Appendix IA. Simple Time Evolution Operator

The simplest nontrivial system is perhaps the spin-1/2 particle in a magnetic field $\boldsymbol{B}$.

The reduced Hamiltonian operator is

$$\hat{H}_0 = -\frac{\hbar}{2}\boldsymbol{B}\cdot\boldsymbol{\sigma} \tag{1A.1a}$$

where σ_i are the Pauli matrices previously denoted as $\mathbb{s}_i$ in (1.445).

Since $\hat{H}_0$ is time-independent, the time evolution operator reduces to

$$\hat{U}(t_b, t_a) = e^{-i\hat{H}_0(t_b - t_a)/\hbar} = e^{i(t_b - t_a)\boldsymbol{B}\cdot\boldsymbol{\sigma}} \tag{1A.1}$$

Combining the well-known properties of $\boldsymbol{\sigma}$,

$$\sigma_i\,\sigma_j + \sigma_j\,\sigma_i = 2\,\delta_{ij}$$

$$\sigma_i\,\sigma_j - \sigma_j\,\sigma_i = 2\,i\,\hbar\,\epsilon_{ijk}\,\sigma_k$$

we have

$$\sigma_i\,\sigma_j = \delta_{ij} + i\epsilon_{ijk}\,\sigma_k \tag{1A.5}$$

Hence

$$(a\cdot\boldsymbol{\sigma})(\boldsymbol{b}\cdot\boldsymbol{\sigma}) = a_i\,b_j\,\sigma_i\,\sigma_j = a_i\,b_i + i\,\epsilon_{ijk}\,a_i\,b_j\,\sigma_k$$

$$= \boldsymbol{a}\cdot\boldsymbol{b} + i\,(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{\sigma} \tag{1A.5a}$$

Therefore,

$$(\boldsymbol{B}\cdot\boldsymbol{\sigma})^2 = B^2$$

so that

$$(\boldsymbol{B}\cdot\boldsymbol{\sigma})^{2n} = B^{2n} \qquad (\boldsymbol{B}\cdot\boldsymbol{\sigma})^{2n+1} = B^{2n}\,\boldsymbol{B}\cdot\boldsymbol{\sigma} = B^{2n+1}\,\hat{\boldsymbol{B}}\cdot\boldsymbol{\sigma}$$

where $\hat{B} = \dfrac{\boldsymbol{B}}{B}$ is the unit vector along $\boldsymbol{B}$.

$$\hat{U}(t_b, t_a) = \sum_{k=0}^{\infty}\frac{\left[i\left(\frac{t_b - t_a}{2}\right)\right]^k}{k!}(\boldsymbol{B}\cdot\boldsymbol{\sigma})^k$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-)^n\left(\frac{t_b-t_a}{2}\right)^{2n}}{(2n)!}B^{2n} + i\,\frac{(-)^n\left(\frac{t_b-t_a}{2}\right)^{2n+1}}{(2n+1)!}B^{2n}\,\boldsymbol{B}\cdot\boldsymbol{\sigma}\right)$$

$$= \cos\left[B\left(\frac{t_b - t_a}{2}\right)\right] + i\,\hat{\boldsymbol{B}}\cdot\boldsymbol{\sigma}\sin\left[B\left(\frac{t_b - t_a}{2}\right)\right] \tag{1A.2}$$

Using

$$\delta\hat{H}_0 = -\frac{\hbar}{2}\,\delta\boldsymbol{B}(t)\cdot\boldsymbol{\sigma}$$

(1.253) becomes

$$\delta\hat{U}(t_b, t_a) = \frac{1}{2}\,i\int_{t_a}^{t_b}dt\,e^{i(t_b - t)\boldsymbol{B}\cdot\boldsymbol{\sigma}}\,\delta\boldsymbol{B}\cdot\boldsymbol{\sigma}\,e^{i(t - t_a)\boldsymbol{B}\cdot\boldsymbol{\sigma}} \tag{1A.3}$$

$$= \frac{1}{2}\,i\int_{t_a}^{t_b}dt\left\{\cos\left[B\left(\frac{t_b - t}{2}\right)\right] + i\,\hat{\boldsymbol{B}}\cdot\boldsymbol{\sigma}\sin\left[B\left(\frac{t_b - t}{2}\right)\right]\right\} \tag{1A.4}$$

$$\times\,\delta\boldsymbol{B}\cdot\boldsymbol{\sigma}\left\{\cos\left[B\left(\frac{t - t_a}{2}\right)\right] + i\,\hat{\boldsymbol{B}}\cdot\boldsymbol{\sigma}\sin\left[B\left(\frac{t - t_a}{2}\right)\right]\right\}$$

Using (1A.5a), we have

$$\hat{\boldsymbol{B}}\cdot\boldsymbol{\sigma}\,\delta\boldsymbol{B}\cdot\boldsymbol{\sigma} = \hat{\boldsymbol{B}}\cdot\delta\boldsymbol{B} + i\left(\hat{\boldsymbol{B}}\times\delta\boldsymbol{B}\right)\cdot\boldsymbol{\sigma} \tag{1A.6}$$

Switching $\hat{\boldsymbol{B}} \leftrightarrow \delta \boldsymbol{B}$ gives

$$\delta \boldsymbol{B} \cdot \boldsymbol{\sigma} \, \hat{\boldsymbol{B}} \cdot \boldsymbol{\sigma} = \hat{\boldsymbol{B}} \cdot \delta \boldsymbol{B} - i \left(\hat{\boldsymbol{B}} \times \delta \boldsymbol{B} \right) \cdot \boldsymbol{\sigma} \tag{1A.6a}$$

$$\begin{aligned} \hat{\boldsymbol{B}} \cdot \boldsymbol{\sigma} \, \delta \boldsymbol{B} \cdot \boldsymbol{\sigma} \, \hat{\boldsymbol{B}} \cdot \boldsymbol{\sigma} &= \left[\hat{\boldsymbol{B}} \cdot \delta \boldsymbol{B} + i \left(\hat{\boldsymbol{B}} \times \delta \boldsymbol{B} \right) \cdot \boldsymbol{\sigma} \right] \hat{\boldsymbol{B}} \cdot \boldsymbol{\sigma} \\ &= \hat{\boldsymbol{B}} \cdot \delta \boldsymbol{B} \, \hat{\boldsymbol{B}} \cdot \boldsymbol{\sigma} + i \left(\hat{\boldsymbol{B}} \times \delta \boldsymbol{B} \right) \cdot \boldsymbol{\sigma} \, \hat{\boldsymbol{B}} \cdot \boldsymbol{\sigma} \\ &= \hat{\boldsymbol{B}} \cdot \delta \boldsymbol{B} \, \hat{\boldsymbol{B}} \cdot \boldsymbol{\sigma} + i \left(\hat{\boldsymbol{B}} \times \delta \boldsymbol{B} \right) \cdot \hat{\boldsymbol{B}} - \left[\left(\hat{\boldsymbol{B}} \times \delta \boldsymbol{B} \right) \times \hat{\boldsymbol{B}} \right] \cdot \boldsymbol{\sigma} && \text{(1A.7)} \\ &= \hat{\boldsymbol{B}} \cdot \delta \boldsymbol{B} \, \hat{\boldsymbol{B}} \cdot \boldsymbol{\sigma} - \left[\delta \boldsymbol{B} - \left(\delta \boldsymbol{B} \cdot \hat{\boldsymbol{B}} \right) \hat{\boldsymbol{B}} \right] \cdot \boldsymbol{\sigma} \\ &= -\delta \boldsymbol{B} \cdot \boldsymbol{\sigma} && \text{(1A.7a)} \end{aligned}$$

The integrand of (1A.4) thus becomes

$$\begin{aligned} \mathcal{I} = {} & \cos\left[B\left(\frac{t_b - t}{2} \right) \right] \cos\left[B\left(\frac{t - t_a}{2} \right) \right] \delta \boldsymbol{B} \cdot \boldsymbol{\sigma} && \text{(1A.8)} \\ & + i \sin\left[B\left(\frac{t_b - t}{2} \right) \right] \cos\left[B\left(\frac{t - t_a}{2} \right) \right] \left[\hat{\boldsymbol{B}} \cdot \delta \boldsymbol{B} + i \left(\hat{\boldsymbol{B}} \times \delta \boldsymbol{B} \right) \cdot \boldsymbol{\sigma} \right] \\ & + i \cos\left[B\left(\frac{t_b - t}{2} \right) \right] \sin\left[B\left(\frac{t - t_a}{2} \right) \right] \left[\hat{\boldsymbol{B}} \cdot \delta \boldsymbol{B} - i \left(\hat{\boldsymbol{B}} \times \delta \boldsymbol{B} \right) \cdot \boldsymbol{\sigma} \right] \\ & + \sin\left[B\left(\frac{t_b - t}{2} \right) \right] \sin\left[B\left(\frac{t - t_a}{2} \right) \right] \delta \boldsymbol{B} \cdot \boldsymbol{\sigma} \\ = {} & \cos\left[B\left(\frac{t_b + t_a}{2} - t \right) \right] \delta \boldsymbol{B} \cdot \boldsymbol{\sigma} + i \sin\left[B\left(\frac{t_b - t_a}{2} \right) \right] \hat{\boldsymbol{B}} \cdot \delta \boldsymbol{B} && \text{(1A.9)} \\ & - \sin\left[B\left(\frac{t_b + t_a}{2} - t \right) \right] \left(\hat{\boldsymbol{B}} \times \delta \boldsymbol{B} \right) \cdot \boldsymbol{\sigma} \end{aligned}$$