1.16. Scattering

1.16.1. Scattering Matrix

In a scattering process, an incident free particle with momentum $\boldsymbol{p}_a$ collides with a localized potential and emerges as a free particle with momentum $\boldsymbol{p}_b$. Energy conservation requires

$$E = E_a = \frac{\boldsymbol{p}_a^2}{2m} = E_b = \frac{\boldsymbol{p}_b^2}{2m}$$

The probability amplitude of the process is

$$(\boldsymbol{p}_b\, t_b \mid \boldsymbol{p}_a\, t_a) = \left\langle \boldsymbol{p}_b \left| e^{-i\hat{H}(t_b - t_a)/\hbar} \right| \boldsymbol{p}_a \right\rangle \tag{1.470}$$

with the assumptions that $(t_b, t_a) \to (\infty, -\infty)$ and the potential is centered at the origin.

In the absence of the potential center, (1.470) becomes

$$\begin{aligned}(\boldsymbol{p}_b\, t_b \mid \boldsymbol{p}_a\, t_a)_0 &= \left\langle \boldsymbol{p}_b \left| e^{-i\hat{H}_0(t_b - t_a)/\hbar} \right| \boldsymbol{p}_a \right\rangle \\ &= \left\langle \boldsymbol{p}_b \left| e^{-i\hat{H}_0 t_b/\hbar} e^{i\hat{H}_0 t_a/\hbar} \right| \boldsymbol{p}_a \right\rangle \\ &= e^{-i(E_b t_b - E_a t_a)/\hbar} (2\pi\hbar)^3\, \delta(\boldsymbol{p}_b - \boldsymbol{p}_a) \end{aligned} \tag{1.470a}$$

where

$$\langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle = (2\pi\hbar)^3\, \delta(\boldsymbol{p}_b - \boldsymbol{p}_a) \tag{1.473}$$

The exponential factor fluctuates wildly as $(t_b, t_a) \to (\infty, -\infty)$. However, since its absolute magnitude is 1, it does not contribute to the probability of the process. We therefore remove it from (1.470) and define the scattering matrix (S-matrix) as

$$\begin{aligned}\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle &= \lim_{(t_b, t_a) \to (\infty, -\infty)} e^{i(E_b t_b - E_a t_a)/\hbar} (\boldsymbol{p}_b\, t_b \mid \boldsymbol{p}_a\, t_a) \\ &= \lim_{(t_b, t_a) \to (\infty, -\infty)} e^{i(E_b t_b - E_a t_a)/\hbar} \left\langle \boldsymbol{p}_b \left| e^{-i\hat{H}(t_b - t_a)/\hbar} \right| \boldsymbol{p}_a \right\rangle \qquad (1.471) \\ &= \lim_{(t_b, t_a) \to (\infty, -\infty)} \left\langle \boldsymbol{p}_b \left| e^{i\hat{H}_0 t_b/\hbar} e^{-i\hat{H}(t_b - t_a)/\hbar} e^{-i\hat{H}_0 t_a/\hbar} \right| \boldsymbol{p}_a \right\rangle \end{aligned}$$

Since $\hat{H} \to \hat{H}_0$ far away from the origin, we expect (1.471) contains (1.470a) as a leading term. We therefore define the T-matrix by

$$\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle = \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle + \left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle' \tag{1.472}$$

$$= (2\pi\hbar)^3\, \delta(\boldsymbol{p}_b - \boldsymbol{p}_a) - 2\pi i\, \delta(E_b - E_a) \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle \tag{1.474}$$

where the prefactor $2\pi i\, \delta(E_b - E_a)$ was introduced for later convenience.

From the definition (1.471), we see that the $\hat{S}$ is unitary if $\hat{H}$ is hermitian. The probability of finding the particle ends up in free state b is

$$\begin{aligned}\mathcal{P}_{a\to b} &= |\, (\boldsymbol{p}_b\, t_b \mid \boldsymbol{p}_a\, t_a)\, |^2 \\ &= \left| \left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle \right|^2 \\ &= \left\langle \boldsymbol{p}_a \mid \hat{S}^+ \mid \boldsymbol{p}_b \right\rangle \left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle \end{aligned} \tag{1.474a}$$

Assuming the momenta can take on only discrete values so that

$$\sum_{\boldsymbol{p}} |\, \boldsymbol{p} \rangle \langle \boldsymbol{p} \,| = \boldsymbol{I} \qquad\qquad \langle \boldsymbol{p} \mid \boldsymbol{p}' \rangle = \delta_{\boldsymbol{p}\boldsymbol{p}'}$$

the total probability of the particle ending up in some free state b is therefore

$$\mathcal{P} = \sum_b \mathcal{P}_{a\to b} = \sum_{\boldsymbol{p}_b} \left\langle \boldsymbol{p}_a \mid \hat{S}^+ \mid \boldsymbol{p}_b \right\rangle \left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle$$
$$= \left\langle \boldsymbol{p}_a \,\middle|\, \hat{S}^+ \hat{S} \,\middle|\, \boldsymbol{p}_a \right\rangle$$
$$= \langle \boldsymbol{p}_a \mid \boldsymbol{p}_a \rangle \qquad \text{if } \hat{H} \text{ is hermitian}$$
$$= 1 \tag{1.475}$$

which means the particle is always ends up in some free particle state and not trapped at the potential center (if $P < 1$) nor inducing other emission particles (if $P > 1$).

If the system is confined in a finite but large volume L^3, the momentum $\boldsymbol{p}$ becomes quasi- continuous so that

$$\frac{L^3}{(2\pi\hbar)^3} \int d^3 p \mid \boldsymbol{p}\rangle\langle \boldsymbol{p} \mid = \boldsymbol{I} \qquad \int d^3 r \mid \boldsymbol{r}\rangle\langle \boldsymbol{r} \mid = \boldsymbol{I}$$

$$\langle \boldsymbol{p} \mid \boldsymbol{p}' \rangle = \delta_{\boldsymbol{p}\boldsymbol{p}'} \qquad \langle \boldsymbol{r} \mid \boldsymbol{r}' \rangle = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$\to \qquad \psi_{\boldsymbol{p}}(\boldsymbol{r}) = \frac{e^{i\,\boldsymbol{p}\cdot\boldsymbol{r}/\hbar}}{\sqrt{L^3}} \tag{1.475a}$$

The total probability of the particle ending up in some free state b becomes

$$\mathcal{P} = \frac{L^3}{(2\pi\hbar)^3} \int d^3 p_b \left\langle \boldsymbol{p}_a \mid \hat{S}^+ \mid \boldsymbol{p}_b \right\rangle \left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle = 1$$

as in (1.475) and with the same interpretation.

1.16.2. Cross Section

From (1.474), we see that the probability $\mathcal{P}_{a\to b}$ of (1.474a) includes a contribution from unscattered particles. We therefore define the probability for scattered particles as

$$P_{a\to b} = \left| 2\pi\hbar\,\delta(E_b - E_a) \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle \right|^2$$
$$= 2\pi\hbar\,\delta(0)\, 2\pi\,\delta(E_b - E_a) \left| \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle \right|^2 \tag{1.478}$$

Using

$$\delta(E) = \lim_{T\to\infty} \int_{-T/2}^{T/2} \frac{dt}{2\pi\hbar} e^{i E t/\hbar}$$

we have

$$2\pi\hbar\,\delta(0) = \lim_{T\to\infty} T$$

so that (1.478) becomes

$$\frac{P_{a\to b}}{T} = 2\pi\,\delta(E_b - E_a) \left| \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle \right|^2 \tag{1.479}$$

For a time-independent $\hat{H}$, the total probability per unit time is therefore

$$\frac{dP}{dt} = \sum_b \frac{P_{a\to b}}{T} = \frac{L^3}{(2\pi\hbar)^3} \int d^3 p_b \frac{P_{a\to b}}{T}$$
$$= \frac{L^3}{(2\pi\hbar)^3} \int d^3 p_b\, 2\pi\,\delta(E_b - E_a) \left| \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle \right|^2 \tag{1.480}$$

Using

$$E_b = \frac{p_b^2}{2M} \qquad \to \qquad dE_b = \frac{p_b}{M} dp_b$$

we have

$$d^3 p_b = p_b^2 \, d p_b \, d\Omega \; = M p_b \, d E_b \, d\Omega \qquad (1.481)$$

where $d\Omega = \sin\theta_b \, d\theta_b \, d\phi_b$ is the solid angle element in the direction of $\boldsymbol{p}_b$.

(1.480) thus becomes

$$\frac{dP}{dt} = \frac{L^3}{(2\pi\hbar)^3} \int d\Omega \, 2\pi M p_b \left| \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle \right|^2 \qquad (1.480a)$$

The differential scattering cross section $\frac{d\sigma}{d\Omega}$ is defined as the probability that a single impinging particle ends up in a solid angle element $d\Omega$ per unit time and unit current density j, i.e.,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{j} \frac{d}{d\Omega} \left(\frac{dP}{dt} \right) \\ &= \frac{1}{j} \frac{L^3}{(2\pi\hbar)^2 \hbar} M p_b \, | T_{\boldsymbol{p}_b \boldsymbol{p}_a} |^2 \end{aligned} \qquad (1.482)$$

where

$$T_{\boldsymbol{p}_b \boldsymbol{p}_a} = \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle \qquad (1.483)$$

Using

$$\boldsymbol{j} = \rho \, \boldsymbol{v} = | \psi_{\boldsymbol{p}_a}(\boldsymbol{r}) |^2 \frac{\boldsymbol{p}_a}{M} = \frac{\boldsymbol{p}_a}{M L^3} \qquad \text{and} \qquad p_b = p_a \qquad (1.484)$$

(1.482) becomes

$$\frac{d\sigma}{d\Omega} = \frac{M^2 L^6}{(2\pi\hbar)^2 \hbar} | T_{\boldsymbol{p}_b \boldsymbol{p}_a} |^2 \qquad (1.485)$$

Note that by (1.475a), the factor L^6 is cancelled by the plane waves contained in $| T_{\boldsymbol{p}_b \boldsymbol{p}_a} |^2$.

For relativistic particles,

$$E = \sqrt{p^2 c^2 + M^2 c^4} \qquad \rightarrow \qquad dE = \frac{c^2}{E} p \, dp$$

(1.481) thus becomes

$$d^3 p_b = p_b^2 \, d p_b \, d\Omega \; = \frac{E p_b}{c^2} dE \, d\Omega \qquad (1.486)$$

$\boldsymbol{v}$ and $\boldsymbol{p}$ are related by

$$\boldsymbol{p} = \frac{M \boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad E = \frac{M c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\rightarrow \qquad \boldsymbol{v} = \boldsymbol{p} \frac{c^2}{E}$$

(1.484) thus becomes

$$\boldsymbol{j} = \boldsymbol{p}_a \frac{c^2}{E L^3} \qquad (1.487)$$

These turn the cross section (1.485) into

$$\frac{d\sigma}{d\Omega} = \frac{E^2 L^6}{(2\pi\hbar)^2 \hbar} | T_{\boldsymbol{p}_b \boldsymbol{p}_a} |^2 \qquad (1.488)$$

1.16.3. Born Approximation

(1.471) suggests the definition

$$\hat{S} = \lim_{(t_b,\, t_a) \to (\infty,\, -\infty)} \hat{S}(t_b, t_a)$$

where

$$\hat{S}(t_b, t_a) = e^{i \hat{H}_0 t_b / \hbar} e^{-i \hat{H}(t_b - t_a) / \hbar} e^{-i \hat{H}_0 t_a / \hbar}$$
$$= e^{i \hat{H}_0 t_b / \hbar} \hat{U}(t_b, t_a) e^{-i \hat{H}_0 t_a / \hbar} \qquad (1.489a)$$

Using

$$i \hbar \frac{\partial}{\partial t} \hat{U}(t, t_a) = \hat{H}(t) \hat{U}(t, t_a) \qquad \hat{H}(t) = \hat{H}_0 + \hat{V}(t)$$

we have

$$i \hbar \frac{\partial}{\partial t} \hat{S}(t, t_a) = e^{i \hat{H}_0 t / \hbar} \hat{V}(t) \hat{U}(t, t_a) e^{-i \hat{H}_0 t_a / \hbar}$$
$$= e^{i \hat{H}_0 t / \hbar} \hat{V}(t) e^{-i \hat{H}_0 t / \hbar} \hat{S}(t, t_a)$$
$$= \hat{V}_{\mathcal{I}}(t) \hat{S}(t, t_a) \qquad (1.489b)$$

where we have introduced the interaction picture of $\hat{O}(t)$ as

$$\hat{O}_{\mathcal{I}}(t) = e^{i \hat{H}_0 t / \hbar} \hat{O}(t) e^{-i \hat{H}_0 t / \hbar} \qquad (1.489c)$$

(1.489a) can be solved iteratively with the initial condition

$$\hat{S}(t_a, t_a) = \hat{1} \qquad (1.498d)$$

$$\to \quad \hat{S}(t, t_a) = \hat{1} + \frac{1}{i \hbar} \int_{t_a}^{t} d t_1 \hat{V}_{\mathcal{I}}(t) \hat{S}(t, t_a)$$
$$= \hat{1} + \frac{1}{i \hbar} \int_{t_a}^{t} d t_1 \hat{V}_{\mathcal{I}}(t_1) \hat{S}(t_1, t_a) \qquad (1.498e)$$
$$+ \frac{1}{(i \hbar)^2} \int_{t_a}^{t} d t_1 \int_{t_a}^{t_2} d t_2 \hat{V}_{\mathcal{I}}(t_1) \hat{V}_{\mathcal{I}}(t_2) \hat{S}(t_2, t_a)$$

To the lowest order, we have

$$\hat{S} \approx \hat{1} + \frac{1}{i \hbar} \int_{-\infty}^{\infty} d t \hat{V}_{\mathcal{I}}(t) \qquad (1.498f)$$

$$\to \quad \left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle \approx \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle + \frac{1}{i \hbar} \int_{-\infty}^{\infty} d t \left\langle \boldsymbol{p}_b \middle| \hat{V}_{\mathcal{I}}(t) \middle| \boldsymbol{p}_a \right\rangle \qquad (1.498g)$$

Assuming a time-independent potential, we have

$$\int_{-\infty}^{\infty} d t \left\langle \boldsymbol{p}_b \middle| \hat{V}_{\mathcal{I}}(t) \middle| \boldsymbol{p}_a \right\rangle = \int_{-\infty}^{\infty} d t\, e^{i E_b t / \hbar} \left\langle \boldsymbol{p}_b \mid \hat{V} \mid \boldsymbol{p}_a \right\rangle e^{-i E_a t / \hbar}$$
$$= 2 \pi \hbar \delta(E_b - E_a) \left\langle \boldsymbol{p}_b \mid \hat{V} \mid \boldsymbol{p}_a \right\rangle$$
$$= 2 \pi \hbar \delta(E_b - E_a) V_{\boldsymbol{p}_b \boldsymbol{p}_a}$$

where

$$V_{\boldsymbol{p}_b \boldsymbol{p}_a} = \left\langle \boldsymbol{p}_b \mid \hat{V} \mid \boldsymbol{p}_a \right\rangle$$
$$= \frac{1}{L^3} \int d^3 x\, e^{i (\boldsymbol{p}_b - \boldsymbol{p}_a) \cdot \boldsymbol{x} / \hbar} V(\boldsymbol{x}) \qquad \text{[See (1.475a).]}$$
$$= \frac{1}{L^3} \tilde{V}(\boldsymbol{p}_b - \boldsymbol{p}_a) \qquad (1.498h)$$

with $\tilde{V}(\boldsymbol{q})$ being the Fourier transform of $V(\boldsymbol{x})$.

(1.498g) thus becomes

$$\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle \approx \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle - 2\pi i\, \delta(E_b - E_a)\, V_{\boldsymbol{p}_b \boldsymbol{p}_a} \qquad (1.498i)$$

Comparing with (1.474), we have

$$T_{\boldsymbol{p}_b \boldsymbol{p}_a} \approx \frac{1}{\hbar} V_{\boldsymbol{p}_b \boldsymbol{p}_a} \qquad (1.490)$$

(1.498i) can also be written in operator form as

$$\hat{S} \approx \hat{1} - 2\pi i \sum_{\boldsymbol{p},\boldsymbol{p}'} \delta(E_{\boldsymbol{p}} - E_{\boldsymbol{p}'}) \mid \boldsymbol{p} \rangle \langle \boldsymbol{p} \mid \hat{V} \mid \boldsymbol{p}' \rangle \langle \boldsymbol{p}' \mid \qquad (1.489)$$

$$= \hat{1} - 2\pi i \sum_{\boldsymbol{p},\boldsymbol{p}'} \delta(E_{\boldsymbol{p}} - E_{\boldsymbol{p}'}) \mid \boldsymbol{p} \rangle V_{\boldsymbol{p}\boldsymbol{p}'} \langle \boldsymbol{p}' \mid$$

(1.489) is applicable to a general $\hat{H}_0$ if one replaces $\mid \boldsymbol{p} \rangle$ with the eigenstates of $\hat{H}_0$.

(1.488) then reduces to the Born approximation

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi\hbar)^2 \hbar^2} \left| \tilde{V}(\boldsymbol{p}_b - \boldsymbol{p}_a) \right|^2 \qquad (1.492)$$

$$\equiv \mid f_{\boldsymbol{p}_b \boldsymbol{p}_a} \mid^2 \qquad (1.493)$$

$$\to \qquad f_{\boldsymbol{p}_b \boldsymbol{p}_a} = -\frac{E}{2\pi\hbar^2} \tilde{V}(\boldsymbol{p}_b - \boldsymbol{p}_a)$$

where $f_{\boldsymbol{p}_b \boldsymbol{p}_a}$ is called the scattering amplitude and the negative sign was chosen to agree with Landau and Lifshitz .

If we start with (1.485), then

$$f_{\boldsymbol{p}_b \boldsymbol{p}_a} = -\frac{M}{2\pi\hbar^2} \tilde{T}(\boldsymbol{p}_b - \boldsymbol{p}_a) \qquad (1.494)$$

where $\tilde{T}(\boldsymbol{q})$ is the Fourier transform of $T(\boldsymbol{x})$.

1.16.4. Partial Wave Expansion and Eikonal Approximation

For a central potential $V = V(r)$, the scattering amplitude $f_{\boldsymbol{p}_b \boldsymbol{p}_a}$ can depend only on the scattering angle

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{p}_b \cdot \boldsymbol{p}_a}{p_b\, p_a}\right) \qquad \text{with} \qquad p_b = p_a = p = \hbar k$$

We can therefore expand it in terms of the Legendre polynomials $P_l(x) \equiv P_l^0(x)$ as

$$f_{\boldsymbol{p}_b \boldsymbol{p}_a} = \sum_{l=0}^{\infty} f_l\, (2l+1)\, P_l(\cos\theta) \qquad (1.495)$$

where the partial scattering amplitude f_l is related to the phase shift $\delta_l(p)$ by

$$f_l = \frac{\hbar}{2ip}\left(e^{2i\delta_l} - 1\right) \qquad (1.495a)$$

(1.495a) is obtained using the expansion for the incoming plane wave

$$e^{ikz} = e^{ikr\cos\theta} = \sum_l (2l+1)\, i^l\, j_l(kr)\, P_l(\cos\theta)$$

$$\xrightarrow[r\to\infty]{} \sum_l (2l+1)\, i^l\, \frac{\sin\left(kr - l\frac{\pi}{2}\right)}{kr} P_l(\cos\theta)$$

Since the scattering potential is central, its effect can only be a phase shift, i.e., replacing

$\sin\left(k\,r - l\,\frac{\pi}{2}\right)$ with $\sin\left(k\,r - l\,\frac{\pi}{2} + \delta_l\right)$. Since the outgoing wave is related to $f_{\mathbf{p}_b\,\mathbf{p}_a}$ by

$$\psi \underset{r\to\infty}{\longrightarrow} e^{i\,k\,z} + \frac{e^{i\,k\,r}}{r}\,f_{\mathbf{p}_b\,\mathbf{p}_a} \tag{1.495b}$$

a little algebraic manipulation then gives (1.495a).

For small θ, we have [see Abramowitz & Stegan, Formula 9.1.71.]

$$P_l^{-m}(\cos\theta) \approx l^{-m}\,J_m(l\,\theta) \tag{1.496}$$

where J_m is a Bessel function.

(1.495) thus becomes

$$f_{\mathbf{p}_b\,\mathbf{p}_a} \approx \frac{\hbar}{2\,i\,p}\sum_{l=0}^{\infty}\left(e^{2\,i\,\delta_l} - 1\right)(2\,l+1)\,J_0(l\,\theta) \tag{1.497a}$$

In terms of the impact parameter

$$b \equiv \frac{L_z}{p} = \frac{l\,\hbar}{p}$$

we have

$$1 = \Delta\,l \approx \frac{p}{\hbar}\,d\,b \qquad l\,\theta = b\,\frac{p}{\hbar}\,\theta \approx b\,\frac{q}{\hbar} \qquad \mathbf{q} = \mathbf{p}_b - \mathbf{p}_a$$

$$\rightarrow \qquad \sum_{l=0}^{\infty}(2\,l+1) \approx \int_0^{\infty} d\,b\,\frac{p}{\hbar}\left(\frac{2}{\hbar}\,b\,p\right) = \frac{2\,p^2}{\hbar^2}\int d\,b\,b$$

(1.497a) thus becomes

$$f_{\mathbf{p}_b\,\mathbf{p}_a} \approx \frac{p}{i\,\hbar}\int d\,b\,b\left(e^{2\,i\,\delta_{p\,b/\hbar}} - 1\right)J_0\left(\frac{q}{\hbar}\,b\right) \tag{1.497}$$

which is known as the eikonal approximation.

Discussion of the example of Coulomb scattering will be postponed until §2.22.3 in which (2.749) is derived.

1.16.5. Scattering Amplitude from Time Evolution Amplitude

(1.474) can be formally inverted to give

$$T_{\mathbf{p}_b\,\mathbf{p}_a} = -\frac{1}{2\,\pi\,i\,\delta(E_b - E_a)}\left[\left\langle \mathbf{p}_b \mid \hat{S} \mid \mathbf{p}_a\right\rangle - \langle \mathbf{p}_b \mid \mathbf{p}_a\rangle\right] \tag{1.506a}$$

provided we know how to write $\frac{1}{\delta(E_b - E_a)}$ in a meaningful manner.

Using the fact that an infinitely narrow gaussian of unit area is a δ-function, we have

$$\delta(x) = \lim_{\sigma\to 0}\frac{1}{\sqrt{\pi\,\sigma}}\,e^{-x^2/\sigma}$$

so that

$$\delta(E_b - E_a) = \frac{M}{p_b}\,\delta(p_b - p_a)$$

$$\delta(p_b - p_a) = \lim_{\sigma\to 0}\frac{1}{\sqrt{\pi\,\sigma}}\,e^{-(p_b - p_a)^2/\sigma} = \lim_{\sigma\to 0}\frac{1}{\sqrt{\pi\,\sigma}}$$

Setting

$$\sigma = \frac{2M\hbar}{i\,t_b}$$

we have

$$\delta(E_b - E_a) = \frac{M}{p_b} \lim_{t_b\to\infty} \sqrt{\frac{i\,t_b}{2\pi\hbar M}} \tag{1.506}$$

(1.494) then becomes

$$f_{\mathbf{p}_b\mathbf{p}_a} = -\frac{L^6}{(2\pi\hbar)^2\, i\, p_b} \lim_{t_b\to\infty} \sqrt{\frac{2\pi\hbar M}{i\,t_b}} \left[\left\langle \mathbf{p}_b \mid \hat{S} \mid \mathbf{p}_a \right\rangle - \langle \mathbf{p}_b \mid \mathbf{p}_a \rangle \right] \tag{1.507}$$

$$= -\frac{L^6}{(2\pi\hbar)^2\, i\, p_b} \lim_{\substack{t_b\to\infty \\ t_a\to-\infty}} \sqrt{\frac{2\pi\hbar M}{i\,t_b}} \left[e^{i(E_b t_b - E_a t_a)/\hbar} (\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a) - \langle \mathbf{p}_b \mid \mathbf{p}_a \rangle \right]$$

which is not an entirely satisfactory result. A better treatment will be given in the path integral formulation in §2.22.

Using the definition [note the slight difference from (1.489c)]

$$\hat{U}_{\mathcal{I}}(t_b, t_a) = e^{i\hat{H}_0 t_b/\hbar}\, \hat{U}(t_b, t_a)\, e^{-i\hat{H}_0 t_a/\hbar} \tag{1.507a}$$

$$= e^{i\hat{H}_0 t_b/\hbar}\, e^{-i\hat{H} t_b/\hbar}\, e^{i\hat{H} t_a/\hbar}\, e^{-i\hat{H}_0 t_a/\hbar}$$

(1.489a) becomes

$$\hat{S}(t_b, t_a) = \hat{U}_{\mathcal{I}}(t_b, t_a) \tag{1.489a}$$

$$\to \qquad \hat{S} = \lim_{(t_b, t_a)\to(\infty, -\infty)} \hat{U}_{\mathcal{I}}(t_b, t_a)$$

$$\therefore \qquad \left\langle \mathbf{p}_b \mid \hat{S} \mid \mathbf{p}_a \right\rangle = \lim_{(t_b, t_a)\to(\infty, -\infty)} \left\langle \mathbf{p}_b \,\middle|\, \hat{U}_{\mathcal{I}}(t_b, t_a) \,\middle|\, \mathbf{p}_a \right\rangle \tag{1.508}$$

1.16.6. Lippmann-Schwinger Equation

Like $\hat{U}$, the interaction picture version $\hat{U}_{\mathcal{I}}$ also possess the group properties

$$\hat{U}_{\mathcal{I}}(t, t') = \hat{U}_{\mathcal{I}}(t, t_1)\, \hat{U}_{\mathcal{I}}(t_1, t') \tag{1.509}$$

Now, from (1.507a), we have

$$e^{-i\hat{H}_0 t/\hbar}\, \hat{U}_{\mathcal{I}}(t, t_a) = e^{-i\hat{H} t/\hbar}\, e^{i\hat{H} t_a/\hbar}\, e^{-i\hat{H}_0 t_a/\hbar} = e^{-iHt/\hbar}\, \hat{U}_{\mathcal{I}}(0, t_a)$$

$$= e^{i\hat{H}(t_a - t)/\hbar}\, e^{-i\hat{H}_0(t_a - t)/\hbar}\, e^{-i\hat{H}_0 t/\hbar}$$

$$= \hat{U}_{\mathcal{I}}(0, t_a - t)\, e^{-i\hat{H}_0 t/\hbar} \tag{1.510}$$

$$\to \qquad e^{-i\hat{H}_0 t/\hbar}\, \hat{U}_{\mathcal{I}}(t, t_a) = e^{-iHt/\hbar}\, \hat{U}_{\mathcal{I}}(0, t_a) \xrightarrow[t_a\to-\infty]{} \hat{U}_{\mathcal{I}}(0, t_a)\, e^{-i\hat{H}_0 t/\hbar} \tag{1.511}$$

$$\therefore \qquad \lim_{t_a\to-\infty} \hat{U}_{\mathcal{I}}(t_b, t_a) = \lim_{t_a\to-\infty} e^{i\hat{H}_0 t_b/\hbar}\, e^{-iHt_b/\hbar}\, \hat{U}_{\mathcal{I}}(0, t_a) \tag{1.512}$$

$$= \lim_{t_a\to-\infty} e^{i\hat{H}_0 t_b/\hbar}\, \hat{U}_{\mathcal{I}}(0, t_a)\, e^{-i\hat{H}_0 t_b/\hbar}$$

(1.508) thus becomes

$$\left\langle \mathbf{p}_b \mid \hat{S} \mid \mathbf{p}_a \right\rangle = \lim_{(t_b, t_a)\to(\infty, -\infty)} e^{i(E_b - E_a)t_b/\hbar} \left\langle \mathbf{p}_b \,\middle|\, \hat{U}_{\mathcal{I}}(0, t_a) \,\middle|\, \mathbf{p}_a \right\rangle \tag{1.513}$$

(1.291) gives

$$\left\langle \boldsymbol{p}_b \middle| \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle = \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle - \frac{i}{\hbar} \int_{t_a}^{0} d\,t\, e^{i E_b t/\hbar} \left\langle \boldsymbol{p}_b \mid V e^{-i \hat{H}_0 t/\hbar} \hat{U}_{\mathcal{I}}(t, t_a) \middle| \boldsymbol{p}_a \right\rangle$$

Setting $t_a \to -\infty$ to use (1.511), we have

$$\left\langle \boldsymbol{p}_b \middle| \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle = \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle - \frac{i}{\hbar} \int_{t_a}^{0} d\,t\, e^{i (E_b - E_a - i\eta) t/\hbar} \left\langle \boldsymbol{p}_b \middle| V \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle \tag{1.514}$$

where $i\,\eta$ was added by hand to ensure the integral remain finite as $t_a \to -\infty$. Doing the integral gives

$$\left\langle \boldsymbol{p}_b \middle| \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle = \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle - \frac{1}{E_b - E_a - i\,\eta} \left\langle \boldsymbol{p}_b \middle| V \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle \tag{1.515}$$

which is the renowned Lippmann-Schwinger eq.

Inserting this into (1.513) gives

$$\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle = \lim_{(t_b, t_a) \to (\infty, -\infty)} e^{i (E_b - E_a) t_b/\hbar} \tag{1.516}$$

$$\times \left[\langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle - \frac{1}{E_b - E_a - i\,\eta} \left\langle \boldsymbol{p}_b \middle| V \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle \right]$$

Using

$$e^{i (E_b - E_a) t_b/\hbar} \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle = \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle$$

and

$$\lim_{t_b \to \infty} \frac{e^{i (E_b - E_a) t_b/\hbar}}{E_b - E_a - i\,\eta} = \lim_{t_b \to \infty} e^{i (E_b - E_a) t_b/\hbar} \left[\mathcal{P} \frac{1}{E_b - E_a} + 2\pi i\, \delta(E_b - E_a) \right]$$

$$= 2\pi i\, \delta(E_b - E_a) \tag{1.518}$$

where we've used

$$\lim_{t_b \to \infty} e^{i (E_b - E_a) t_b/\hbar} = \begin{cases} 0 & \text{for } E_b \neq E_a \\ 1 & \text{for } E_b = E_a \end{cases} \tag{1.517}$$

(1.516) becomes

$$\left\langle \boldsymbol{p}_b \mid \hat{S} \mid \boldsymbol{p}_a \right\rangle = \langle \boldsymbol{p}_b \mid \boldsymbol{p}_a \rangle - 2\pi i\, \delta(E_b - E_a) \left\langle \boldsymbol{p}_b \middle| V \hat{U}_{\mathcal{I}}(0, -\infty) \middle| \boldsymbol{p}_a \right\rangle]$$

Comparing with (1.474) gives

$$\left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle = \left\langle \boldsymbol{p}_b \middle| V \hat{U}_{\mathcal{I}}(0, -\infty) \middle| \boldsymbol{p}_a \right\rangle \tag{1.521}$$

$$\to \qquad T_{\boldsymbol{p}_b \boldsymbol{p}_a} = \left\langle \boldsymbol{p}_b \mid \hat{T} \mid \boldsymbol{p}_a \right\rangle$$

$$= \int \frac{d^3 p_c}{(2\pi\hbar)^3} V_{\boldsymbol{p}_b \boldsymbol{p}_c} \left\langle \boldsymbol{p}_c \middle| \hat{U}_{\mathcal{I}}(0, -\infty) \middle| \boldsymbol{p}_a \right\rangle$$

Using (1.515), we have

$$T_{\boldsymbol{p}_b \boldsymbol{p}_a} = V_{\boldsymbol{p}_b \boldsymbol{p}_a} - \int \frac{d^3 p_c}{(2\pi\hbar)^3} V_{\boldsymbol{p}_b \boldsymbol{p}_c} \frac{1}{E_c - E_a - i\,\eta} \left\langle \boldsymbol{p}_c \middle| V \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle$$

$$= V_{\boldsymbol{p}_b \boldsymbol{p}_a} - \int \frac{d^3 p_c}{(2\pi\hbar)^3} V_{\boldsymbol{p}_b \boldsymbol{p}_c} \frac{1}{E_c - E_a - i\,\eta} T_{\boldsymbol{p}_c \boldsymbol{p}_a} \tag{1.522}$$

Using (1.515), the outgoing wave is given by

$$\left\langle \boldsymbol{x} \middle| \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle = \int \frac{d^3 p}{(2\pi\hbar)^3} \langle \boldsymbol{x} \mid \boldsymbol{p} \rangle \left\langle \boldsymbol{p} \middle| \hat{U}_{\mathcal{I}}(0, t_a) \middle| \boldsymbol{p}_a \right\rangle$$

$$= \int \frac{d^3 p}{(2\pi\hbar)^3} \langle \boldsymbol{x} \mid \boldsymbol{p} \rangle \left[\langle \boldsymbol{p} \mid \boldsymbol{p}_a \rangle - \frac{1}{E_{\boldsymbol{p}} - E_a - i\eta} \left\langle \boldsymbol{p} \,\middle|\, V \hat{U}_{\mathcal{I}}(0, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle \right]$$
$$= \langle \boldsymbol{x} \mid \boldsymbol{p}_a \rangle + \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{E_a - E_{\boldsymbol{p}} + i\eta} \langle \boldsymbol{x} \mid \boldsymbol{p} \rangle \left\langle \boldsymbol{p} \,\middle|\, V \hat{U}_{\mathcal{I}}(0, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle$$
$$= \langle \boldsymbol{x} \mid \boldsymbol{p}_a \rangle + \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}}{E_a - E_{\boldsymbol{p}} + i\eta} \left\langle \boldsymbol{p} \,\middle|\, V \hat{U}_{\mathcal{I}}(0, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle$$
$$= \langle \boldsymbol{x} \mid \boldsymbol{p}_a \rangle + \int d^3 x' \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}}{E_a - E_{\boldsymbol{p}} + i\eta} \langle \boldsymbol{p} \mid \boldsymbol{x}' \rangle \left\langle \boldsymbol{x}' \,\middle|\, V \hat{U}_{\mathcal{I}}(0, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle$$
$$= \langle \boldsymbol{x} \mid \boldsymbol{p}_a \rangle + \int d^3 x' \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')/\hbar}}{E_a - E_{\boldsymbol{p}} + i\eta} \left\langle \boldsymbol{x}' \,\middle|\, V \hat{U}_{\mathcal{I}}(0, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle$$
$$= \langle \boldsymbol{x} \mid \boldsymbol{p}_a \rangle + \int d^3 x' \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')/\hbar}}{E_a - \frac{\boldsymbol{p}^2}{2M} + i\eta} V(\boldsymbol{x}') \left\langle \boldsymbol{x}' \,\middle|\, \hat{U}_{\mathcal{I}}(0, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle$$
(1.523)

where we’ve used

$$\left\langle \boldsymbol{x}' \,\middle|\, V \hat{U}_{\mathcal{I}}(0, t_a) = \int d^3 x'' \langle \boldsymbol{x}' \mid V \mid \boldsymbol{x}'' \rangle \left\langle \boldsymbol{x}'' \,\middle|\, \hat{U}_{\mathcal{I}}(0, t_a)\right.\right.$$
$$= \int d^3 x'' \, V(\boldsymbol{x}') \, \delta(\boldsymbol{x}' - \boldsymbol{x}'') \left\langle \boldsymbol{x}'' \,\middle|\, \hat{U}_{\mathcal{I}}(0, t_a)\right.$$
$$= V(\boldsymbol{x}') \left\langle \boldsymbol{x}' \,\middle|\, \hat{U}_{\mathcal{I}}(0, t_a)\right.$$

Also, $\boldsymbol{p}$ is assumed to be continuous so that

$$\int \frac{d^3 p}{(2\pi\hbar)^3} \mid \boldsymbol{p} \rangle \langle \boldsymbol{p} \mid = \hat{1} \qquad \to \qquad \langle \boldsymbol{x} \mid \boldsymbol{p} \rangle = e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}.$$

According to the definition (1.340) of the fixed-energy amplitude of a free particle, we set

$$(\boldsymbol{x} \mid \boldsymbol{x}')_{E_a} = \int \frac{d^3 p}{(2\pi\hbar)^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')/\hbar} \frac{i\hbar}{E_a - \frac{\boldsymbol{p}^2}{2M} + i\eta} \qquad (1.524)$$
$$= \frac{i}{(2\pi\hbar)^2} \int_{-1}^{1} d\cos\theta \int_0^\infty d p\, p^2\, e^{ip\,|\boldsymbol{x}-\boldsymbol{x}'|\cos\theta/\hbar} \frac{1}{\frac{p_a^2}{2M} - \frac{p^2}{2M} + i\eta}$$
$$= \frac{1}{(2\pi)^2 \hbar \mid \boldsymbol{x} - \boldsymbol{x}' \mid} \mathcal{I}$$

where

$$\mathcal{I} = \int_0^\infty d p\, p \frac{e^{ip\,|\boldsymbol{x}-\boldsymbol{x}'|/\hbar} - e^{-ip\,|\boldsymbol{x}-\boldsymbol{x}'|/\hbar}}{\frac{1}{2M}\left(p_a^2 - p^2\right) + i\eta}$$

Setting $p \to -p$ gives

$$\mathcal{I} = \int_0^{-\infty} d p\, p \frac{e^{-ipR} - e^{ipR}}{\frac{1}{2M}\left(p_a^2 - p^2\right) + i\eta} \qquad R = \frac{\mid \boldsymbol{x} - \boldsymbol{x}' \mid}{\hbar}$$
$$= \int_{-\infty}^{0} d p\, p \frac{e^{ipR} - e^{-ipR}}{\frac{1}{2M}\left(p_a^2 - p^2\right) + i\eta}$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} d p\, p \frac{e^{ipR} - e^{-ipR}}{\frac{1}{2M}\left(p_a^2 - p^2\right) + i\eta}$$

Using

$$\frac{1}{\frac{1}{2M}\left(p_a^2 - p^2\right) + 2 i p_a \eta} = -\frac{M}{p}\left(\frac{1}{p - p_a - i\eta} + \frac{1}{p + p_a + i\eta}\right) \qquad (\eta \to 0)$$

we have

$$\mathcal{I} = -\frac{M}{2}\int_{-\infty}^{\infty} d\,p\,(e^{i\,p\,R} - e^{-i\,p\,R})\left(\frac{1}{p - p_a - i\,\eta} + \frac{1}{p + p_a + i\,\eta}\right)$$

Contour closed in upper-half plane:

$$\int_{-\infty}^{\infty} d\,p\,\frac{e^{i\,p\,R}}{p - p_a - i\,\eta} = 2\,\pi\,i\,e^{i\,p_a\,R} \qquad \int_{-\infty}^{\infty} d\,p\,\frac{e^{i\,p\,R}}{p + p_a + i\,\eta} = 0$$

Contour closed in lower-half plane:

$$\int_{-\infty}^{\infty} d\,p\,\frac{e^{-i\,p\,R}}{p - p_a - i\,\eta} = 0 \qquad \int_{-\infty}^{\infty} d\,p\,\frac{e^{-i\,p\,R}}{p + p_a + i\,\eta} = 2\,\pi\,i\,e^{i\,p_a\,R}$$

$$\therefore \qquad \mathcal{I} = -2\,M\,\pi\,i\;e^{i\,p_a\,|\,\boldsymbol{x} - \boldsymbol{x}'\,|\,/\,\hbar}$$

$$(\,\boldsymbol{x}\mid\boldsymbol{x}'\,)_{E_a} = -\frac{2\,M\,i}{\hbar}\;\frac{e^{i\,p_a\,|\,\boldsymbol{x} - \boldsymbol{x}'\,|\,/\,\hbar}}{4\,\pi\,|\,\boldsymbol{x} - \boldsymbol{x}'\,|} \qquad p_a = \sqrt{2\,M\,E_a} \qquad (1.525)$$

(1.523) thus becomes

$$\left\langle \boldsymbol{x} \,\middle|\, \hat{U}_{\mathcal{I}}(0,\,t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle = \langle\,\boldsymbol{x}\mid\boldsymbol{p}_a\,\rangle + \frac{1}{i\,\hbar}\int d^3\,x'\;(\,\boldsymbol{x}\mid\boldsymbol{x}'\,)_{E_a}\,V(\boldsymbol{x}')\left\langle \boldsymbol{x}' \,\middle|\, \hat{U}_{\mathcal{I}}(0,\,t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle$$

$$= e^{i\,\boldsymbol{p}_a\cdot\boldsymbol{x}/\hbar} - \frac{2\,M}{\hbar^2}\int d^3\,x'\;\frac{e^{i\,p_a\,|\,\boldsymbol{x} - \boldsymbol{x}'\,|\,/\,\hbar}}{4\,\pi\,|\,\boldsymbol{x} - \boldsymbol{x}'\,|}\,V(\boldsymbol{x}')\left\langle \boldsymbol{x}' \,\middle|\, \hat{U}_{\mathcal{I}}(0,\,t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle \qquad (1.525a)$$

Let $\boldsymbol{x}$ be the observation point. For scattering by a localized potential, the integrand in (1.525a) is non-zero only for $r = |\,\boldsymbol{x}\,| >> |\,\boldsymbol{x}'\,|$ so that

$$|\,\boldsymbol{x} - \boldsymbol{x}'\,| = \sqrt{r^2 - 2\,\hat{\boldsymbol{x}}\cdot\boldsymbol{x}' + |\,\boldsymbol{x}'\,|^2}$$

$$\approx r - \frac{1}{r}\,\boldsymbol{x}\cdot\boldsymbol{x}' = r - \hat{\boldsymbol{x}}\cdot\boldsymbol{x}'$$

(1.525a) thus becomes

$$\left\langle \boldsymbol{x} \,\middle|\, \hat{U}_{\mathcal{I}}(0,\,t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle \qquad (1.526)$$

$$\approx e^{i\,\boldsymbol{p}_a\cdot\boldsymbol{x}/\hbar} - \frac{e^{i\,p_a\,r/\hbar}}{4\,\pi\,r}\int d^3\,x'\;e^{-i\,p_a\,\hat{\boldsymbol{x}}\cdot\boldsymbol{x}'/\hbar}\,\frac{2\,M}{\hbar^2}\,V(\boldsymbol{x}')\left\langle \boldsymbol{x}' \,\middle|\, \hat{U}_{\mathcal{I}}(0,\,t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle$$

Comparing with (1.495b), we have

$$f_{\boldsymbol{p}_b\,\boldsymbol{p}_a} = -\frac{2\,M}{\hbar^2}\int d^3\,x'\;e^{-i\,\boldsymbol{p}_b\cdot\boldsymbol{x}'/\hbar}\,\frac{2\,M}{\hbar^2}\,V(\boldsymbol{x}')\left\langle \boldsymbol{x}' \,\middle|\, \hat{U}_{\mathcal{I}}(0,\,-\infty) \,\middle|\, \boldsymbol{p}_a \right\rangle \qquad (1.527)$$

where $\boldsymbol{p}_b = p_a\,\hat{\boldsymbol{x}}$ since the scattering is elastic. Thus, we have related the scattering amplitude to the evolution amplitude without having to deal with the reciprocal of a δ-function encountered in §1.16.5.

By the definition (1.507a), we have

$$\left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}_{\mathcal{I}}(0,\,-\infty) \,\middle|\, \boldsymbol{p}_a \right\rangle = \lim_{t_a\to-\infty}\left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(0,\,t_a)\,e^{-i\,\hat{H}_0\,t_a/\hbar} \,\middle|\, \boldsymbol{p}_a \right\rangle$$

$$= \lim_{t_a\to-\infty}\left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}\,(0,\,t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle e^{-i\,E_a\,t_a/\hbar} \qquad (1.528a)$$

Now,

$$(\,\boldsymbol{x}_b\,t_b\mid\boldsymbol{x}_a\,t_a\,) = \left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(t_b,\,t_a) \,\middle|\, \boldsymbol{x}_a \right\rangle$$

$$= \int \frac{d^3\,p_a}{(2\,\pi\,\hbar)^3}\left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(t_b,\,t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle\langle\,\boldsymbol{p}_a\mid\boldsymbol{x}_a\,\rangle$$

$$= \int \frac{d^3 p_a}{(2\pi\hbar)^3} \left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(t_b, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle e^{-i\,\boldsymbol{p}_a\cdot\boldsymbol{x}_a/\hbar}$$

$$\rightarrow \qquad \left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(t_b, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle = \int d^3 x_a\, e^{i\,\boldsymbol{p}_a\cdot\boldsymbol{x}_a/\hbar} \left(\boldsymbol{x}_b\, t_b \,\middle|\, \boldsymbol{x}_a\, t_a \right)$$

With $E_a = \frac{p_a^2}{2M}$, we have

$$\left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(0, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle e^{-i E_a t_a/\hbar} = \int d^3 x_a\, e^{i\left(\boldsymbol{p}_a\cdot\boldsymbol{x}_a - \frac{p_a^2}{2M} t_a\right)/\hbar} \left(\boldsymbol{x}_b\, 0 \,\middle|\, \boldsymbol{x}_a\, t_a \right) \qquad (1.529)$$

Setting $\boldsymbol{x}_a = \frac{\boldsymbol{p}\, t_a}{M}$, this becomes

$$\left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(0, t_a) \,\middle|\, \boldsymbol{p}_a \right\rangle e^{-i E_a t_a/\hbar} = \left(\frac{t_a}{M}\right)^3 \int d^3 p\, e^{i\left(2\,\boldsymbol{p}_a\cdot\boldsymbol{p} - \boldsymbol{p}_a^2\right) t_a/2M\hbar} \left(\boldsymbol{x}_b\, 0 \,\middle|\, \frac{\boldsymbol{p}\, t_a}{M}\, t_a \right) \qquad (1.530)$$

For $t_a \rightarrow -\infty$, the exponential factor fluctuates wildly so that only the term

$$\boldsymbol{p}_a \cdot \boldsymbol{p} - p_a^2 = 0 \quad \rightarrow \qquad \boldsymbol{p} = \boldsymbol{p}_a$$

survives. Adapting the δ-function representation (1.338) to read

$$\delta(\boldsymbol{p} - \boldsymbol{p}_a) = \lim_{t_a \rightarrow -\infty} \frac{(-t_a)^{3/2}}{(2\pi i \hbar M)^{3/2}}\, e^{-i\,(\boldsymbol{p} - \boldsymbol{p}_a)^2 t_a/2M\hbar} \qquad (1.531)$$

Using

$$2\,\boldsymbol{p}_a \cdot \boldsymbol{p} - \boldsymbol{p}_a^2 = -(\boldsymbol{p} - \boldsymbol{p}_a)^2 + \boldsymbol{p}^2$$

(1.530) becomes

$$\left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}_{\mathcal{I}}(0, -\infty) \,\middle|\, \boldsymbol{p}_a \right\rangle = \lim_{t_a \rightarrow -\infty} \left(\frac{-2\pi i \hbar\, t_a}{M}\right)^{3/2} e^{i p_a^2 t_a/2M\hbar} \left(\boldsymbol{x}_b\, 0 \,\middle|\, \frac{\boldsymbol{p}_a\, t_a}{M}\, t_a \right) \qquad (1.528)$$

which serves as a reliable starting point for extracting the scattering amplitude, via (1.527), from the time evolution amplitude in $\boldsymbol{x}$-space at $\boldsymbol{x}_a = \frac{\boldsymbol{p}_a\, t_a}{M}$.