2.1. Path Integral Representation of Time Evolution Amplitudes

The time evolution amplitude, or propagator, is defined as

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) \equiv \left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(t_b, t_a) \,\middle|\, \boldsymbol{x}_a \right\rangle \tag{2.1}$$

where $\hat{U}$ is the time displacement operator & $\boldsymbol{x}$ are Cartesian coordinates.

2.1.1. Sliced Time Evolution Amplitude

We shall deal only with the causal, or retarded, propagator defined as

$$\begin{aligned}(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) &\equiv \Theta(t_b - t_a)\left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(t_b, t_a) \,\middle|\, \boldsymbol{x}_a \right\rangle \\ &= \left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(t_b, t_a) \,\middle|\, \boldsymbol{x}_a \right\rangle \qquad \text{with} \quad t_b > t_a\end{aligned} \tag{1.299}$$

Using the group property

$$\hat{U}(t_b, t_a) = \hat{U}(t_b, t)\, \hat{U}(t, t_a) \qquad\qquad \forall\ t \in (t_a, t_b)$$

we can insert N points between t_a and t_b so that

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left\langle \boldsymbol{x}_b \,\middle|\, \hat{U}(t_b, t_N) \ldots \hat{U}(t_n, t_{n-1}) \ldots \hat{U}(t_1, t_a) \,\middle|\, \boldsymbol{x}_a \right\rangle \tag{2.2}$$

where

$$t_n = t_0 + n\,\epsilon \qquad\qquad \epsilon = \frac{t_b - t_a}{N+1} \qquad\qquad t_{N+1} = t_b t_0 = t_a \tag{2.2a}$$

Inserting the identity operators

$$\hat{1} = \int d\,\boldsymbol{x}_n \ \left| \boldsymbol{x}_n \right\rangle \left\langle \boldsymbol{x}_n \right| \qquad\qquad n = 1,\ \ldots,\ N \tag{2.3}$$

in front of each $\hat{U}(t_n, t_{n-1})$, we get

$$\begin{aligned}(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) &= \left(\prod_{k=1}^{N} \int d\,\boldsymbol{x}_k \right) \left\langle \boldsymbol{x}_{N+1} \,\middle|\, \hat{U}(t_{N+1}, t_N) \,\middle|\, \boldsymbol{x}_N \right\rangle \ldots \\ &\quad \times \left\langle \boldsymbol{x}_n \,\middle|\, \hat{U}(t_n, t_{n-1}) \ \middle|\, \boldsymbol{x}_{n-1} \right\rangle \ldots \left\langle \boldsymbol{x}_1 \,\middle|\, \hat{U}(t_1, t_0) \,\middle|\, \boldsymbol{x}_0 \right\rangle \\ &= \left(\prod_{k=1}^{N} \int d\,\boldsymbol{x}_k \right) \prod_{n=1}^{N+1} (\boldsymbol{x}_n,\, t_n \mid \boldsymbol{x}_{n-1}\, t_{n-1})\end{aligned} \tag{2.4}$$

which contains N integration over $N+1$ propagators.

Let

$$\hat{H}(t) \equiv H(\hat{\boldsymbol{x}},\, \hat{\boldsymbol{p}},\, t) \tag{2.6}$$

For $\epsilon \to 0$, we have

$$\begin{aligned}\hat{U}(t_n, t_{n-1}) &\approx \hat{1} + \frac{1}{i\hbar} \int_{t_{n-1}}^{t_n} d\,t\, \hat{H}(t) \\ &\approx \hat{1} - \frac{i}{\hbar}\, \epsilon\, \hat{H}(t_n) \\ &\approx e^{-i\epsilon \hat{H}(t_n)/\hbar}\end{aligned}$$

so that

$$(\boldsymbol{x}_n\, t_n \mid \boldsymbol{x}_{n-1}\, t_{n-1}) \approx \left\langle \boldsymbol{x}_n \,\middle|\, e^{-i\epsilon \hat{H}(t_n)/\hbar} \,\middle|\, \boldsymbol{x}_{n-1} \right\rangle \tag{2.5}$$

Assume now

$$H(\hat{\boldsymbol{x}},\, \hat{\boldsymbol{p}},\, t) = T(\hat{\boldsymbol{p}}) + V(\hat{\boldsymbol{x}},\, t) \tag{2.7}$$

then

$$e^{-i\epsilon\hat{H}(t_n)/\hbar} = e^{-i\epsilon\left[\hat{T}+\hat{V}(t_n)\right]/\hbar} \tag{2.8}$$

can be factored using the Baker-Campbell-Hausdorff formula (see App. 2A for proof):

$$e^{-i\epsilon\left(\hat{T}+\hat{V}\right)/\hbar} = e^{-i\epsilon\hat{V}/\hbar}\, e^{-i\epsilon\hat{T}/\hbar}\, e^{-i\epsilon^2\hat{X}/\hbar} \tag{2.9}$$

where

$$\hat{X} = \frac{i}{2}\left[\hat{V},\hat{T}\right] - \frac{\epsilon}{\hbar}\left(\frac{1}{6}\left[\hat{V},\left[\hat{V},\hat{T}\right]\right] - \frac{1}{3}\left[\left[\hat{V},\hat{T}\right],\hat{T}\right]\right) + O(\epsilon^2) \tag{2.10}$$

As $\epsilon\to 0$, we may drop the $e^{-i\epsilon^2\hat{X}/\hbar}$ term so that (2.5) becomes

$$\begin{aligned}
(\boldsymbol{x}_n t_n \mid \boldsymbol{x}_{n-1} t_{n-1}) &\approx \left\langle \boldsymbol{x}_n \left| e^{-i\epsilon V(\hat{\boldsymbol{x}},t_n)/\hbar} e^{-i\epsilon\hat{T}/\hbar} \right| \boldsymbol{x}_{n-1}\right\rangle \\
&= \int d\boldsymbol{x} \left\langle \boldsymbol{x}_n \left| e^{-i\epsilon V(\hat{\boldsymbol{x}},t_n)/\hbar} \right| \boldsymbol{x}\right\rangle \left\langle \boldsymbol{x} \left| e^{-i\epsilon\hat{T}/\hbar} \right| \boldsymbol{x}_{n-1}\right\rangle \\
&= \int d\boldsymbol{x}\, \delta(\boldsymbol{x}-\boldsymbol{x}_n)\, e^{-i\epsilon V(\boldsymbol{x},t_n)/\hbar} \left\langle \boldsymbol{x} \left| e^{-i\epsilon\hat{T}/\hbar} \right| \boldsymbol{x}_{n-1}\right\rangle \\
&= e^{-i\epsilon V(\boldsymbol{x}_n,t_n)/\hbar} \left\langle \boldsymbol{x}_n \left| e^{-i\epsilon\hat{T}/\hbar} \right| \boldsymbol{x}_{n-1}\right\rangle
\end{aligned} \tag{2.11a}$$

where

$$\langle \boldsymbol{x} \mid \boldsymbol{x}'\rangle = \delta(\boldsymbol{x}-\boldsymbol{x}') \tag{2.10a}$$

Using

$$\begin{aligned}
\left\langle \boldsymbol{x}_n \left| e^{-i\epsilon\hat{T}/\hbar} \right| \boldsymbol{x}_{n-1}\right\rangle &= \int \frac{d\boldsymbol{p}_n}{(2\pi\hbar)^3} \left\langle \boldsymbol{x}_n \left| e^{-i\epsilon\hat{T}/\hbar} \right| \boldsymbol{p}_n\right\rangle \langle \boldsymbol{p}_n \mid \boldsymbol{x}_{n-1}\rangle \\
&= \int \frac{d\boldsymbol{p}_n}{(2\pi\hbar)^3} e^{-i\epsilon T(\boldsymbol{p}_n)/\hbar} \langle \boldsymbol{x}_n \mid \boldsymbol{p}_n\rangle \langle \boldsymbol{p}_n \mid \boldsymbol{x}_{n-1}\rangle \\
&= \int \frac{d\boldsymbol{p}_n}{(2\pi\hbar)^3} e^{-i\epsilon T(\boldsymbol{p}_n)/\hbar} e^{i\boldsymbol{p}_n\cdot(\boldsymbol{x}_n-\boldsymbol{x}_{n-1})/\hbar}
\end{aligned} \tag{2.11b}$$

(2.11a) becomes

$$\begin{aligned}
&(\boldsymbol{x}_n t_n \mid \boldsymbol{x}_{n-1} t_{n-1}) \\
&\approx \int \frac{d\boldsymbol{p}_n}{(2\pi\hbar)^3} \exp\left(\frac{i}{\hbar}\boldsymbol{p}_n\cdot(\boldsymbol{x}_n-\boldsymbol{x}_{n-1}) - \frac{i}{\hbar}\epsilon\left[T(\boldsymbol{p}_n)+V(\boldsymbol{x}_n,t_n)\right]\right) \\
&= \int \frac{d\boldsymbol{p}_n}{(2\pi\hbar)^3} \exp\left\{\frac{i}{\hbar}\left[\boldsymbol{p}_n\cdot(\boldsymbol{x}_n-\boldsymbol{x}_{n-1}) - \epsilon H(\boldsymbol{p}_n,\boldsymbol{x}_n,t_n)\right]\right\}
\end{aligned} \tag{2.13}$$

in which all terms are ordinary functions.

Note that we've used the normalization

$$\langle \boldsymbol{x} \mid \boldsymbol{p}\rangle = e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar} \tag{2.13a}$$

The completeness (2.3) then implies

$$\langle \boldsymbol{p} \mid \boldsymbol{p}'\rangle = (2\pi\hbar)^3\,\delta(\boldsymbol{p}-\boldsymbol{p}') \tag{2.13b}$$

(2.4) thus becomes the Feynman's path integral formula:

$$(\boldsymbol{x}_b t_b \mid \boldsymbol{x}_a t_a) = \left(\prod_{k=1}^{N}\int d\boldsymbol{x}_k\right)\left(\prod_{n=1}^{N+1}\int\frac{d\boldsymbol{p}_n}{(2\pi\hbar)^3}\right)\exp\left(\frac{i}{\hbar}\mathcal{A}^N\right) \tag{2.14}$$

where

$$\mathcal{A}^N = \sum_{n=1}^{N+1}\left[\boldsymbol{p}_n\cdot(\boldsymbol{x}_n-\boldsymbol{x}_{n-1}) - \epsilon H(\boldsymbol{p}_n,\boldsymbol{x}_n,t_n)\right] \tag{2.15}$$

2.1.2. Zero-Hamiltonian Path Integral

Setting $H = 0$ turns (2.14) into

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left(\prod_{k=1}^{N} \int d\,\boldsymbol{x}_k \right) \left(\prod_{n=1}^{N+1} \int \frac{d\,\boldsymbol{p}_n}{(2\pi\hbar)^3} \right) \tag{2.16}$$

$$\times \exp\left(\frac{i}{\hbar} \sum_{n=1}^{N+1} \boldsymbol{p}_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) \right)$$

$$= \left(\prod_{k=1}^{N} \int d\,\boldsymbol{x}_k \right) \left[\prod_{n=1}^{N+1} \int \frac{d\,\boldsymbol{p}_n}{(2\pi\hbar)^3} \exp\left(\frac{i}{\hbar} \boldsymbol{p}_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) \right) \right]$$

$$= \left(\prod_{k=1}^{N} \int d\,\boldsymbol{x}_k \right) \left[\prod_{n=1}^{N+1} \delta(\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) \right]$$

$$= \delta(\boldsymbol{x}_{N+1} - \boldsymbol{x}_0)$$

$$= \delta(\boldsymbol{x}_b - \boldsymbol{x}_a) \tag{2.18a}$$

As $N \to \infty$, the discrete times t_n becomes a continuous variable t so that $\boldsymbol{x}_n$, $\boldsymbol{p}_n$ become $\boldsymbol{x}(t)$, $\boldsymbol{p}(t)$. Defining

$$\int \mathcal{D}'\boldsymbol{x} = \lim_{N\to\infty} \prod_{k=1}^{N} \int d\,\boldsymbol{x}_k$$

$$\int \frac{\mathcal{D}\,\boldsymbol{p}}{(2\pi\hbar)^3} = \lim_{N\to\infty} \prod_{n=1}^{N+1} \int \frac{d\,\boldsymbol{p}_n}{(2\pi\hbar)^3} \tag{2.16a}$$

(2.14) becomes

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \int \mathcal{D}'\boldsymbol{x} \int \frac{\mathcal{D}\,\boldsymbol{p}}{(2\pi\hbar)^3} \exp\left(\frac{i}{\hbar} \mathcal{A} \right) \tag{2.16b}$$

where

$$\mathcal{A} = \lim_{N\to\infty} \sum_{n=1}^{N+1} \epsilon \left[\boldsymbol{p}_n \cdot \left(\frac{\boldsymbol{x}_n - \boldsymbol{x}_{n-1}}{\epsilon} \right) - H(\boldsymbol{p}_n, \boldsymbol{x}_n, t_n) \right]$$

$$= \int_{t_a}^{t_b} d\,t \left[\boldsymbol{p}(t) \cdot \dot{\boldsymbol{x}}(t) - H(t) \right] \tag{2.16c}$$

Note that the ' in $\mathcal{D}'$ indicates the number of integrals is one less than that in $\mathcal{D}$.

(2.18a) can be written as

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \int \mathcal{D}'\boldsymbol{x} \int \frac{\mathcal{D}\,\boldsymbol{p}}{(2\pi\hbar)^3} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} d\,t\; \boldsymbol{p} \cdot \dot{\boldsymbol{x}} \right)$$

$$= \delta(\boldsymbol{x}_b - \boldsymbol{x}_a) \tag{2.18}$$

Alternatively, since

$$H = 0 \qquad \to \qquad \hat{U} = \hat{1}$$

(2.1) gives

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \langle \boldsymbol{x}_b \mid \boldsymbol{x}_a \rangle$$

in agreement with (2.18).

Consider now inserting a factor $\boldsymbol{p}_m$, with $N \geq m \geq 1$, into the integrand of (2.16):

$$\mathcal{I} = \left(\prod_{k=1}^{N} \int d\,\boldsymbol{x}_k \right) \left(\prod_{n=1}^{N+1} \int \frac{d\,\boldsymbol{p}_n}{(2\pi\hbar)^3} \right) \boldsymbol{p}_m \exp\left(\frac{i}{\hbar} \sum_{n=1}^{N+1} \boldsymbol{p}_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) \right)$$

$$= \left(\prod_{k=1}^{N} \int d\,\boldsymbol{x}_k \right) \left(\prod_{n=1}^{N+1} \int \frac{d\,\boldsymbol{p}_n}{(2\pi\hbar)^3} \right) \frac{\hbar}{i} \frac{\partial}{\partial\,\boldsymbol{x}_m} \exp\left(\frac{i}{\hbar} \sum_{n=1}^{N+1} \boldsymbol{p}_n \cdot (\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) \right)$$

$$= \left(\prod_{k=1}^{N} \int d\, \mathbf{x}_k \right) \left(\prod_{n=1}^{N+1} \int \frac{d\, \mathbf{p}_n}{(2\pi\hbar)^3} \right) \hat{\mathbf{p}}_m \exp\left(\frac{i}{\hbar} \sum_{n=1}^{N+1} \mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) \right) \tag{2.19a}$$

where

$$\hat{\mathbf{p}}_m = \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_m}$$

In the continuum limit, we have

$$\hat{\mathbf{p}}(t) = \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}(t)} \tag{2.19b}$$

and

$$\int \mathcal{D}' \mathbf{x} \int \frac{\mathcal{D}\mathbf{p}}{(2\pi\hbar)^3} f[\mathbf{p}(t')] \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \mathbf{p} \cdot \dot{\mathbf{x}} \right)$$
$$= \int \mathcal{D}' \mathbf{x}\; f\left[\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}(t')} \right] \int \frac{\mathcal{D}\mathbf{p}}{(2\pi\hbar)^3} \exp\left(\frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \mathbf{p} \cdot \dot{\mathbf{x}} \right) \tag{2.19c}$$

which will be used in deriving the Schrodinger eq. from the path integral.

(2.19b) implies the equal-time commutation relations

$$[\, x_a(t),\ \hat{p}_b(t)\,] = i\hbar\, \delta_{ab} \tag{2.19}$$

where a, b denote Cartesian components.

2.1.3. Schrodinger Equation for Time Evolution Amplitude

By inserting only one time-point t_N, one gets

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int d\, \mathbf{x}_N\, (\mathbf{x}_b\, t_b \mid \mathbf{x}_N\, t_N)\, (\mathbf{x}_N\, t_N \mid \mathbf{x}_a\, t_a) \tag{2.20}$$

where, according to (2.13),

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_N\, t_N) \approx \int \frac{d\, \mathbf{p}_b}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar} \left(\mathbf{p}_b \cdot (\mathbf{x}_b - \mathbf{x}_N) - \epsilon H(\mathbf{x}_b,\ \mathbf{p}_b,\ t_b) \right) \right] \tag{2.21}$$

Using (2.19c), we have

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_N\, t_N) \approx \exp\left[-\frac{i}{\hbar} \epsilon H\left(\mathbf{x}_b,\ \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b},\ t_b \right) \right] \int \frac{d\, \mathbf{p}_b}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar} \mathbf{p}_b \cdot (\mathbf{x}_b - \mathbf{x}_N) \right]$$
$$= \exp\left[-\frac{i}{\hbar} \epsilon H\left(\mathbf{x}_b,\ \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b},\ t_b \right) \right] \delta(\mathbf{x}_b - \mathbf{x}_N) \tag{2.22}$$

(2.20) then becomes, with $t_N = t_b - \epsilon$,

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \exp\left[-\frac{i}{\hbar} \epsilon H\left(\mathbf{x}_b,\ \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b},\ t_b \right) \right] \int d\, \mathbf{x}_N\, \delta(\mathbf{x}_b - \mathbf{x}_N)\, (\mathbf{x}_N,\ t_N \mid \mathbf{x}_a\, t_a)$$
$$= \exp\left[-\frac{i}{\hbar} \epsilon H\left(\mathbf{x}_b,\ \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b},\ t_b \right) \right] (\mathbf{x}_b,\ t_b - \epsilon \mid \mathbf{x}_a\, t_a) \tag{2.23}$$
$$\approx \left[1 - \frac{i}{\hbar} \epsilon H\left(\mathbf{x}_b,\ \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b},\ t_b \right) \right] (\mathbf{x}_b,\ t_b - \epsilon \mid \mathbf{x}_a\, t_a)$$
$$= (\mathbf{x}_b,\ t_b - \epsilon \mid \mathbf{x}_a\, t_a) - \frac{i}{\hbar} \epsilon H\left(\mathbf{x}_b,\ \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b},\ t_b \right)] (\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) + O(\epsilon^2)$$

Hence,

$$\frac{(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) - (\mathbf{x}_b,\ t_b - \epsilon \mid \mathbf{x}_a\, t_a)}{\epsilon} \approx -\frac{i}{\hbar} H\left(\mathbf{x}_b,\ \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b},\ t_b \right) (\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a)$$

$$\rightarrow \qquad i\hbar \frac{\partial}{\partial t_b} (\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = H\left(\mathbf{x}_b, \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b}, t_b\right) (\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) \qquad (2.25)$$

$$i\hbar \frac{\partial}{\partial t_b} \left\langle \mathbf{x}_b \,\middle|\, \hat{U}(t_b, t_a) \,\middle|\, \mathbf{x}_a \right\rangle = H\left(\mathbf{x}_b, \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_b}, t_b\right) \left\langle \mathbf{x}_b \,\middle|\, \hat{U}(t_b, t_a) \,\middle|\, \mathbf{x}_a \right\rangle$$

Since (2.25) is true for all $|\, \mathbf{x}_a \rangle$, we can replace it with any state $|\, \psi \rangle$ so that

$$i\hbar \frac{\partial}{\partial t} \left\langle \mathbf{x} \,\middle|\, \hat{U}(t, 0) \,\middle|\, \psi \right\rangle = H\left(\mathbf{x}, \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}}, t\right) \left\langle \mathbf{x} \,\middle|\, \hat{U}(t, 0) \,\middle|\, \psi \right\rangle$$

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = H\left(\mathbf{x}, \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}}, t\right) \psi(\mathbf{x}, t)$$

which is just the Schrodinger equation (1.297).

2.1.4. Convergence of the Time-Sliced Evolution Amplitude

Dividing the interval (t_b, t_a) into $N + 1$ segments of length $\epsilon = \dfrac{t_b - t_a}{N+1}$, we have

$$e^{-i(t_b - t_a)\hat{H}/\hbar} = \left(e^{-i\epsilon \hat{H}/\hbar}\right)^{N+1}$$

$$= \left(e^{-i\epsilon(\hat{T} + \hat{V})/\hbar}\right)^{N+1} \qquad \hat{H} = \hat{T} + \hat{V}$$

$$= \left(e^{-i\epsilon \hat{T}/\hbar}\, e^{-i\epsilon \hat{V}/\hbar}\, e^{-i\epsilon^2 \hat{X}/\hbar}\right)^{N+1} \qquad [\text{ (2.9) used. }]$$

As $N \to \infty$, $\epsilon \to 0$ and it seems reasonable to drop the ϵ^2 term and get

$$e^{-i(t_b - t_a)\hat{H}/\hbar} = \lim_{N \to \infty} \left(e^{-i\epsilon \hat{T}/\hbar}\, e^{-i\epsilon \hat{V}/\hbar}\right)^{N+1} \qquad (2.26)$$

which is known as the Trotter product formula. Rigorous proof of (2.26) requires functional analysis too technical to be presented here [see reference list at the end of Chapter]. We'll state without proof that (2.26) is valid for operators that are bounded below.

The time-sliced path integral (2.14) was derived using (2.26) in §2.1.1. Unfortunately, the technique fails for many potentials of interest, e.g., the Coulomb potential and the centrifugal barrier. Since $\hat{X}$ is too singular to be ignored, new technique is required to define the path integral [see Chapter 12].

For the time being, we shall assume (2.14) is valid. The relevant formulae are listed below for later convenience.

$$\lim_{N \to \infty} \prod_{k=1}^{N} \int d\mathbf{x}_k = \int_{x(t_a) = x_a}^{x(t_b) = x_b} \mathcal{D}'\mathbf{x} \qquad (2.28a)$$

$$\int \frac{\mathcal{D}\mathbf{p}}{(2\pi\hbar)^3} = \lim_{N \to \infty} \prod_{n=1}^{N+1} \int \frac{d\mathbf{p}_n}{(2\pi\hbar)^3} \qquad (2.28)$$

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int_{\mathbf{x}(t_a) = \mathbf{x}_a}^{\mathbf{x}(t_b) = \mathbf{x}_b} \mathcal{D}'\mathbf{x} \int \frac{\mathcal{D}\mathbf{p}}{(2\pi\hbar)^3} \exp\left(\frac{i}{\hbar} \mathcal{A}\right) \qquad (2.29)$$

$$\mathcal{A} = \int_{t_a}^{t_b} dt \left\{ \mathbf{p}(t) \cdot \dot{\mathbf{x}}(t) - H[\,\mathbf{x}(t), \mathbf{p}(t), t\,] \right\} \qquad (2.27)$$

A more vivid interpretation of the time-sliced approach is that

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \sum_{\substack{\text{all histories} \\ (\mathbf{x}_a, t_a) \to (\mathbf{x}_b, t_b)}} \exp\left(\frac{i}{\hbar} \mathcal{A}\right) \qquad (2.30)$$

where $\exp\left(\frac{i}{\hbar} \mathcal{A}\right)$ acts like a Boltzmann factor in statistical mechanics.

2.1.5. Time Evolution Amplitude in Momentum Space

Working in momentum space, we have

$$(\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a) = \left\langle \mathbf{p}_b \,\middle|\, \hat{U}(t_b, t_a) \,\middle|\, \mathbf{p}_a \right\rangle \tag{2.31}$$

The time-slicing procedure follows exactly that of §2.1.1 but with $\boldsymbol{x}$ and $\boldsymbol{p}$ switching places. We need therefore point out the difference between the key equations.

Thus, (2.3) is replaced by

$$\hat{1} = \int \frac{d\,\mathbf{p}_n}{(2\pi\hbar)^3} \mid \mathbf{p}_n \rangle\langle \mathbf{p}_n \mid \qquad n = 1, \ldots, N \tag{2.32}$$

which follows (2.13a) and (2.10a).

(2.4) by

$$(\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a) = \left(\prod_{k=1}^{N} \int \frac{d\,\mathbf{p}_k}{(2\pi\hbar)^3} \right) \prod_{n=1}^{N+1} (\mathbf{p}_n,\, t_n \mid \mathbf{p}_{n-1}\, t_{n-1}) \tag{2.32a}$$

(2.11a) by

$$(\mathbf{p}_n,\, t_n \mid \mathbf{p}_{n-1}\, t_{n-1}) \approx e^{-i\epsilon T(\mathbf{p}_{n-1})/\hbar} \left\langle \mathbf{p}_n \,\middle|\, e^{-i\epsilon V(\hat{\mathbf{x}},\, t_{n-1})/\hbar} \,\middle|\, \mathbf{p}_{n-1} \right\rangle \tag{2.32b}$$

(2.11b) by

$$\left\langle \mathbf{p}_n \,\middle|\, e^{-i\epsilon V(\hat{\mathbf{x}},\, t_n)/\hbar} \,\middle|\, \mathbf{p}_{n-1} \right\rangle = \int d\,\mathbf{x}_{n-1}\, e^{-i\epsilon V(\mathbf{x}_{n-1},\, t_n)/\hbar}\, e^{-i\mathbf{x}_{n-1}\cdot(\mathbf{p}_n - \mathbf{p}_{n-1})/\hbar} \tag{2.32c}$$

(2.13) by

$$\begin{aligned} &(\mathbf{p}_n,\, t_n \mid \mathbf{p}_{n-1}\, t_{n-1}) \\ \approx &\int d\,x_{n-1} \exp\left\{ \frac{i}{\hbar} \left(-\mathbf{x}_{n-1}\cdot(\mathbf{p}_n - \mathbf{p}_{n-1}) - \epsilon H(\mathbf{x}_{n-1},\, \mathbf{p}_{n-1},\, t_{n-1}) \right) \right\} \end{aligned} \tag{2.32d}$$

(2.14) by

$$(\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a) = \left(\prod_{k=1}^{N} \int \frac{d\,\mathbf{p}_k}{(2\pi\hbar)^3} \right) \left(\prod_{n=0}^{N} \int d\,\mathbf{x}_n \right) \exp\left(\frac{i}{\hbar} \overline{\mathcal{A}}^N \right) \tag{2.34}$$

(2.15) by

$$\overline{\mathcal{A}}^N = \sum_{n=0}^{N} \left[-\mathbf{x}_n\cdot(\mathbf{p}_{n+1} - \mathbf{p}_n) - \epsilon H(\mathbf{x}_n,\, \mathbf{p}_n,\, t_n) \right] \tag{2.34a}$$

The relation between (2.14) and (2.34) can be obtained as follows. Consider the sum in (2.15):

$$\begin{aligned} &\sum_{n=1}^{N+1} \mathbf{p}_n\cdot(\mathbf{x}_n - \mathbf{x}_{n-1}) \\ &= \mathbf{p}_{N+1}\cdot(\mathbf{x}_{N+1} - \mathbf{x}_N) + \mathbf{p}_N\cdot(\mathbf{x}_N - \mathbf{x}_{N-1}) + \ldots + \mathbf{p}_2\cdot(\mathbf{x}_2 - \mathbf{x}_1) + \mathbf{p}_1\cdot(\mathbf{x}_1 - \mathbf{x}_0) \\ &= \mathbf{p}_{N+1}\cdot\mathbf{x}_{N+1} - (\mathbf{p}_{N+1} - \mathbf{p}_N)\cdot\mathbf{x}_N - \ldots - (\mathbf{p}_2 - \mathbf{p}_1)\cdot\mathbf{x}_1 - \mathbf{p}_1\cdot\mathbf{x}_0 \\ &= \mathbf{p}_{N+1}\cdot\mathbf{x}_{N+1} - \mathbf{p}_0\cdot\mathbf{x}_0 - \sum_{n=0}^{N} (\mathbf{p}_{n+1} - \mathbf{p}_n)\cdot\mathbf{x}_n \end{aligned} \tag{2.35}$$

(2.15) thus becomes

$$\mathcal{A}^N = \mathbf{p}_{N+1}\cdot\mathbf{x}_{N+1} - \mathbf{p}_0\cdot\mathbf{x}_0 + \overline{\mathcal{A}}^N \tag{2.35a}$$

and (2.14):

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \left(\prod_{k=1}^{N} \int d\,\mathbf{x}_k \right) \left(\prod_{n=1}^{N+1} \int \frac{d\,\mathbf{p}_n}{(2\pi\hbar)^3} \right) e^{i(\mathbf{p}_{N+1}\cdot\mathbf{x}_{N+1} - \mathbf{p}_0\cdot\mathbf{x}_0)/\hbar} \exp\left(\frac{i}{\hbar} \overline{\mathcal{A}}^N \right) \tag{2.35b}$$

Comparing with (2.34), we have

$$\int d\mathbf{x}_0\, e^{i\,\mathbf{p}_0\cdot\mathbf{x}_0/\hbar}\,(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int \frac{d\,\mathbf{p}_{N+1}}{(2\pi\hbar)^3}\, e^{i\,\mathbf{p}_{N+1}\cdot\mathbf{x}_{N+1}/\hbar}\,(\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a)$$

or

$$\int d\mathbf{x}_a\, e^{i\,\mathbf{p}_a\cdot\mathbf{x}_a/\hbar}\,(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int \frac{d\,\mathbf{p}_b}{(2\pi\hbar)^3}\, e^{i\,\mathbf{p}_b\cdot\mathbf{x}_b/\hbar}\,(\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a) \qquad (2.36a)$$

Taking the inverse transforms, we have

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int \frac{d\,\mathbf{p}_a}{(2\pi\hbar)^3}\, e^{-i\,\mathbf{p}_a\cdot\mathbf{x}_a/\hbar} \int \frac{d\,\mathbf{p}_b}{(2\pi\hbar)^3}\, e^{i\,\mathbf{p}_b\cdot\mathbf{x}_b/\hbar}\,(\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a) \qquad (2.36)$$

$$(\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a) = \int d\mathbf{x}_b\, e^{-i\,\mathbf{p}_b\cdot\mathbf{x}_b/\hbar} \int d\mathbf{x}_a\, e^{i\,\mathbf{p}_a\cdot\mathbf{x}_a/\hbar}\,(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) \qquad (2.37)$$

In the continuum limit, (2.35a-b & 2.34a) become

$$(\mathbf{p}_b\, t_b \mid \mathbf{p}_a\, t_a) = \int_{\mathbf{p}(t_a)=\mathbf{p}_a}^{\mathbf{p}(t_b)=\mathbf{p}_b} \frac{\mathcal{D}'\mathbf{p}}{(2\pi\hbar)^3} \int \mathcal{D}\mathbf{x}\, \exp\left(\frac{i}{\hbar}\overline{\mathcal{A}}\right) \qquad (2.38)$$

$$\overline{\mathcal{A}} = \mathcal{A} - \mathbf{p}_b\cdot\mathbf{x}_b + \mathbf{p}_a\cdot\mathbf{x}_a$$

$$= \int_{t_a}^{t_b} dt\left\{-\mathbf{x}(t)\cdot\dot{\mathbf{p}}(t) - H[\mathbf{x}(t), \mathbf{p}(t), t]\right\} \qquad (2.39)$$

Reminder: The ' in $\mathcal{D}'$ indicates the number of integrals is one less than that in $\mathcal{D}$.

2.1.6. Quantum-Mechanical Partition Function

With $\hbar\beta = i(t_b - t_a)$, the quantum partition function of §1.17

$$Z_{\text{QM}}(t_b, t_a) = \text{Tr}\, e^{-i(t_b-t_a)\hat{H}/\hbar} \qquad (2.40)$$

can be written as a path integral symmetric in $\mathbf{x}$ & $\mathbf{p}$.

In the local basis $|\,\mathbf{x}\rangle$, (2.40) becomes

$$Z_{\text{QM}}(t_b, t_a) = \int d\mathbf{x}\left\langle \mathbf{x}\,\middle|\, e^{-i(t_b-t_a)\hat{H}/\hbar}\,\middle|\,\mathbf{x}\right\rangle$$

$$= \int d\mathbf{x}_b\,(\mathbf{x}_b\, t_b \mid \mathbf{x}_b\, t_a) \qquad (2.41)$$

$$= \left(\prod_{k=1}^{N+1}\int_{\mathbf{x}_{N+1}=\mathbf{x}_0} d\mathbf{x}_k\right)\left(\prod_{n=1}^{N+1}\int\frac{d\,\mathbf{p}_n}{(2\pi\hbar)^3}\right)\exp\left(\frac{i}{\hbar}\mathcal{A}^N\right) \quad [\text{ (2.14) used. }]$$

$$= \oint \mathcal{D}\mathbf{x} \int \frac{\mathcal{D}\mathbf{p}}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\mathcal{A}\right) \qquad (2.45a)$$

where the circle in $\oint$ denotes the identity, $\mathbf{x}_b = \mathbf{x}_a$, of the end points. Note also that the number of integrals in $\mathcal{D}\mathbf{x}$ & $\mathcal{D}\mathbf{p}$ are now equal.

Alternatively, in the basis $|\,\mathbf{p}\rangle$, we have

$$Z_{\text{QM}}(t_b, t_a) = \int \frac{d\,\mathbf{p}_b}{(2\pi\hbar)^3}\,(\mathbf{p}_b\, t_b \mid \mathbf{p}_b\, t_a)$$

$$= \int \mathcal{D}\mathbf{x} \oint \frac{\mathcal{D}\mathbf{p}}{(2\pi\hbar)^3}\exp\left(\frac{i}{\hbar}\overline{\mathcal{A}}\right) \qquad (2.46)$$

Comparing with (2.45a), we have

$$\mathcal{A}\Big|_{\mathbf{x}_b=\mathbf{x}_a} = \overline{\mathcal{A}}\Big|_{\mathbf{p}_b=\mathbf{p}_a}$$

or, using (2.16c) & (2.34a),

$$\sum_{n=1}^{N+1} \mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) \Big|_{\mathbf{x}_{N+1}=\mathbf{x}_0} = -\sum_{n=0}^{N} \mathbf{x}_n \cdot (\mathbf{p}_{n+1} - \mathbf{p}_n) \Big|_{\mathbf{p}_{N+1}=\mathbf{p}_0} \tag{2.47}$$

Using (2.35), we have

$$L.H.S. = (\mathbf{p}_{N+1} - \mathbf{p}_0) \cdot \mathbf{x}_0 - \sum_{n=0}^{N} (\mathbf{p}_{n+1} - \mathbf{p}_n) \cdot \mathbf{x}_n$$

Setting $\mathbf{p}_{N+1} = \mathbf{p}_0$ then gives us the $R.H.S.$

See Kleinert's comment in the last paragraph.

2.1.7. Feynman's Configuration Space Path Integral

In Feynman's original derivation of the path integral, he started with the Hamiltonian

$$H = \frac{1}{2M} \mathbf{p}^2 + V(\mathbf{x}, t) \tag{2.48}$$

The time-sliced action (2.15) thus becomes

$$\mathcal{A}^N = \sum_{n=1}^{N+1} \left[\mathbf{p}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \epsilon \frac{\mathbf{p}_n^2}{2M} - \epsilon V(\mathbf{x}_n, t_n) \right] \tag{2.49}$$

$$= \sum_{n=1}^{N+1} \epsilon \left[-\frac{1}{2M} \left(\mathbf{p}_n - M \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\epsilon} \right)^2 + \frac{M}{2} \left(\frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\epsilon} \right)^2 - V(\mathbf{x}_n, t_n) \right] \tag{2.50}$$

Using the Fresnel formula (1.333), we have

$$\int \frac{d\,\mathbf{p}_n}{(2\pi\hbar)^3} \exp\left[-\frac{i\epsilon}{2M\hbar} \left(\mathbf{p}_n - M \frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\epsilon} \right)^2 \right] = \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{3/2} \tag{2.51}$$

(2.14) thus becomes

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{3(N+1)/2} \left(\prod_{k=1}^{N} \int d\,\mathbf{x}_k \right) \exp\left(\frac{i}{\hbar} \mathbb{A}^N \right) \tag{2.52}$$

where

$$\mathbb{A}^N = \epsilon \sum_{n=1}^{N+1} \left[\frac{1}{2} M \left(\frac{\mathbf{x}_n - \mathbf{x}_{n-1}}{\epsilon} \right)^2 - V(\mathbf{x}_n, t_n) \right] \tag{2.53}$$

$$\to \qquad \mathbb{A}[\mathbf{x}] = \lim_{N\to\infty} \mathbb{A}^N = \int_{t_a}^{t_b} d\,t \left[\frac{1}{2} M \dot{\mathbf{x}}^2 - V(\mathbf{x}, t) \right] \tag{2.54}$$

$$= \int_{t_a}^{t_b} d\,t\, L$$

where L is the classical Lagrangian, which makes $\mathbb{A}$ the classical action.

Since only coordinates $\mathbf{x}_k$ appear in (2.52), it is called the configuration space path integral.

To derive the Schrodinger, we start again with (2.20)

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_a\, t_a) = \int d\,\mathbf{x}_N (\mathbf{x}_b\, t_b \mid \mathbf{x}_N\, t_N) (\mathbf{x}_N\, t_N \mid \mathbf{x}_a\, t_a)$$

$$= \int d\,\Delta\mathbf{x}\, (\mathbf{x}_b\, t_b \mid \mathbf{x}_b - \Delta\,\mathbf{x},\, t_b - \epsilon) (\mathbf{x}_b - \Delta\,\mathbf{x},\, t_b - \epsilon \mid \mathbf{x}_a\, t_a) \tag{2.55}$$

Using (2.51), we have

$$(\mathbf{x}_b\, t_b \mid \mathbf{x}_b - \Delta\,\mathbf{x},\, t_b - \epsilon) \approx \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{3/2} \exp\left\{ \epsilon \frac{i}{\hbar} \left[\frac{1}{2} M \left(\frac{\Delta\,\mathbf{x}}{\epsilon} \right)^2 - V(\mathbf{x}_b, t_b) \right] \right\} \tag{2.56}$$

Taylor expansion gives

$$(\boldsymbol{x}_b - \Delta \boldsymbol{x},\, t_b - \epsilon \mid \boldsymbol{x}_a\, t_a) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\Delta \boldsymbol{x} \cdot \frac{\partial}{\partial \boldsymbol{x}_b} \right)^k (\boldsymbol{x}_b,\, t_b - \epsilon \mid \boldsymbol{x}_a\, t_a) \qquad (2.57)$$

(2.55) thus becomes

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{3/2} \exp\left[-\epsilon \frac{i}{\hbar} V(\boldsymbol{x}_b,\, t_b) \right]$$
$$\times \sum_{k=0}^{\infty} \frac{1}{k!} \int d\,\Delta \boldsymbol{x} \, \exp\left[\epsilon \frac{iM}{2\hbar} \left(\frac{\Delta \boldsymbol{x}}{\epsilon} \right)^2 \right] \left(-\Delta \boldsymbol{x} \cdot \frac{\partial}{\partial \boldsymbol{x}_b} \right)^k (\boldsymbol{x}_b,\, t_b - \epsilon \mid \boldsymbol{x}_a\, t_a)$$

Using

$$\left(\frac{M}{2\pi i \hbar \epsilon} \right)^{3/2} \int d\,\Delta \boldsymbol{x} \, \exp\left[\epsilon \frac{iM}{2\hbar} \left(\frac{\Delta \boldsymbol{x}}{\epsilon} \right)^2 \right] (-\Delta \boldsymbol{x} \cdot \boldsymbol{a})^k$$
$$= \begin{cases} \frac{1}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) \left(\frac{2 i \hbar \epsilon}{M} \right)^n \boldsymbol{a}^{2n} & \text{for } k = 2n \\ 0 & \text{for } k = 2n+1 \end{cases} \qquad (2.58)$$

we have

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a)$$
$$= \exp\left[-\epsilon \frac{i}{\hbar} V(\boldsymbol{x}_b,\, t_b) \right] \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}\, (2n)!} \left(\frac{2 i \hbar \epsilon}{M} \right)^n \frac{\partial^{2n}}{\partial \boldsymbol{x}_b^{2n}} (\boldsymbol{x}_b,\, t_b - \epsilon \mid \boldsymbol{x}_a\, t_a)$$
$$= \left[1 - \epsilon \frac{i}{\hbar} V(\boldsymbol{x}_b,\, t_b) + \ldots \right] \left[1 + \frac{1}{2} \left(\frac{i \hbar \epsilon}{M} \right) \frac{\partial^2}{\partial \boldsymbol{x}_b^2} + \ldots \right] (\boldsymbol{x}_b,\, t_b - \epsilon \mid \boldsymbol{x}_a\, t_a)$$
$$= (\boldsymbol{x}_b,\, t_b - \epsilon \mid \boldsymbol{x}_a\, t_a) - \epsilon \frac{i}{\hbar} \left[V(\boldsymbol{x}_b,\, t_b) - \frac{\hbar^2}{2M} \frac{\partial^2}{\partial \boldsymbol{x}_b^2} \right] (\boldsymbol{x}_b,\, t_b \mid \boldsymbol{x}_a\, t_a) + O\left(\epsilon^2\right)$$

$$\rightarrow \qquad \frac{\partial}{\partial t_b} (\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \lim_{\epsilon \to 0} \frac{(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) - (\boldsymbol{x}_b,\, t_b - \epsilon \mid \boldsymbol{x}_a\, t_a)}{\epsilon}$$
$$= -\frac{i}{\hbar} \left[V(\boldsymbol{x}_b,\, t_b) - \frac{\hbar^2}{2M} \frac{\partial^2}{\partial \boldsymbol{x}_b^2} \right] (\boldsymbol{x}_b,\, t_b \mid \boldsymbol{x}_a\, t_a)$$
$$= -\frac{i}{\hbar} H\left(\boldsymbol{x}_b,\, \hat{\boldsymbol{p}}_b,\, t_b\right) (\boldsymbol{x}_b,\, t_b \mid \boldsymbol{x}_a\, t_a) \qquad (2.59a)$$

which, as shown in §2.1.3, is equivalent to the Schrodinger equation.

In the continuum limit, (2.52) becomes

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \int_{\boldsymbol{x}(t_a) = \boldsymbol{x}_a}^{\boldsymbol{x}(t_b) = \boldsymbol{x}_b} \mathcal{D} \boldsymbol{x} \, \exp\left(\frac{i}{\hbar} \mathbb{A}[\boldsymbol{x}] \right) \qquad (2.60)$$

with $\mathbb{A}[\boldsymbol{x}]$ given by (2.54) and

$$\int_{\boldsymbol{x}(t_a) = \boldsymbol{x}_a}^{\boldsymbol{x}(t_b) = \boldsymbol{x}_b} \mathcal{D} \boldsymbol{x} = \lim_{N \to \infty} \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{3(N+1)/2} \left(\prod_{k=1}^{N} \int d\, \boldsymbol{x}_k \right) \qquad (2.60a)$$
$$= \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{3/2} \lim_{N \to \infty} \left(\prod_{k=1}^{N} \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{3/2} \int d\, \boldsymbol{x}_k \right)$$

Similarly, for the quantum partition function, we have

$$Z_{\mathrm{QM}} = \oint \mathcal{D} \boldsymbol{x} \, \exp\left(\frac{i}{\hbar} \mathbb{A}[\boldsymbol{x}] \right) \qquad (2.61)$$

where

$$\oint \mathcal{D}\boldsymbol{x} = \lim_{N\to\infty}\left(\prod_{k=1}^{N+1}\left(\frac{M}{2\pi i\hbar\epsilon}\right)^{3/2}\int d\boldsymbol{x}_k\right)_{\boldsymbol{x}_b=\boldsymbol{x}_a} \tag{2.62}$$

In case one wishes to use time-slices of unequal lengths

$$\epsilon_n = t_n - t_{n-1} \tag{2.63}$$

(2.52-3) become

$$(\boldsymbol{x}_b\, t_b \mid \boldsymbol{x}_a\, t_a) = \left(\frac{M}{2\pi i\hbar\epsilon_b}\right)^{3/2}\left(\prod_{k=1}^{N}\left(\frac{M}{2\pi i\hbar\epsilon_k}\right)^{3N/2}\int d\boldsymbol{x}_k\right)\exp\left(\frac{i}{\hbar}\mathbb{A}^N\right) \tag{2.64}$$

where

$$\mathbb{A}^N = \sum_{n=1}^{N+1}\epsilon_n\left[\frac{1}{2}M\left(\frac{\boldsymbol{x}_n-\boldsymbol{x}_{n-1}}{\epsilon_n}\right)^2 - V(\boldsymbol{x}_n, t_n)\right] \tag{2.64a}$$

Obviously, they give the same continuum limits (2.60-a) as before.

Finally, we point out that it is possible to define the amplitude as the solution of the Schrodinger equation

$$\left[i\hbar\frac{\partial}{\partial t} - H\left(\boldsymbol{x}, \frac{\hbar}{i}\partial_{\boldsymbol{x}}\right)\right](\boldsymbol{x}\, t \mid \boldsymbol{x}_a\, t_a) = i\hbar\,\delta(t-t_a)\,\delta(\boldsymbol{x}-\boldsymbol{x}_a) \tag{2.65}$$

with the appropriate initial (retarded) conditions.

Let

$$\hat{H}\,\psi_n(\boldsymbol{x}) = E_n\,\psi_n(\boldsymbol{x}) \tag{2.65a}$$

then it is straightforward to show that

$$(\boldsymbol{x}\, t \mid \boldsymbol{x}_a\, t_a) = \Theta(t_b - t_a)\sum_n \psi_n(\boldsymbol{x}_b)\,\psi_n^*(\boldsymbol{x}_a)\,e^{-iE_n(t_b-t_a)/\hbar} \tag{2.66}$$

However, the path integral approach provides much more physical insights. See Kleinert's comments in the last paragraph of the section in the text.