Appendix 1C. The Asymmetric Top

This section relies heavily on the symbolic manipulation software *Mathematica.* Most formulae are copied from the *Mathematica* output in the file “A1_Code.nb”.

For an asymmetric top, the classical Lagrangian is

$$L = \frac{1}{2}\left(\mathcal{I}_\xi\, \omega_\xi^2 + \mathcal{I}_\eta\, \omega_\eta^2 + \mathcal{I}_\zeta\, \omega_\zeta^2 \right) \qquad (1C.1)$$

The elements of the metric tensor $\mathbb{g}$ are

$$\begin{aligned}
g_{1,1} &= \mathcal{I}_\zeta \cos^2(\beta) + \sin^2(\beta) \left(\mathcal{I}_\eta \sin^2(\gamma) + \mathcal{I}_\xi \cos^2(\gamma)\right) \\
g_{1,2} &= \sin(\beta) \sin(\gamma) \cos(\gamma) \left(\mathcal{I}_\eta - \mathcal{I}_\xi\right) \\
g_{1,3} &= \mathcal{I}_\zeta \cos(\beta) \\
g_{2,2} &= \mathcal{I}_\eta \cos^2(\gamma) + \mathcal{I}_\xi \sin^2(\gamma) \\
g_{2,3} &= 0 \\
g_{3,3} &= \mathcal{I}_\zeta
\end{aligned} \qquad (1C.2)$$

The determinant of $\mathbb{g}$ is

$$g = \mathcal{I}_\zeta\ \mathcal{I}_\eta\ \mathcal{I}_\xi\ \sin^2(\beta) \qquad (1C.3)$$

The elements of the inverse metric tensor $\mathbb{g}^{-1}$ are

$$\begin{aligned}
g^{\{1,1\}} &= \frac{\csc^2(\beta) \left(\mathcal{I}_\eta \cos^2(\gamma) + \mathcal{I}_\xi \sin^2(\gamma)\right)}{\mathcal{I}_\eta\, \mathcal{I}_\xi} \\
g^{\{1,2\}} &= \frac{\csc(\beta) \sin(\gamma) \cos(\gamma) \left(\mathcal{I}_\xi - \mathcal{I}_\eta\right)}{\mathcal{I}_\eta\, \mathcal{I}_\xi} \\
g^{\{1,3\}} &= -\frac{\cot(\beta) \csc(\beta) \left(\mathcal{I}_\eta \cos^2(\gamma) + \mathcal{I}_\xi \sin^2(\gamma)\right)}{\mathcal{I}_\eta\, \mathcal{I}_\xi} \\
g^{\{2,2\}} &= \frac{\cos^2(\gamma)}{\mathcal{I}_\eta} + \frac{\sin^2(\gamma)}{\mathcal{I}_\xi} \\
g^{\{2,3\}} &= \frac{\cot(\beta) \sin(\gamma) \cos(\gamma) \left(\mathcal{I}_\eta - \mathcal{I}_\xi\right)}{\mathcal{I}_\eta\, \mathcal{I}_\xi} \\
g^{\{3,3\}} &= \frac{1}{\mathcal{I}_\zeta} + \frac{\cot^2(\beta) \sin^2(\gamma)}{\mathcal{I}_\eta} + \frac{\cot^2(\beta) \cos^2(\gamma)}{\mathcal{I}_\xi}
\end{aligned} \qquad (1C.4)$$

Elements of the Riemann connection (or Christopher symbol) (1.70) are

$$\begin{aligned}
\overline{\Gamma}_{1,1}{}^{1} &= \frac{\cos(\beta) \sin(\gamma) \cos(\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (-\mathcal{I}_\zeta + \mathcal{I}_\eta + \mathcal{I}_\xi)}{\mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{2,1}{}^{1} &= \frac{\cot(\beta) (-\mathcal{I}_\zeta + \mathcal{I}_\eta + \mathcal{I}_\xi) (\cos(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) + \mathcal{I}_\eta + \mathcal{I}_\xi)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{2,2}{}^{1} &= 0 \\
\overline{\Gamma}_{3,1}{}^{1} &= \frac{\sin(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (-\mathcal{I}_\zeta + \mathcal{I}_\eta + \mathcal{I}_\xi)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{3,2}{}^{1} &= -\frac{\csc(\beta) \left(\cos(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (\mathcal{I}_\zeta - \mathcal{I}_\eta - \mathcal{I}_\xi) + \mathcal{I}_\zeta (\mathcal{I}_\eta + \mathcal{I}_\xi) - (\mathcal{I}_\eta - \mathcal{I}_\xi)^2\right)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{3,3}{}^{1} &= 0 \\
\overline{\Gamma}_{1,1}{}^{2} &= -\frac{\sin(2\beta) (\cos(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (\mathcal{I}_\zeta - \mathcal{I}_\eta - \mathcal{I}_\xi) - \mathcal{I}_\zeta (\mathcal{I}_\eta + \mathcal{I}_\xi) + \mathcal{I}_\eta^2 + \mathcal{I}_\xi^2)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{2,1}{}^{2} &= -\frac{\cos(\beta) \sin(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (-\mathcal{I}_\zeta + \mathcal{I}_\eta + \mathcal{I}_\xi)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{2,2}{}^{2} &= 0 \\
\overline{\Gamma}_{3,1}{}^{2} &= -\frac{\sin(\beta) \left(\cos(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (\mathcal{I}_\zeta - \mathcal{I}_\eta - \mathcal{I}_\xi) - \mathcal{I}_\zeta (\mathcal{I}_\eta + \mathcal{I}_\xi) + (\mathcal{I}_\eta - \mathcal{I}_\xi)^2\right)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{3,2}{}^{2} &= -\frac{\sin(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (-\mathcal{I}_\zeta + \mathcal{I}_\eta + \mathcal{I}_\xi)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{3,3}{}^{2} &= 0 \\
\overline{\Gamma}_{1,1}{}^{3} &= \frac{\sin(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (\mathcal{I}_\zeta^2 + \cos(2\beta) (\mathcal{I}_\zeta - \mathcal{I}_\eta) (\mathcal{I}_\zeta - \mathcal{I}_\xi) - \mathcal{I}_\zeta (\mathcal{I}_\eta + \mathcal{I}_\xi) - \mathcal{I}_\eta\, \mathcal{I}_\xi)}{4\, \mathcal{I}_\zeta\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{2,1}{}^{3} &= \frac{\frac{1}{2} \mathcal{I}_\zeta \cos(\beta) \cot(\beta) (\mathcal{I}_\zeta - \mathcal{I}_\eta - \mathcal{I}_\xi) (\cos(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) + \mathcal{I}_\eta + \mathcal{I}_\xi) - \mathcal{I}_\eta\, \mathcal{I}_\xi \sin(\beta) (\mathcal{I}_\zeta + \cos(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi))}{2\, \mathcal{I}_\zeta\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{2,2}{}^{3} &= \frac{\sin(\gamma) \cos(\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi)}{\mathcal{I}_\zeta} \\
\overline{\Gamma}_{3,1}{}^{3} &= -\frac{\cos(\beta) \sin(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (-\mathcal{I}_\zeta + \mathcal{I}_\eta + \mathcal{I}_\xi)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{3,2}{}^{3} &= \frac{\cot(\beta) \left(\cos(2\gamma) (\mathcal{I}_\eta - \mathcal{I}_\xi) (\mathcal{I}_\zeta - \mathcal{I}_\eta - \mathcal{I}_\xi) + \mathcal{I}_\zeta (\mathcal{I}_\eta + \mathcal{I}_\xi) - (\mathcal{I}_\eta - \mathcal{I}_\xi)^2\right)}{4\, \mathcal{I}_\eta\, \mathcal{I}_\xi} \\
\overline{\Gamma}_{3,3}{}^{3} &= 0
\end{aligned}$$

(1C.5)

The curvature tensor is defined as [see I.D.Lawrie, “A Unified Grand Tour of Theoretical Physics”]

$$R_{ijk}{}^{m} = \frac{\partial}{\partial q^{j}} \Gamma_{ik}{}^{m} - \frac{\partial}{\partial q^{k}} \Gamma_{ij}{}^{m} + \Gamma_{nj}{}^{m} \Gamma_{ik}{}^{n} - \Gamma_{nk}{}^{m} \Gamma_{ij}{}^{n} \qquad (1C.5a)$$

Its contraction

$$\begin{aligned} R_{ik} &= R_{ijk}{}^{j} \\ &= \frac{\partial}{\partial q^{j}} \Gamma_{ik}{}^{j} - \frac{\partial}{\partial q^{k}} \Gamma_{ij}{}^{j} + \Gamma_{nj}{}^{j} \Gamma_{ik}{}^{n} - \Gamma_{nk}{}^{j} \Gamma_{ij}{}^{n} \end{aligned} \qquad (1C.5b)$$

is called the Ricci tensor.

For the asymmetric top,

$$\begin{aligned} R_{2,1} &= \frac{\sin(\beta) \sin(2\gamma) (I_\eta - I_\xi)(-I_\zeta + I_\eta + I_\xi)(I_\zeta + I_\eta + I_\xi)}{4 I_\zeta I_\eta I_\xi} \\ R_{2,2} &= \frac{(-I_\zeta + I_\eta + I_\xi)\left(\cos(2\gamma)(I_\eta - I_\xi)(I_\zeta + I_\eta + I_\xi) + I_\zeta (I_\eta + I_\xi) + (I_\eta - I_\xi)^2\right)}{4 I_\zeta I_\eta I_\xi} \\ R_{3,1} &= \frac{\cos(\beta)(I_\zeta + I_\eta - I_\xi)(I_\zeta - I_\eta + I_\xi)}{2 I_\eta I_\xi} \\ R_{3,2} &= 0 \\ R_{3,3} &= \frac{(I_\zeta + I_\eta - I_\xi)(I_\zeta - I_\eta + I_\xi)}{2 I_\eta I_\xi} \end{aligned} \qquad (1C.6)$$

where we’ve omitted R_{11} since *Mathematica* has failed to simplify it to a reasonably short form.

The Ricci curvature scalar is defined as

$$R = g^{ij} R_{ji} = R^{i}{}_{i} \qquad (1C.7a)$$

For the asymmetric top,

$$R = -\frac{I_\zeta^2 - 2 I_\zeta (I_\eta + I_\xi) + (I_\eta - I_\xi)^2}{2 I_\zeta I_\eta I_\xi} \qquad (1C.7)$$