilon}}_m = \sum_{m'} \hat{\boldsymbol{\epsilon}}_{m'}\, d^1_{m'\,m}(\beta)$$

Canonical Approach

Many of the results in this section are derived using the *Mathematica* code in file “1.15._Code.nb”.

Using

$$L_k^2 = \mathcal{I}_k\,\omega_k^2$$

the Hamiltonian (1.423) corresponds to the classical Lagrangian

$$L = \frac{1}{2}\left[\mathcal{I}_\xi \left(\omega_\xi^2 + \omega_\eta^2 \right) + \mathcal{I}_\zeta \, \omega_\zeta^2 \right] \tag{1.454}$$

where ω_ξ , ω_η , ω_ζ are the angular velocities around the principal axes.

From the definition of the Euler angles, we see that

1. $R_3(\alpha)$ produces an angular velocity $\dot{\alpha}\,\hat{\boldsymbol{x}}_3$.
2. $R_\eta(\beta)$ produces $\dot{\beta}\,\hat{\boldsymbol{\eta}} = \dot{\beta}\,\mathbb{R}_3(\alpha)\,\hat{\boldsymbol{x}}_2 = \dot{\beta}\left(-\sin\alpha\,\hat{\boldsymbol{x}}_1 + \cos\alpha\,\hat{\boldsymbol{x}}_2\right)$.
3. $R_\zeta(\gamma)$ produces $\dot{\gamma}\,\hat{\boldsymbol{\zeta}} = \dot{\gamma}\,\mathbb{R}\,\hat{\boldsymbol{x}}_3 = \dot{\gamma}\left[\sin\beta\left(\cos\alpha\,\hat{\boldsymbol{x}}_1 + \sin\alpha\,\hat{\boldsymbol{x}}_2\right) + \cos\beta\,\hat{\boldsymbol{x}}_3\right]$.

The total angular velocity therefore has components in the rest frame

$$\begin{aligned} \omega_1 &= -\dot{\beta}\sin\alpha + \dot{\gamma}\sin\beta\cos\alpha \\ \omega_2 &= \dot{\beta}\cos\alpha + \dot{\gamma}\sin\beta\sin\alpha \\ \omega_3 &= \dot{\alpha} + \dot{\gamma}\cos\beta \end{aligned} \tag{1.456}$$

By definition, the final (red) body axes are related to the fixed frame axes by

$$\hat{\boldsymbol{\mu}}_a = \hat{\boldsymbol{x}}_j\,\mathbb{R}_{j\,a} \tag{1.456a}$$

where, with the color red suppressed,

$$(\hat{\boldsymbol{\mu}}_a) = \left(\hat{\boldsymbol{\mu}}_1,\ \hat{\boldsymbol{\mu}}_2,\ \hat{\boldsymbol{\mu}}_3\right) = \left(\hat{\boldsymbol{\xi}},\ \hat{\boldsymbol{\eta}},\ \hat{\boldsymbol{\zeta}}\right) \tag{1.456b}$$

and we shall allow the dummy index a to assume two sets of equivalent values:

$$a = 1,\ 2,\ 3\,, \quad \text{or} \quad \xi,\ \eta,\ \zeta$$

Inverting (1.456a), we have

$$\hat{\boldsymbol{x}}_j = \hat{\boldsymbol{\mu}}_a\,\mathbb{R}^{-1}_{a j} \tag{1.456c}$$

Using

$$\boldsymbol{\omega} = \omega_j\,\hat{\boldsymbol{x}}_j = \omega_a\,\hat{\boldsymbol{\mu}}_a = \omega_j\,\hat{\boldsymbol{\mu}}_a\,\mathbb{R}^{-1}_{a j}$$

$$\rightarrow \qquad \omega_a = \omega_j\,\mathbb{R}^{-1}_{a j}$$

we have

$$\begin{aligned} \omega_\xi &= \dot{\beta}\sin\gamma - \dot{\alpha}\sin\beta\cos\gamma \\ \omega_\eta &= \dot{\beta}\cos\gamma + \dot{\alpha}\sin\beta\sin\gamma \\ \omega_\zeta &= \dot{\alpha}\cos\beta + \dot{\gamma} \end{aligned} \tag{1.457}$$

(1.454) thus becomes

$$\begin{aligned} L &= \frac{1}{2}\left[\mathcal{I}_\xi \left(\dot{\beta}^2 + \dot{\alpha}^2 \sin^2\beta \right) + \mathcal{I}_\zeta (\,\dot{\alpha}\cos\beta + \dot{\gamma})^2 \right] \\ &\equiv g_{\mu\,\nu}\,\dot{q}^\mu\,\dot{q}^\nu \end{aligned} \tag{1.458}$$

where

$$(\,q^\mu\,) = (\,\alpha,\ \beta,\ \gamma)$$

and

$$\mathbb{g} = (\,g_{\mu\nu}) = \begin{pmatrix} \mathcal{I}_\xi \sin^2\beta + \mathcal{I}_\zeta \cos^2\beta & 0 & \mathcal{I}_\zeta \cos\beta \\ 0 & \mathcal{I}_\xi & 0 \\ \mathcal{I}_\zeta \cos\beta & 0 & \mathcal{I}_\zeta \end{pmatrix} \tag{1.459}$$

with inverse

$$\mathbb{g}^{-1} = (\,g^{\mu\nu}) = \frac{1}{\mathcal{I}_\xi \sin^2\beta} \begin{pmatrix} 1 & 0 & -\cos\beta \\ 0 & \sin^2\beta & 0 \\ -\cos\beta & 0 & \cos^2\beta + \frac{\mathcal{I}_\xi}{\mathcal{I}_\zeta}\sin^2\beta \end{pmatrix} \tag{1.462}$$

and

$$g = \det\mathbb{g} = \mathcal{I}_\xi^2\,\mathcal{I}_\zeta \sin^2\beta \tag{1.460}$$

Hence, the measure for the inner product is

$$\int d\tau = \int d^3 q \sqrt{g} = \mathcal{I}_\xi \sqrt{\mathcal{I}_\zeta} \int d\alpha \int d\beta \sin\beta \int d\gamma$$

in agreement with (1.452) aside from the constant factor $\mathcal{I}_\xi \sqrt{\mathcal{I}_\zeta}$.

The canonical momenta are

$$\begin{aligned}
p_\alpha &= \frac{\partial L}{\partial \dot{\alpha}} = \mathcal{I}_\xi \dot{\alpha} \sin^2\beta + \mathcal{I}_\zeta \cos\beta (\dot{\alpha}\cos\beta + \dot{\gamma}) \\
p_\beta &= \frac{\partial L}{\partial \dot{\beta}} = \mathcal{I}_\xi \dot{\beta} \\
p_\gamma &= \frac{\partial L}{\partial \dot{\gamma}} = \mathcal{I}_\zeta(\dot{\alpha}\cos\beta + \dot{\gamma})
\end{aligned} \tag{1.461}$$

The classical Hamiltonian associated with the Lagrangian of (1.458) is

$$H_{\text{can}} = \frac{1}{2\mathcal{I}_\xi}\left[\frac{1}{\sin^2\beta} p_\alpha^2 + p_\beta^2 + \left(\cot^2\beta + \frac{\mathcal{I}_\xi}{\mathcal{I}_\zeta}\right) p_\gamma^2 - \frac{2\cos\beta}{\sin^2\beta} p_\alpha p_\gamma\right] \tag{1.463}$$

Since there is no apparent ordering problem, one follows (1.387) and set

$$\begin{aligned}
\hat{p}_\alpha &= \frac{\hbar}{i}\frac{\partial}{\partial\alpha} \\
\hat{p}_\beta &= \frac{\hbar}{i}\frac{1}{\sqrt{\sin\beta}}\frac{\partial}{\partial\beta}\sqrt{\sin\beta} = \frac{\hbar}{i}\left(\frac{\partial}{\partial\beta} + \frac{1}{2}\cot\beta\right) \\
\hat{p}_\gamma &= \frac{\hbar}{i}\frac{\partial}{\partial\gamma}
\end{aligned} \tag{1.464}$$

Hence,

$$\hat{p}_\beta^2 = -\hbar^2\left(\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} - \frac{1}{4}\left(1 + \csc^2\beta\right)\right) \tag{1.464a}$$

and (1.463) becomes

$$\begin{aligned}
H_{\text{can}} = -\frac{\hbar^2}{2\mathcal{I}_\xi}\Bigg[& \frac{1}{\sin^2\beta}\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} - \frac{1}{4}\left(1 + \csc^2\beta\right) \\
& + \left(\cot^2\beta + \frac{\mathcal{I}_\xi}{\mathcal{I}_\zeta}\right)\frac{\partial^2}{\partial\gamma^2} - \frac{2\cos\beta}{\sin^2\beta}\frac{\partial^2}{\partial\alpha\,\partial\gamma}\Bigg]
\end{aligned} \tag{1.466a}$$

Using the operator form of $\left(\hat{L}_\xi, \hat{L}_\eta, \hat{L}_\zeta\right)$ in (1.437) on the correct Hamiltonian (1.423), we get

$$\begin{aligned}
\hat{H} = -\frac{\hbar^2}{2\mathcal{I}_\xi}\Bigg[& \frac{1}{\sin^2\beta}\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + \left(\frac{\mathcal{I}_\xi}{\mathcal{I}_\zeta} + \cot^2\beta\right)\frac{\partial^2}{\partial\gamma^2} \\
& - \frac{2\cos\beta}{\mathcal{I}_\xi\sin^2\beta}\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\gamma}\Bigg]
\end{aligned} \tag{1.466}$$

which can also be obtained by quantizing p_β^2 as $g^{-1/4}\,\hat{p}_\beta\, g^{1/2}\,\hat{p}_\beta\, g^{-1/4}$ instead of (1.464a) [see (1.394)]. The discrepancy is

$$\hat{H}_{\text{disc}} = \hat{H}_{\text{can}} - \hat{H} = \frac{\hbar^2}{8\mathcal{I}_\xi}\left(1 + \csc^2\beta\right) \tag{1.467}$$

See p.66 of Kleinert’s text for further discussions.