Appendix IB. Convergence of the Fresnel Integral

Consider the Fresnel integral

$$\int_{-\infty}^{\infty} dx\, e^{i a x^2/2} = \sqrt{\frac{2\pi i}{a}}$$

$$= \sqrt{\frac{2\pi}{|a|}} \begin{cases} \sqrt{i} & a>0 \\ \frac{1}{\sqrt{i}} & a<0 \end{cases} \qquad (1.333)$$

It can be obtained by setting $\alpha = -i a$ to the Gauss integral

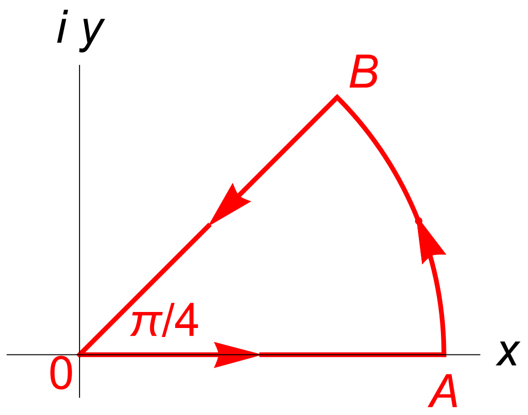
$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2/2} = \sqrt{\frac{2\pi}{\alpha}} \qquad (1.B1a)$$

This extension of the domain of α from the real line to the complex plane is called analytic continuation.

Alternatively, one can derive (1.333) by evaluating the integral

$$\mathcal{I} = \oint dz\, e^{-z^2}$$

for the contour shown belown.



Since e^{-z^2} is holomorphic (no poles in the finite z-plane)

$$\mathcal{I} = \left(\int_0^A + \int_A^B + \int_B^0 \right) dz = 0 \qquad (1.B1)$$

Let R be the radius of the arc. Along the segments of the contour, we have

$0A$:	$z = x$	$dz = dx$	$z^2 = x^2$
AB:	$z = R e^{i\phi}$	$dz = i R e^{i\phi}\, d\phi$	$z^2 = R^2 e^{2i\phi}$
$B0$:	$z = r e^{i\pi/4}$	$dz = e^{i\pi/4}\, dr$	$z^2 = i r^2$

For the AB segment, the absolute value of the integral is

$$R \left| \int_0^{\pi/4} d\phi\, e^{i\phi} e^{-R^2(\cos 2\phi + i \sin 2\phi)} \right|$$

$$\leq R \int_0^{\pi/4} d\phi \left| \int_0^{\pi/4} d\phi\, e^{i\phi} e^{-R^2 \cos 2\phi} e^{-i R^2 \sin 2\phi} \right|$$

$$= R \int_0^{\pi/4} d\phi\, e^{-R^2 \cos 2\phi}$$

$$= 0 \qquad \text{as} \quad R \to \infty$$

Thus,

$$\mathcal{I} = \int_0^\infty d\,x\, e^{-x^2} + e^{i\,\pi/4} \int_\infty^0 d\,r\; e^{-i\,r^2} = 0$$

$$\therefore \qquad \int_0^\infty d\,r\; e^{-i\,r^2} = \frac{1}{\sqrt{i}} \int_0^\infty d\,x\, e^{-x^2} = \sqrt{-i\,\pi}$$

Setting $r = \sqrt{\frac{a}{2}}\; x$, we have

$$\sqrt{\frac{a}{2}} \int_0^\infty d\,x\; e^{-i\,a\,x^2/2} = \sqrt{-i\,\pi}$$

which is (1.333) with $a \to -a$.